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Research Article

Hyperbolic Cosines and Sines Theorems for the Triangle Formed by Arcs of Intersecting Semicircles on Euclidean Plane

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The hyperbolic cosines and sines theorems for the curvilinear triangle bounded by circular arcs of three intersecting circles are formulated and proved by using the general complex calculus. The method is based on a key formula establishing a relationship between exponential function and the cross-ratio. The proofs are carried out on Euclidean plane.

1. Introduction

Classically, the models of the hyperbolic plane are regarded as based on Euclidean geometry. One starts with a piece of a Euclidean plane, a half-plane, or a circular disk and then, in that half-plane or disk the notions of points, lines, distances, angles are defined as things that could be described in terms of Euclidean geometry [1–3]. The model of the hyperbolic plane is the half-plane model. The underlying space of this model is the *upper half-plane* model H in the complex plane C, defined to be

$$H = \{z \in C : \operatorname{Im}(z) > 0\}. \tag{1}$$

In coordinates (x, y) the line element is defined as

$$ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right). \tag{2}$$

The geodesics of this space are semicircles centered on the x-axis and vertical half-lines. The geometrical properties of the figures on the half-plane are studied by considering quantities invariant under an action of the general *Möbius group*, which consists of compositions of Möbius transformations and reflections [4]. The curvilinear triangle formed by circular arcs of three intersecting semicircles is one of the principal figures of the upper half-plane model H. The hyperbolic laws of sines-cosines for that triangle are proved by using properties of the Möbius group and the upper half-plane H.

In this paper we suggest another way of construction of proofs of the sines-cosines theorems of the Poincaré model. The curvilinear triangle formed by circular arcs is the figure of the Euclidean plane; consequently, on the Euclidean plane we have to find relationships antecedent to the sines-cosines hyperbolic laws. Therefore, first of all, we establish these relationships by making use of axioms of the Euclidean plane, only. Secondly, we proove that these relationships can be formulated as the hyperbolic sine-cosine theorems. For that purpose we refer to the general complex calculus and within its framework establish a relationship between exponential function and the cross-ratio. In this way the hyperbolic trigonometry emerges on Euclidean plane in a natural way.

The paper is organized as follows. In Section 2 *a key formula* connecting the hyperbolic calculus with the cross-ratio is established. By employing the key formula and the Pythagoras theorem the elements of the right-angled triangle are expressed as functions of the hyperbolic trigonometry. In Section 3 we explore Euclidian properties of the curvilinear triangle bounded by circular arcs of three intersecting semicircles. Two main relationships connecting three intersecting semicircles are established. In Section 4, we prove the theorem of cosines for the curvilinear triangle bounded by the

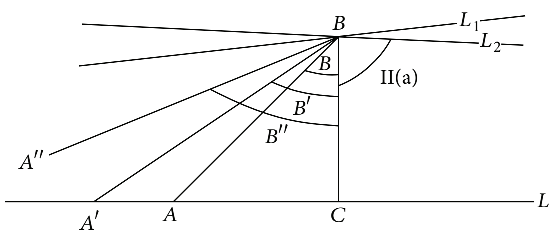

Figure 1: Motion of the hypothenuse of the right angled triangle.

circular arcs. In Section 5, the hyperbolic law of sines and second hyperbolic law of cosines are derived on the basis of the main relationships between these semicircles.

2. Hyperbolic Trigonometry in Euclidean Geometry

2.1. Trigonometry Induced by General Complex Algebra. The simplest generalization of the complex algebra, denominated as *General Complex Algebra*, is defined by unique generator **e** satisfying the quadratic equation [5]

$$\mathbf{e}^2 - b_1\mathbf{e} + b_0 = 0, \tag{3}$$

where coefficients b_0, b_1 are given by real positive numbers and $b_1^2 - 4b_0 > 0$.

The following Euler formula holds true:

$$\exp(\mathbf{e}\phi) = g_0(\phi) + \mathbf{e}g_1(\phi). \tag{4}$$

Denote by $x_1, x_2 \in R$ roots of the quadratic equation (3). The Euler formula (4) is decomposed into two independent equations:

$$\exp(x_k\phi) = g_0(\phi) + x_k g_1(\phi), \quad k = 1, 2, \tag{5}$$

from which one may find explicit formulae for g-functions. These functions are linear combinations of the exponential functions $\exp(x_k\phi)$, $k = 1, 2$. Geometrical interpretation of the general complex algebra is done in [6].

Form the following ratio

$$\exp((x_2 - x_1)\phi) = \frac{x_2 - D(\phi)}{x_1 - D(\phi)}, \tag{6}$$

where

$$D(\phi) = -\frac{g_0(\phi)}{g_1(\phi)}. \tag{7}$$

Introduce a pair of variables X_1, X_2 by

$$X_k(\phi) = x_k - D(\phi), \quad k = 1, 2. \tag{8}$$

Then, (6) is written as follows:

$$\exp((X_2 - X_1)\phi) = \frac{X_2(\phi)}{X_1(\phi)}. \tag{9}$$

This formula we denominate as the *key formula* and use it in order to introduce hyperbolic trigonometry on Euclidean plane. Notice that the argument of the exponential function is proportional to the difference between the numerator and the denominator, $X_1 - X_2 = x_1 - x_2$, and this difference does not depend on the parameter ϕ.

