

Research Article

Shrinking Projection Methods for Split Common Fixed-Point Problems in Hilbert Spaces

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Inspired by Moudafi (2011) and Takahashi et al. (2008), we present the shrinking projection method for the split common fixed-point problem in Hilbert spaces, and we obtain the strong convergence theorem. As a special case, the split feasibility problem is also considered.

1. Introduction

Let C and Q be nonempty closed convex sets in real Hilbert spaces H_1 and H_2 , respectively. The split feasibility problem (SFP) is to find

$$x \in C, \quad \text{such that } Ax \in Q, \quad (1)$$

where $A : H_1 \rightarrow H_2$ is a bounded linear operator. We use Φ to denote the solution set of the SFP (1). The SFP in finite-dimensional Hilbert space was first introduced by Censor and Elfving [1]. In 2010, Xu [2] considered the SFP in the setting of infinite-dimensional Hilbert space. The SFP has received much attention due to its wide applications in signal processing, image reconstruction, intensity-modulated radiation therapy, and so on (see [3–6]). Several iterative methods can be used to solve the SFP (1). Censor and Elfving [1] constructed the iterative process which involves the computation of the inverse of a matrix. A more popular algorithm that solves the SFP is the CQ algorithm of Byrne [3, 4]; that is, let x_0 be an arbitrary point in H_1 :

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \quad (2)$$

where $\gamma > 0$ is a parameter and P_C and P_Q are metric projections onto C and Q , respectively.

Let K be a nonempty closed convex subset of a real Hilbert space H and let $T : K \rightarrow K$ be a mapping. We denote by $\text{Fix}(T)$ the fixed-point set of T ; that is, $\text{Fix}(T) =$

$\{x \in K : Tx = x\}$. A mapping $T : K \rightarrow K$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. A mapping $T : K \rightarrow K$ is quasinonexpansive if $\text{Fix}(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in K$ and $y \in \text{Fix}(T)$. It is known that the fixed-point set of a quasinonexpansive mapping is closed and convex (see [7, 8]). There are some quasinonexpansive mappings which are not nonexpansive (see [9–11]). For example, the level set of a continuous convex function is characterized as the fixed-point set of a nonlinear mapping called the subgradient projection, which is not nonexpansive but quasinonexpansive.

Now we focus our attention on the following two-operator split common fixed-point problem (SCFP):

$$\text{find } x^* \in C, \quad \text{such that } Ax^* \in Q, \quad (3)$$

where $A : H_1 \rightarrow H_2$ is a bounded linear operator and $U : H_1 \rightarrow H_1$ and $T : H_2 \rightarrow H_2$ are two quasinonexpansive mappings with $\text{Fix}(U) = C$ and $\text{Fix}(T) = Q$. The solution set of the SCFP (3) is denoted by

$$\Gamma = \{x^* \in C : Ax^* \in Q\}. \quad (4)$$

As far as we know, the SCFP is introduced by Censor and Segal [12]. By taking $U = P_C$ and $T = P_Q$, the SCFP reduces to the SFP. Hence, the SCFP is a generalization of the SFP. Moudafi [13] considered the following algorithm for the

SCFP: let $x_0 \in H_1$ be arbitrary, $u_k = x_k - \gamma\beta A^*(I - T)Ax_k$ and

$$x_{k+1} = (1 - \alpha_k)u_k + \alpha_k U(u_k), \tag{5}$$

where $\beta \in (0, 1)$, $\alpha_k \in (0, 1)$, and $\gamma \in (0, 1/\lambda\beta)$, with λ being the spectral radius of the operator A^*A . He obtained the weak convergence of the algorithm (5).

In 2008, Takahashi et al. [14] developed the shrinking projection method for the nonexpansive mapping. Let T be a nonexpansive mapping of K into itself such that $\text{Fix}(T) \neq \emptyset$. Let $x_0 \in H, C_1 = K$ and $u_1 = P_{C_1}x_0$;

$$\begin{aligned} y_n &= \alpha_n u_n + (1 - \alpha_n)Tu_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} &= P_{C_{n+1}}x_0, \end{aligned} \tag{6}$$

where $0 \leq \alpha_n \leq a < 1$. They proved that the sequence $\{u_n\}$ converges strongly to $P_{\text{Fix}(T)}x_0$.