2.2. Elements of Right-Angled Triangle as Functions of a Hyperbolic Trigonometry. Let $\triangle ABC$ be a right-angled triangle with right angle at C. Denote the sides by a, b, the hypotenuse by c, and the angles opposite to a, b, c by A, B, C, correspondingly. Traditionally interrelations between angles and sides of a triangle are described by the trigonometry via periodical sine-cosine functions. The periodical functions of the circular angles are defined via the ratios

$$\sin B = \frac{b}{c}, \qquad \tan B = \frac{b}{a}. \tag{10}$$

Notice that these two ratios are functions of the same angle B. On making use of formula (9) we are able to introduce the hyperbolic trigonometry besides the circular trigonometry. Define the following relationships:

$$\frac{c + a}{c - a} = \exp(2\xi). \tag{11}$$

From this equation it follows that

$$\frac{a}{c} = \tanh(\xi), \qquad \frac{a}{b} = \sinh(\xi). \tag{12}$$

In this way we arrive to the following interrelations between circular and hyperbolic functions:

$$\cos B = \frac{a}{c} = \tanh(\xi), \qquad \tan B = \frac{b}{a} = \frac{1}{\sinh(\xi)}. \tag{13}$$

Now let us introduce the following geometrical motion; namely, change position of the point A along the line AC (see, Figure 1). Then, the sides b and c will change while the side a will not change. Since the length a is a constant of this evolution, in agreement with key formula (9) we rewrite (11) as follows:

$$\frac{c + a}{c - a} = \exp(2a\phi), \tag{14}$$

that is, the argument of exponential function is proportional to a: $\xi = a\phi$, where ϕ is a parameter of the evolution. Formulae in (13) are rewritten as

$$\cos B = \frac{a}{c} = \tanh(a\phi), \qquad \tan B = \frac{b}{a} = \frac{1}{\sinh(a\phi)}. \tag{15}$$

From these formulas it is derived that

$$\cot B = \sinh(a\phi), \tag{16}$$

or,

$$\tan\frac{B}{2} = \exp(-a\phi). \tag{17}$$

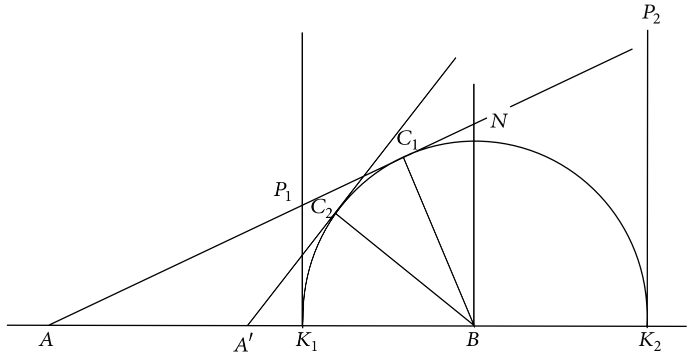

FIGURE 2: Semicircle and lines tangent to the semicircle.

Install the triangle $\triangle ABC$ in such a way that the side $b = AC$ lies on the line L, and the side $a = BC$ is perpendicular to this line at the point C

In Figure 1 the line AB moves in such a way that cuts line L at points A, A', A'', and so forth. Now, let us recall the problem of parallel lines in geometry [7]. Let the ray AB tend to a definite limiting position; in Figure 1 this position is given by line L_1. As the point A moves along line L away from point C there are two possibilities to consider.

(1) In Euclidean geometry, the angle between lines L_1 and BC is equal to right angle.

(2) The hypothesis of hyperbolic geometry is that this angle is less than the right angle.

The most fundamental formula of the hyperbolic geometry is the formula connecting the angle of parallelism $\Pi(a)$ and the length a of the perpendicular from the given point to the given line. In order to establish those relationships the concept of horocycles, some circles with center and axis at infinity, was introduced [8]. The great theorem which enables one to introduce the circular functions, sines, and cosines of an angle is that the geometry of shortest lines (horocycles) traced on horosphere is the same as plane Euclidean geometry. The function connecting the angle of parallelism with the distance a is given by

$$\exp\left(-\frac{a}{\kappa}\right) = \tan\frac{\Pi(a)}{2}. \tag{18}$$

Now, we introduce the value inverse to ϕ by $\mathbf{K} = 1/\phi$ and rewrite (17) in the form

$$\exp\left(-\frac{a}{\mathbf{K}}\right) = \tan\frac{B}{2}. \tag{19}$$

Let the point A run away from the point C to infinity. The following two cases can be considered.

(1) ϕ tends to zero, and $\mathbf{K}$ tends to infinity; then the angle B will tend to right angle.

This is true in Euclidean plane.

(2) Suppose that ϕ, and $\mathbf{K}$ and the angle B go to some limited values,

$$\lim_{AC\longrightarrow\infty}\mathbf{K} = \kappa, \qquad \lim B = \Pi(a). \tag{20}$$

In this way, we have established connection of formula (17) with the main formula of hyperbolic geometry (19).

2.3. Rotational Motion of a Line Tangent to the Semicircle. The concept of the circular angle in Euclidean plane is intimately related to the figure of a circle and to motion of a point along the circumference. The hyperbolic angle is also related to the circle because of a motion along the circumference coherence with the motion along the hyperbola [9].

Consider semicircle $\mathbf{C}$ (Figure 2) with end-points and the center on x-axis. Denote by B the center and by K_1, K_2 the end-points of the semicircle. Through end-points of the semicircle, K_1, K_2 erect the lines parallel to vertical axis, Y-axis. Draw a line tangent to the semicircle at the point C; this line crosses x-axis at the point A and intersects with the vertical lines at points P_1 and P_2. Draw a line parallel to Y-axis from the center B which crosses $\mathbf{C}$ at the top point N.