Motivated by the above results, especially by Moudafi [13] and Takahashi et al. [14], in this paper, we present the shrinking projection methods for the split common fixed-point problems. As a special case, the split feasibility problem is also discussed.

2. Preliminaries

Throughout this paper, let \mathbb{N} and \mathbb{R} be the sets of positive integers and real numbers, respectively. For any $x \in H$, there exists a unique point $P_Kx \in K$ such that

$$\|x - P_Kx\| \leq \|x - y\| \quad \forall y \in K, \tag{7}$$

where K is a nonempty closed convex subset of a real Hilbert space H . The mapping P_K is called the metric projection of H onto K . Note that P_K is a nonexpansive mapping. For $x \in H$ and $z \in K$, we have

$$z = P_Kx \iff \langle x - z, y - z \rangle \leq 0 \quad \text{for every } y \in K. \tag{8}$$

We say that a mapping $T : K \rightarrow K$ is demiclosed at zero if for any sequence $\{x_n\} \subset K$ which converges weakly to x , the strong convergence of the sequence $\{Tx_n\}$ to zero implies that $Tx = 0$. It is well known that $I - T$ is demiclosed whenever T is nonexpansive. In fact, this property is satisfied for more general mappings (see [15, 16]).

We will use the following notations:

- (1) $x_n \rightarrow x$ stands for the strong convergence of $\{x_n\}$ to x ;
- (2) $x_n \rightharpoonup x$ stands for the weak convergence of $\{x_n\}$ to x ;
- (3) $\omega_\omega(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω -limit set of $\{x_n\}$.

Here are two useful lemmas.

Lemma 1. *Let $x, y \in H$ and let $\lambda \in \mathbb{R}$. One has*

$$\begin{aligned} &\|\lambda x + (1 - \lambda)y\|^2 \\ &= \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \end{aligned} \tag{9}$$

Lemma 2 (see [17]). *Let K be a closed convex subset of a real Hilbert space H and let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_Ku$. If $\{x_n\}$ satisfies the following conditions:*

- (1) $\omega_\omega(x_n) \subset K$,
- (2) $\|x_n - u\| \leq \|u - q\|$ for all $n \in \mathbb{N}$,

then one has $x_n \rightarrow q$.

3. Shrinking Projection Methods

Now we are in a position to give the shrinking projection method for split common fixed-point problem (3).

Theorem 3. *Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Let $U : H_1 \rightarrow H_1$ and $T : H_2 \rightarrow H_2$ be two quasicontractive mappings with $\text{Fix}(U) = C$ and $\text{Fix}(T) = Q$. Suppose that $I - U$ and $I - T$ are demiclosed at zero and solution set Γ of the SCFP (3) is nonempty. For $u \in H_1$ chosen arbitrarily, $C_1 = H_1, h_1 = P_{C_1}u$, define a sequence $\{h_n\}$ by the following algorithm:*

$$\begin{aligned} w_n &= h_n - \gamma A^*(I - T)Ah_n, \\ y_n &= \alpha_n w_n + (1 - \alpha_n)Uw_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|h_n - z\|\}, \\ h_{n+1} &= P_{C_{n+1}}u. \end{aligned} \tag{10}$$

If the following are satisfied:

- (1) $\{\alpha_n\} \subset (0, 1)$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$,
- (2) $0 < \gamma < (1/\lambda_{AA^*})$, where λ_{AA^*} denotes the spectral radius of the operator AA^* ,

then the sequence $\{h_n\}$ converges strongly to $P_\Gamma u$.

Proof. We first show that $\Gamma \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that Γ is contained in $C_1 = H_1$. Suppose that $\Gamma \subset C_k$ for some $k \in \mathbb{N}$. We have, for any $p \in \Gamma \subset C_k$,

$$\begin{aligned} &\|y_k - p\|^2 \\ &= \|\alpha_k w_k + (1 - \alpha_k)Uw_k - p\|^2 \\ &\leq \alpha_k\|w_k - p\|^2 + (1 - \alpha_k)\|w_k - p\|^2 \\ &= \|h_k - \gamma A^*(I - T)Ah_k - p\|^2 \\ &= \|h_k - p\|^2 - 2\gamma \langle A^*(I - T)Ah_k, h_k - p \rangle \\ &\quad + \gamma^2 \|A^*(I - T)Ah_k\|^2 \\ &= \|h_k - p\|^2 + 2\gamma \langle TAH_k - Ah_k, Ah_k - Ap \rangle \end{aligned}$$

$$\begin{aligned}
 & + \gamma^2 \langle (I - T) Ah_k, AA^* (I - T) Ah_k \rangle \\
 \leq & \|h_k - p\|^2 + \gamma^2 \lambda_{AA^*} \|(I - T) Ah_k\|^2 \\
 & + \gamma [\|TAh_k - Ap\|^2 - \|TAh_k - Ah_k\|^2 - \|Ah_k - Ap\|^2] \\
 \leq & \|h_k - p\|^2 + \gamma^2 \lambda_{AA^*} \|(I - T) Ah_k\|^2 \\
 & - \gamma \|TAh_k - Ah_k\|^2 \\
 = & \|h_k - p\|^2 + \gamma (\gamma \lambda_{AA^*} - 1) \|(I - T) Ah_k\|^2 \\
 \leq & \|h_k - p\|^2.
 \end{aligned} \tag{11}$$

It follows that $p \in C_{k+1}$. Thus, we get $\Gamma \subset C_n$ for all $n \in \mathbb{N}$.

Next we show that C_n is closed and convex for all $n \in \mathbb{N}$. The set $C_1 = H_1$ is obviously closed and convex. Suppose that C_k is closed and convex. We see that C_{k+1} is closed and convex since $\|y_n - z\| \leq \|h_n - z\|$ is equivalent to

$$\|y_n\|^2 - \|h_n\|^2 - 2 \langle y_n - h_n, z \rangle \leq 0. \tag{12}$$

It follows that C_n is closed and convex for all $n \in \mathbb{N}$. Therefore, we obtain that the sequence $\{h_n\}$ is well defined.

From $h_n = P_{C_n} u$, we have

$$\langle u - h_n, h_n - y \rangle \geq 0 \quad \forall y \in C_n. \tag{13}$$

Recalling that $\Gamma \subset C_n$, one has

$$\langle u - h_n, h_n - p \rangle \geq 0 \quad \forall p \in \Gamma. \tag{14}$$

Hence,

$$\begin{aligned}
 0 & \leq \langle u - h_n, h_n - p \rangle \\
 & = \langle u - h_n, h_n - u + u - p \rangle \\
 & \leq -\|u - h_n\|^2 + \|u - h_n\| \|u - p\|.
 \end{aligned} \tag{15}$$

This implies that

$$\|u - h_n\| \leq \|u - p\|, \tag{16}$$

which yields that $\{h_n\}$ is bounded.

From $h_n = P_{C_n} u$ and $h_{n+1} = P_{C_{n+1}} u \in C_{n+1} \subset C_n$, we get

$$\begin{aligned}
 0 & \leq \langle u - h_n, h_n - h_{n+1} \rangle \\
 & \leq -\|u - h_n\|^2 + \|u - h_n\| \|u - h_{n+1}\|,
 \end{aligned} \tag{17}$$

which gives that

$$\|u - h_n\| \leq \|u - h_{n+1}\|. \tag{18}$$

Hence,

$$\text{the limit } \lim_{n \rightarrow \infty} \|u - h_n\| \text{ exists.} \tag{19}$$

It follows from (17) that

$$\begin{aligned}
 & \|h_n - h_{n+1}\|^2 \\
 & = \|h_n - u\|^2 + 2 \langle h_n - u, u - h_{n+1} \rangle \\
 & \quad + \|u - h_{n+1}\|^2 \\
 & = \|h_n - u\|^2 + 2 \langle h_n - u, u - h_n + h_n - h_{n+1} \rangle \\
 & \quad + \|u - h_{n+1}\|^2 \\
 & = -\|h_n - u\|^2 + 2 \langle h_n - u, h_n - h_{n+1} \rangle \\
 & \quad + \|u - h_{n+1}\|^2 \\
 & \leq -\|h_n - u\|^2 + \|u - h_{n+1}\|^2.
 \end{aligned} \tag{20}$$