Denote by r radius of the circle, so that $r = BN = BC$ and $r = K_1K_2$. Denote by B the angle $\angle ABC$. The triangle $\triangle ABC$ is a right triangle, so that

$$(AB)^2 - (AC)^2 = r^2. \tag{21}$$

Consider rotational motion of the line tangent to the semicircle at the point $\mathbf{C}$. In Figure 2, two positions of this line are given by lines $A'C_2$ and AC_1. When the point Cruns from end point K_1 to the top-point N, the point A runs along x-axis from point K_1 to infinity. During this motion the triangle formed by the line tangent to the semicircle, the x-axis and the radius of the circle remains to be right angled. This is exactly the case considered in Section 2.2, consequently, we can apply formula (13). According to (13) we write

$$AB = r\coth(r\phi) = \frac{r}{\cos B}, \qquad AC_1 = \frac{r}{\sinh(r\phi)} = r\tan B. \tag{22}$$

Since $\triangle ABC_1 \sim \triangle AK_1P_1$, we have

$$\frac{P_1K_1}{AK_1} = \frac{r}{AC_1} = \sinh(r\phi), \tag{23}$$

$$\begin{aligned} P_1K_1 &= AK_1\sinh(r\phi) = (AB - r)\sinh(r\phi) \\ &= r\exp(-r\phi), \quad P_2K_2 = r\exp(r\phi). \end{aligned} \tag{24}$$

On the basis of obtained formulae the following relationships between circular and hyperbolic trigonometric functions are established:

$$\sin(B) = \frac{1}{\cosh(r\phi)}, \qquad \cot(B) = \sinh(r\phi), \qquad \cos(B) = \tanh(r\phi). \tag{25}$$

If we make the point C tend to the top of semicircle N, the hyperbolic angle ϕ will tend to zero. When the point C tends to end-point K_1, the hyperbolic angle tends to infinity. Thus, the hyperbolic angle is measured from the point N, the top of the semicircle. Consider two different positions of the tangent line, corresponding to two positions C_1, C_2, with hyperbolic angles ϕ_2 and ϕ_1. According to key formula (6) we write

$$\exp((x_2 - x_1)\phi_2) = \frac{x_2 - D_2}{x_1 - D_2}, \qquad \exp((x_2 - x_1)\phi_1) = \frac{x_2 - D_1}{x_1 - D_1}. \tag{26}$$

The hyperbolic angle ϕ_2 corresponds to the arc $\smile NC_2$, and the hyperbolic angle ϕ_1 corresponds to the arc $\smile NC_1$. Then we suppose that the difference in the hyperbolic angles $\phi_2 - \phi_1$ will correspond to the arc $\smile C_2C_1$. From (26) it follows that

$$\exp(x_2 - x_1)(\phi_2 - \phi_1) = \frac{x_2 - D_2}{x_1 - D_2}\frac{x_1 - D_1}{x_2 - D_1}, \qquad D_k = D(\phi_k), \quad k = 1, 2. \tag{27}$$

It is seen, in the right-hand side we have the cross-ratio. Now, let recall definition of the distance between two points of the geodesic line in the Poincaré model. Let $z, w \in H$, and let z_1 and w_1 be end points of a geodesic line passing through z and w. Then the distance between these points is defined by formula

$$\rho(z_1, z_3) = \log(z_1, z_2; z_3, z_4), \tag{28}$$

where

$$(z_1, z_2; z_3, z_4) := \frac{z_2 - z_1}{z_4 - z_1}\frac{z_4 - z_3}{z_2 - z_3} \tag{29}$$

is the cross-ratio. Comparing (29) with (27) we come to the following correspondence: $z_2 = x_2$, $z_4 = x_1$, $z_1 = D_2$, $z_3 = D_1$. Notice, however, in our construction D_k, $k = 1, 2$ are projections of z_1, z_3 on X-axis. The semicircle $\mathbf{C}$ is the geodesic line, and x_1, x_2 are end-points of the geodesic line.

3. Relationships between Elements of Three Intersecting Semicircles

3.1. Hyperbolic Cosine-Sine Functions of Arcs of the Semicircle. The key formula (6) admits to define hyperbolic trigonometric functions of the arcs originated from the top N of the semicircle $\mathbf{C}$. In order to determine trigonometric functions of the arcs with arbitrary end-points on the semicircle we have to use formula (27). Consider arc $\smile C_1C_2$ is defined in the first quadrant of the semicircle with end-points at C_1 and C_2. The arc $\smile C_1C_2$ can be presented as difference of two arcs, both originated from the top of the semicircle:

$$\smile C_1C_2 = \smile NC_1 - \smile NC_2. \tag{30}$$

Denote by a_1, a_2 the angles formed by radiuses BC_1 and BC_2 with x-axis, correspondingly, where B is a center of the circle. Denote the hyperbolic angle corresponding to the arcs $\smile NC_k$, $k = 1, 2$ by $\xi(NC_k)$, $k = 1, 2$. Then the functions

$$\cosh\xi(NC_1), \quad \cosh\xi(NC_2), \quad \sinh\xi(NC_1), \quad \sinh\xi(NC_2) \tag{31}$$

are expressed via circular trigonometric functions according to formulae (25):

$$\cosh\xi(NC_1) = \frac{1}{\sin a_1}, \quad \cosh\xi(NC_2) = \frac{1}{\sin a_2}, \quad \sinh\xi(NC_1) = \cot a_1, \quad \sinh\xi(NC_2) = \cot a_2. \tag{32}$$