Thus, we get

$$\lim_{n \rightarrow \infty} \|h_n - h_{n+1}\| = 0. \tag{21}$$

The fact that $h_{n+1} = P_{C_{n+1}} u \in C_{n+1}$ gives

$$\|y_n - h_{n+1}\| \leq \|h_n - h_{n+1}\| \rightarrow 0. \tag{22}$$

The expressions (21) and (22) yield

$$\lim_{n \rightarrow \infty} \|y_n - h_n\| = 0. \tag{23}$$

We will prove that $\omega_\omega(h_n) \subset \Gamma$. Without loss of generality, we assume that $h_n \rightarrow h^*$. It follows from (11) that

$$\begin{aligned}
 & \gamma (1 - \gamma \lambda_{AA^*}) \|(I - T) Ah_n\|^2 \\
 & \leq \|h_n - p\|^2 - \|y_n - p\|^2 \\
 & \leq (\|h_n - p\| + \|y_n - p\|) \|h_n - y_n\|.
 \end{aligned} \tag{24}$$

This together with (23) implies that

$$\lim_{n \rightarrow \infty} \|(I - T) Ah_n\| = 0. \tag{25}$$

We have $Ah^* \in \text{Fix}(T) = Q$ since $I - T$ is demiclosed at zero.

Using (??) and (25), we get $w_n \rightarrow h^*$. For any $p \in \Gamma$, one has

$$\begin{aligned}
 & \|y_n - p\|^2 \\
 & = \|\alpha_n w_n + (1 - \alpha_n) U w_n - p\|^2 \\
 & = \alpha_n \|w_n - p\|^2 + (1 - \alpha_n) \|U w_n - p\|^2 \\
 & \quad - \alpha_n (1 - \alpha_n) \|U w_n - w_n\|^2 \\
 & \leq \|w_n - p\|^2 - \alpha_n (1 - \alpha_n) \|U w_n - w_n\|^2 \\
 & \leq \|h_n - p\|^2 - \alpha_n (1 - \alpha_n) \|U w_n - w_n\|^2,
 \end{aligned} \tag{26}$$

which implies that

$$\begin{aligned} & \alpha_n (1 - \alpha_n) \|Uw_n - w_n\|^2 \\ & \leq \|h_n - p\|^2 - \|y_n - p\|^2 \\ & \leq (\|h_n - p\| + \|y_n - p\|) \|h_n - y_n\|. \end{aligned} \quad (27)$$

Therefore, one has

$$\lim_{n \rightarrow \infty} \|Uw_n - w_n\| = 0. \quad (28)$$

It follows that $h^* \in \text{Fix}(U) = C$ since $I - U$ is demiclosed at zero. Thus, we have obtained $\omega_\omega(h_n) \in \Gamma$. According to Lemma 2, we see that $h_n \rightarrow P_\Gamma u$. \square

By Theorem 3, we immediately obtain the shrinking projection method for the split feasibility problem.

Theorem 4. *Let H_1 and H_2 be real Hilbert spaces and let C and Q be nonempty closed convex subsets of H_1 and H_2 , respectively. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Suppose that the solution set Φ of the SFP (1) is nonempty. For $u \in H_1$ chosen arbitrarily, $C_1 = H_1$, $h_1 = P_{C_1} u$, define a sequence $\{h_n\}$ by the following algorithm:*

$$\begin{aligned} w_n &= h_n - \gamma A^* (I - P_Q) Ah_n, \\ y_n &= \alpha_n w_n + (1 - \alpha_n) P_C w_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|h_n - z\|\}, \\ h_{n+1} &= P_{C_{n+1}} u. \end{aligned} \quad (29)$$

If the following are satisfied:

- (1) $\{\alpha_n\} \subset (0, 1)$ and $\liminf_{n \rightarrow \infty} \alpha_n (1 - \alpha_n) > 0$,
- (2) $0 < \gamma < 1/\lambda_{AA^*}$, where λ_{AA^*} denotes the spectral radius of the operator AA^* ,

then the sequence $\{h_n\}$ converges strongly to $P_\Phi u$.

Remark 5. Letting $u = 0$ in Theorems 3 and 4, we obtain the shrinking projection methods for minimum-norm solutions of corresponding problems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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