Then, by taking into account additional formulae

$$\begin{aligned}\cosh\xi(C_1C_2) &= \cosh\xi(NC_1)\cosh\xi(NC_2) - \sinh\xi(NC_1)\sinh\xi(NC_2),\\ \sinh\xi(C_1C_2) &= \sinh\xi(NC_1)\cosh\xi(NC_2) - \cosh\xi(NC_1)\sinh\xi(NC_2),\end{aligned} \tag{33}$$

we get hyperbolic trigonometric functions of the arcs with arbitrary end-points on the semicircle expressed via periodic trigonometric functions:

$$\cosh\xi(C_1C_2) = \frac{1 - \cos a_1 \cos a_2}{\sin a_1 \sin a_2}, \qquad \sinh\xi(C_1C_2) = \frac{\cos a_1 - \cos a_2}{\sin a_1 \sin a_2}. \tag{34a}$$

Compare with analogous formula from Poincaré model ([3], formula (1.2.6)) given by

$$\cosh\rho(z, w) = 1 + \frac{|z - w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}. \tag{34b}$$

Two complex numbers

$$z = v_0 + iv, \qquad w = w_0 + iw \tag{35}$$

are related to the radius of the semicircle and the angles a_1, a_2 by

$$v = r\sin a_2, \quad v_0 = r\cos a_2, \quad w = r\sin a_1, \quad w_0 = r\sin a_1. \tag{36}$$

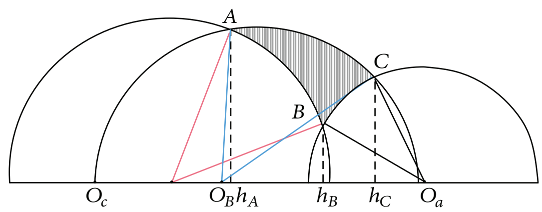

FIGURE 3: Triangle formed by arcs of intersecting semicircles.

Then,

$$\cosh \xi (C_1C_2) = \frac{1 - \cos a_1 \cos a_2}{\sin a_1 \sin a_2} = \frac{r^2 - v_0 w_0}{wv} = 1 + \frac{(v_0 - w_0)^2 + (v - w)^2}{2wv} = \cosh \rho (z, w) . \quad (37)$$

3.2. Relationships between Elements of the Triangle Bounded by Arcs of the Intersecting Semicircles. In Figure 3 three intersecting semicircles with centers installed on horizontal axis at the points O_a, O_b, and O_c are presented. Intersections of the semicircles form triangle $\widetriangle{ABC}$ bounded by the arcs $\smile BC$, $\smile AB$, $\smile AC$. For each arc we can put in correspondence the hyperbolic angle. Denote the hyperbolic angles by a, b, c, where a, b, c consecutively will correspond to the arcs $\smile BC$, $\smile AC$, $\smile AB$.

Connect vertices of the triangle with centers of the circle by corresponding radiuses. Denote by a_k, b_k, c_k, $k = 1, 2$ the angles bounded by the radiuses and x-axis, where $a_1 > a_2 > 0$, $b_1 > b_2 > 0$, $c_1 > c_2 > 0$. By making use of (34a) define hyperbolic cosine-sine functions corresponding to bounding segments:

$$\cosh a = \frac{1 - \cos a_1 \cos a_2}{\sin a_1 \sin a_2}, \qquad \sinh a = \frac{\cos a_1 - \cos a_2}{\sin a_1 \sin a_2}, \quad (38a)$$

$$\cosh b = \frac{1 - \cos b_1 \cos b_2}{\sin b_1 \sin b_2}, \qquad \sinh b = \frac{\cos b_1 - \cos b_2}{\sin b_1 \sin b_2}, \quad (38b)$$

$$\cosh c = \frac{1 - \cos c_1 \cos c_2}{\sin c_1 \sin c_2}, \qquad \sinh c = \frac{\cos c_1 - \cos c_2}{\sin c_1 \sin c_2}. \quad (38c)$$

The usual notion of the angle is used that is, the angle between two curves is defined as an angle between their tangent lines. Let the angles α, β, δ be angles at the verteices A, B, C, correspondingly. For these angles we can define thier proper cosine and sine functions. The angles of the triangle $\widetilde{\triangle}ABC$ α, β, δ are closely related to angles $a_1, a_2, b_1, b_2, c_1, c_2$. From Figure 3 we find the following relationships between them:

$$\beta = a_2 + c_2, \qquad \delta = \pi - a_1 - b_2, \qquad \alpha = b_1 - c_1. \quad (39)$$

Then,

$$\cos \alpha = \cos b_1 \cos c_1 + \sin b_1 \sin c_1, \quad (40a)$$

$$\cos \beta = \cos a_2 \cos c_2 - \sin a_2 \sin c_2, \quad (40b)$$

$$\cos \delta = - \cos b_2 \cos a_1 + \sin b_2 \sin a_1, \quad (40c)$$

$$\sin \alpha = \sin b_1 \cos c_1 - \cos b_1 \sin c_1, \quad (41a)$$

$$\sin \beta = \sin a_2 \cos c_2 + \cos a_2 \sin c_2, \quad (41b)$$

$$\sin \delta = \sin b_2 \cos a_1 + \cos b_2 \sin a_1. \quad (41c)$$

Denote distances between centers by

$$O_{cb} = O_cO_b, \qquad O_{ba} = O_bO_a, \qquad O_{ac} = O_aO_c. \quad (42)$$

The theorem of sines employed for triangles O_cAO_b, O_bCO_a, O_cBO_a gives six relations of type

$$\frac{\sin \alpha}{O_{cb}} = \frac{\sin c_1}{r_b} = \frac{\sin b_1}{r_c}, \quad (43a)$$

$$\frac{\sin \delta}{O_{ba}} = \frac{\sin a_1}{r_b} = \frac{\sin b_2}{r_a}, \quad (43b)$$

$$\frac{\sin \beta}{O_{ac}} = \frac{\sin c_2}{r_a} = \frac{\sin a_2}{r_c}. \quad (43c)$$

From these relations the first set of main relationships follows.

Relation 1. Consider

$$r_a \sin a_1 = r_b \sin b_2, \qquad r_c \sin c_2 = r_a \sin a_2, \qquad r_c \sin c_1 = r_b \sin b_1. \quad (44)$$

From the draught in Figure 3 it is seen that

$$O_{ac} = O_{ba} + O_{cb}, \quad (45)$$

where

$$O_{ac} = r_c \cos c_2 + r_a \cos a_2, \qquad O_{ba} = r_a \cos a_1 + r_b \cos b_2. \quad (46)$$

Hence,

$$O_{cb} = r_c \cos c_2 + r_a \cos a_2 - r_a \cos a_1 - r_b \cos b_2. \quad (47)$$

From vertices of $\widetriangle{ABC}$ erect lines perpendicular to horizontal line intersect with x-axis at points h_A, h_B, h_C, correspondingly. From Figure 3 we find that

$$O_{cb} = O_ch_A - h_AO_b = r_c \cos c_1 - r_b \cos b_1. \quad (48)$$

By equating (47) with (48) we arrive to another main relationship between radii and angles.

Relation 2. One has

$$r_c \cos c_1 - r_b \cos b_1 = r_c \cos c_2 + r_a \cos a_2 - r_a \cos a_1 - r_b \cos b_2. \tag{49}$$

We will effect a simplification by using the following designations:

$$\begin{aligned} r_a \cos a_1 &= w_{01}, & r_a \sin a_1 &= w_1, \\ r_a \cos a_2 &= v_{01}, & r_a \sin a_2 &= v_1, \\ r_b \cos b_1 &= w_{02}, & r_b \sin b_1 &= w_2, \\ r_b \cos b_2 &= v_{02}, & r_b \sin b_2 &= v_2, \\ r_c \cos c_1 &= w_{03}, & r_c \sin c_1 &= w_3, \\ r_c \cos c_2 &= v_{03}, & r_c \sin c_2 &= v_3. \end{aligned} \tag{50}$$

In these designations formulae (40a), (40b), (40c) and (41a), (41b), (41c) are written as follows:

$$\begin{aligned} r_b r_c \sin \alpha &= w_2 w_{03} - w_{02} w_3, \\ r_b r_c \cos \alpha &= w_{02} w_{03} + w_2 w_3, \end{aligned} \tag{51a}$$

$$\begin{aligned} r_a r_c \sin \beta &= v_1 v_{03} + v_{01} v_3, \\ r_a r_c \cos \beta &= v_{01} v_{03} - v_1 v_3, \end{aligned} \tag{51b}$$

$$\begin{aligned} r_a r_b \sin \delta &= v_2 w_{01} + v_{02} w_1, \\ r_a r_b \cos \delta &= v_2 w_1 - v_{02} w_{01}. \end{aligned} \tag{51c}$$

Formulae (38a), (38b), (38c) for hyperbolic sines and cosines are rewritten as follows:

$$\begin{aligned} \cosh a &= \frac{r_a^2 - w_{01} v_{01}}{w_1 v_1}, & \sinh a &= r_a \frac{w_{01} - v_{01}}{w_1 v_1}, \\ \cosh b &= \frac{r_b^2 - w_{02} v_{02}}{w_2 v_2}, & \sinh b &= r_b \frac{w_{02} - v_{02}}{w_2 v_2}, \\ \cosh c &= \frac{r_c^2 - w_{03} v_{03}}{w_3 v_3}, & \sinh c &= r_c \frac{w_{03} - v_{03}}{w_3 v_3}. \end{aligned} \tag{52}$$

In the designations equations of Relation 1 are written as

$$x := w_2 = w_3, \qquad y := v_1 = v_3, \qquad z := w_1 = v_2. \tag{53}$$

Equations (43a), (43b), (43c)–(46) are rewritten as follows:

$$O_{cb} = w_{03} - w_{02}, \qquad O_{ac} = v_{03} + v_{01}, \qquad O_{ba} = w_{01} + v_{02}. \tag{54}$$

Correspondingly, Relation 2 takes the form

$$w_{03} - w_{02} = v_{03} + v_{01} - w_{01} - v_{02}. \tag{55}$$

This expresses the fact that O_{ca} is a sum of O_{cb} and O_{ba}. Notice that (55) can be rewritten also in another equivalent form namely,

$$w_{03} - v_{03} = w_{02} - v_{02} - (w_{01} - v_{01}). \tag{56}$$

Denote the segments projections of sides of $\widetilde{\triangle} ABC$ on x-axis by $\mathbf{P}(AC) = h_A h_C$, $\mathbf{P}(AB) = h_A h_B$, $\mathbf{P}(BC) = h_B h_C$. From Figure 3 it is seen that

$$\mathbf{P}(AC) = \mathbf{P}(AB) + \mathbf{P}(BC), \tag{57}$$

where

$$\mathbf{P}(BC) = w_{01} - v_{01}, \qquad \mathbf{P}(AC) = w_{02} - v_{02}, \qquad \mathbf{P}(AB) = w_{03} - v_{03}. \tag{58}$$

4. Hyperbolic Law of Cosines I for the Triangle Formed by Intersection of Three Semicircles

The main aim of this section is to prove the hyperbolic law theorem of cosines I for the triangle $\widetilde{\triangle} ABC$ formed by arcs of intersecting semicircles with centers installed on x-axis (Figure 3). This law is given by the following set of equations:

$$\begin{aligned} \cosh c &= \cosh a \cosh b - \sinh a \sinh b \cos \delta, \\ \cosh b &= \cosh a \cosh c - \sinh c \sinh a \cos \beta, \\ \cosh a &= \cosh c \cosh b - \sinh c \sinh b \cos \alpha. \end{aligned} \tag{59}$$

Theorem 3 (theorem of cosines I). *The following equation for elements of the triangle $\widetilde{\triangle} ABC$ formed by arcs of three intersecting semicircles holds true*

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \delta, \tag{60}$$

where $\cosh c, \cosh a, \cosh b, \sinh a, \sinh b, \cos \delta$ *are defined by formulae* (51a), (51b), (51c)-(52).

Proof. Square both sides of Relation 2 to obtain

$$(w_{03} - v_{03})^2 = (w_{02} - v_{02})^2 + (w_{01} - v_{01})^2 - 2(w_{02} - v_{02})(w_{01} - v_{01}) \tag{61}$$

and evaluate this equality by taking into account formulae (51a), (51b), (51c)-(52). First of all, evaluate the left-hand side of this equation as follows:

$$\begin{aligned} (w_{03} - v_{03})^2 &= v_{03}^2 + w_{03}^2 - 2 v_{03} w_{03} \\ &= v_{03}^2 + w_{03}^2 - 2 r_c^2 + 2 (r_c^2 - v_{03} w_{03}), \end{aligned} \tag{62}$$

where

$$2 r_c^2 = w_{03}^2 + w_3^2 + v_{03}^2 + v_3^2. \tag{63}$$

Use (63) and write (62) as follows:

$$\begin{aligned} v_{03}^2 + w_{03}^2 - 2 r_c^2 &= v_{03}^2 + w_{03}^2 - (w_{03}^2 + w_3^2 + v_{03}^2 + v_3^2) \\ &= -(w_3^2 + v_3^2). \end{aligned} \tag{64}$$

Equation (61) takes the following form:

$$-\underbrace{(w_3^2 + v_3^2)} + 2(r_c^2 - v_{03} w_{03}) = (w_{02} - v_{02})^2 + (w_{01} - v_{01})^2 - 2(w_{02} - v_{02})(w_{01} - v_{01}). \tag{65}$$

The underlined term passes to the right-hand side of the equation. Then in the left-hand side remains the expression

$$2\left(r_c^2 - v_{03}w_{03}\right) = 2v_3w_3\cosh c. \tag{66}$$

Dividing both sides of the obtained equation by v_3w_3, we get

$$2\cosh c = \frac{1}{v_3w_3}\left(w_3^2 + v_3^2 + \left(v_{02} - w_{02}\right)^2 + \left(v_{01} - w_{01}\right)^2 - 2\left(v_{02} - w_{02}\right)\left(v_{01} - w_{01}\right)\right), \tag{67}$$

according to Relation 1 $w_3 = w_2$, $v_3 = v_1$. These relations make true the following equation

$$\frac{1}{w_3v_3} = \frac{1}{w_2v_2}\frac{1}{w_1v_1}v_2w_1. \tag{68}$$

On making use this formula in the right-hand side of (67) we come to the following equation:

$$\begin{aligned}2\cosh c = &\frac{1}{w_2v_2}\frac{1}{w_1v_1}\\ &\times\left\{v_2w_1\left(w_3^2 + v_3^2 + \left(v_{02} - w_{02}\right)^2 + \left(v_{01} - w_{01}\right)^2\right)\right\}\\ &\underbrace{-2\frac{1}{w_2v_2}\frac{1}{w_1v_1}v_2w_1\left(v_{02} - w_{02}\right)\left(v_{01} - w_{01}\right)}.\end{aligned} \tag{69}$$

Evaluate now the underlined term, which we firstly write as follows:

$$\begin{aligned}&2\left(v_{02} - w_{02}\right)\left(v_{01} - w_{01}\right)\frac{1}{w_2v_2}\frac{1}{w_1v_1}v_2w_1\\ &\quad = 2\frac{v_2w_1}{r_ar_b}\frac{r_b\left(v_{02} - w_{02}\right)}{w_2v_2}\frac{r_a\left(v_{01} - w_{01}\right)}{w_1v_1}.\end{aligned} \tag{70}$$

From the second equation of (51c) we have

$$\frac{v_2w_1}{r_ar_b} = \cos\delta + \frac{v_{02}w_{01}}{r_ar_b}. \tag{71}$$

By making use of (71) in (70), evaluate (70) as follows:

$$\begin{aligned}&2\frac{v_2w_1}{r_ar_b}\frac{r_b\left(v_{02} - w_{02}\right)}{w_2v_2}\frac{r_a\left(v_{01} - w_{01}\right)}{w_1v_1}\\ &\quad = 2\left(\cos\delta + \frac{v_{02}w_{01}}{r_ar_b}\right)\frac{r_b\left(v_{02} - w_{02}\right)}{w_2v_2}\frac{r_a\left(v_{01} - w_{01}\right)}{w_1v_1}\\ &\quad = 2\cos\delta\frac{r_b\left(v_{02} - w_{02}\right)}{w_2v_2}\frac{r_a\left(v_{01} - w_{01}\right)}{w_1v_1}\\ &\qquad - 2v_{02}w_{01}\frac{\left(v_{02} - w_{02}\right)}{w_2v_2}\frac{\left(v_{01} - w_{01}\right)}{w_1v_1}\\ &\quad = 2\cos\delta\sinh a\sinh b - 2v_{02}w_{01}\frac{\left(v_{02} - w_{02}\right)}{w_2v_2}\frac{\left(v_{01} - w_{01}\right)}{w_1v_1}.\end{aligned} \tag{72}$$

Replace the underlined term of (69) by (72), and pass expression containing sines and cosines to the left-hand side of the obtained equation. As a result, we come to the following equation:

$$\begin{aligned}&2\cosh c - 2\cos\delta\sinh b\sinh a\\ &\quad = \frac{1}{w_2v_2w_1v_1}\left\{v_2w_1\left(w_2^2 + v_1^2 + \left(w_{02} - v_{02}\right)^2 + \left(w_{01} - v_{01}\right)^2\right) - 2v_{02}w_{01}\right.\\ &\qquad\left.\times\left(w_{02} - v_{02}\right)\left(w_{01} - v_{01}\right)\right\}\end{aligned} \tag{73}$$

By applying the elementary algebra one may show that (see, [10])

$$\begin{aligned}&v_2w_1\left(w_2^2 + v_1^2 + \left(w_{02} - v_{02}\right)^2 + \left(w_{01} - v_{01}\right)^2\right)\\ &\quad - 2v_{02}w_{01}\left(w_{02} - v_{02}\right)\left(w_{01} - v_{01}\right)\\ &\quad = 2\left(r_a^2 - v_{01}w_{01}\right)\left(r_b^2 - v_{02}w_{02}\right).\end{aligned} \tag{74}$$

Thus, in the right-hand side of (73) we have

$$2\frac{1}{w_1v_1v_2w_2}\left(r_a^2 - v_{01}w_{01}\right)\left(r_b^2 - v_{02}w_{02}\right), \tag{75}$$

which according to formulae (52) is nothing else than

$$2\cosh a\cosh b. \tag{76}$$

By using it in (73) we arrive to the following equation:

$$2\cosh c - 2\cosh a\cosh b = 2\cos\delta\sinh a\sinh b. \tag{77}$$

□

5. Hyperbolic Laws of Sines and Cosines II

The main task of this section is to prove *hyperbolic law (theorem) of sines*, which is given by the formulae

$$\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\delta}, \tag{78}$$

and the hyperbolic law (theorem) of cosines II given by the formulae

$$\cos\delta = -\cos\alpha\cos\beta - \sin\alpha\sin\beta\cosh c, \tag{79}$$

$$\cos\beta = -\cos\alpha\cos\delta - \sin\alpha\sin\delta\cosh b, \tag{80}$$

$$\cos\alpha = -\cos\beta\cos\delta - \sin\delta\sin\beta\cosh a. \tag{81}$$

5.1. Hyperbolic Theorem of Sines and Its Geometrical Interpretation on Euclidean Plane

Theorem 4. *The ratios of projections of the sides of triangle $\widetilde{\triangle}ABC$ on x-axis to corresponding distances between centers of the semicircles are equal to each other.*

Proof. Projections of the sides of $\widetilde{\triangle}ABC$ are given by formulae

$$\mathbf{P}_{bc} = r_a\cos a_1 - r_a\cos a_2, \qquad \mathbf{P}_{ca} = r_b\cos b_1 - r_b\cos b_2,$$
$$\mathbf{P}_{ab} = r_c\cos c_1 - r_c\cos c_2. \tag{82}$$

Distances between centers of the circles have been defined as (see, (46) and (47))

$$O_{ca} = r_c\cos c_2 + r_a\cos a_2, \qquad O_{ba} = r_a\cos a_1 + r_b\cos b_2,$$
$$O_{cb} = r_c\cos c_1 - r_b\cos b_1, \tag{83}$$

$$\mathbf{P}_{ca} = \mathbf{P}_{bc} + \mathbf{P}_{ab}, \qquad O_{ca} = O_{bc} + O_{ab}. \tag{84}$$

Relation 1 given by the set of equations

$$r_a\sin a_1 = r_b\sin b_2, \qquad r_c\sin c_2 = r_a\sin a_2,$$
$$r_c\sin c_1 = r_b\sin b_1 \tag{85}$$

raises to the second power,

$$r_a^2 - r_a^2\cos^2 a_1 = r_b^2 - r_b^2\cos^2 b_2,$$
$$r_c^2 - r_c^2\cos^2 c_2 = r_a^2 - r_a^2\cos^2 a_2, \tag{86}$$
$$r_c^2 - r_c^2\cos^2 c_1 = r_b^2 - r_b^2\cos^2 b_1.$$

Then, square distances $O_{ik}^2, i, k = a, b, c$ and use (86). We get

$$O_{ca}^2 = r_c^2\cos^2 c_2 + r_a^2\cos^2 a_2 + 2r_c r_a\cos c_2\cos a_2,$$
$$O_{ba}^2 = r_a^2\cos^2 a_1 + r_b^2\cos^2 b_2 + 2r_a r_b\cos a_1\cos b_2, \tag{87}$$
$$O_{cb}^2 = r_c^2\cos^2 c_1 + r_b^2\cos^2 b_1 - 2r_c r_b\cos c_1\cos b_1.$$

Combine (86) with (87); in this way we come to the following system of equations:

$$(a)\ O_{ca}^2 = r_c^2 - r_a^2 + 2r_a\cos a_2 O_{ac},$$
$$(b)\ O_{ca}^2 = r_a^2 - r_c^2 + 2r_c\cos c_2 O_{ac}, \tag{88a}$$

$$(a)\ O_{ba}^2 = r_a^2 - r_b^2 + 2r_b\cos b_2 O_{ba},$$
$$(b)\ O_{ba}^2 = r_b^2 - r_a^2 + 2r_a\cos a_1 O_{ba}, \tag{88b}$$

$$(a)\ O_{cb}^2 = r_b^2 - r_c^2 + 2r_c\cos c_1 O_{cb},$$
$$(b)\ O_{cb}^2 = r_c^2 - r_b^2 - 2r_b\cos b_1 O_{cb}. \tag{88c}$$

From these equations the cosines of the angles $a_1, a_2, b_1, b_2, c_1, c_2$ are expressed:

$$\frac{O_{ba}^2 - r_b^2 + r_a^2}{2r_a O_{ba}} = \cos a_1, \qquad \frac{O_{ca}^2 - r_c^2 + r_a^2}{2r_a O_{ac}} = \cos a_2,$$
$$\frac{O_{cb}^2 - r_b^2 + r_c^2}{2r_c O_{cb}} = \cos c_1, \qquad \frac{O_{ca}^2 - r_a^2 + r_c^2}{2r_c O_{ac}} = \cos c_2, \tag{89}$$
$$\frac{-O_{cb}^2 + r_c^2 - r_b^2}{2r_b O_{cb}} = \cos b_1, \qquad \frac{O_{ba}^2 - r_a^2 + r_b^2}{2r_b O_{ba}} = \cos b_2.$$

Having these formulae we may present the projection $\mathbf{P}_{ca}$ as follows

$$\begin{aligned}\mathbf{P}_{ac} &= r_b\cos b_1 - r_b\cos b_2 \\ &= \frac{-O_{cb}^2 + r_c^2 - r_b^2}{2O_{cb}} - \frac{O_{ba}^2 - r_a^2 + r_b^2}{2O_{ba}} \\ &= \frac{-O_{cb}^2 O_{ba} + r_c^2 O_{ba} - r_b^2 O_{ba} - O_{ba}^2 O_{cb} + r_a^2 O_{cb} - r_b^2 O_{cb}}{2O_{cb}O_{ba}}\end{aligned} \tag{90}$$

The first ratio is presented as follows:

$$\begin{aligned}\frac{\mathbf{P}_{ca}}{O_{ca}} &= \frac{-O_{cb}O_{ba}\left(O_{cb}+O_{ba}\right) + r_c^2 O_{ba} - r_b^2\left(O_{ba}-O_{cb}\right) + r_a^2 O_{cb}}{2O_{cb}O_{ba}O_{ca}} \\ &= \frac{-O_{cb}O_{ba}\left(O_{ca}\right) + r_c^2 O_{ba} - r_b^2\left(O_{ca}\right) + r_a^2 O_{cb}}{2O_{cb}O_{ba}O_{ca}}.\end{aligned} \tag{91}$$

Now, in the same way let us calculate the next ratio, namely, $\mathbf{P}_{bc}/O_{bc}$. Formula for the projection is evaluated as follows:

$$\begin{aligned}\mathbf{P}_{bc} &= r_a\cos a_1 - r_a\cos a_2 \\ &= \frac{O_{ba}^2 - r_b^2 + r_a^2}{2O_{ba}} - \frac{O_{ca}^2 - r_c^2 + r_a^2}{2O_{ac}} \\ &= \frac{\left(\left(O_{ba}O_{ac}\left(O_{ba}-O_{ca}\right) - r_b^2 O_{ac} + r_a^2 O_{ac}\right) + r_c^2 O_{ba} - r_a^2 O_{ba}\right)}{2O_{ac}O_{ba}}.\end{aligned} \tag{92}$$

By taking into account equation $-O_{cb} = O_{ba} - O_{ac}$, we get

$$\mathbf{P}_{bc} = \frac{\left(O_{ba}O_{ac}O_{cb} - r_b^2 O_{ac} + r_a^2 O_{cb} + r_c^2 O_{ba}\right)}{2O_{ac}O_{ba}}. \tag{93}$$

Next, let us calculate the ratio

$$\frac{\mathbf{P}_{bc}}{O_{bc}} = \frac{\left(O_{ba}O_{ac}\left(O_{cb}\right) - r_b^2 O_{ac} + r_a^2\left(O_{cb}\right) + r_c^2 O_{ba}\right)}{2O_{ac}O_{ba}O_{bc}}. \tag{94}$$

This expression coincides with (91); hence,

$$\frac{\mathbf{P}_{bc}}{O_{bc}} = \frac{\mathbf{P}_{ac}}{O_{ac}}. \tag{95}$$

By taking into account (84), we arrive to desired relations

$$\frac{\mathbf{P}_{bc}}{O_{bc}} = \frac{\mathbf{P}_{ac}}{O_{ac}} = \frac{\mathbf{P}_{ab}}{O_{ab}}. \tag{96}$$

□

In the sequel come back to designations introduced in Section 3. In these designations (96) is written as follows:

$$\frac{w_{01} - v_{01}}{w_{03} - w_{02}} = \frac{w_{02} - v_{02}}{v_{03} + v_{01}} = \frac{w_{03} - v_{03}}{w_{01} + v_{02}}. \tag{97}$$

Theorem 5. *The sides and the angles of triangle $\widetilde{\triangle}ABC$ satisfy* (78).

Proof. By using formulas (38a), (38b), and (38c) for $\sinh a$, $\sinh b$, $\sinh c$ and formulas (41a), (41b), and (41c) for $\sin\alpha$, $\sin\beta$, $\sin\delta$, we write:

$$\frac{\sinh a}{\sin\alpha} = \frac{r_a(w_{01}-v_{01})}{yz} : \frac{x(w_{03}-w_{02})}{r_b r_c} = \frac{w_{01}-v_{01}}{w_{03}-w_{02}} \frac{xyz}{r_a r_b r_c},$$
$$\frac{\sinh b}{\sin\beta} = \frac{w_{02}-v_{02}}{r_b xz} : \frac{y(v_{03}+v_{01})}{r_a r_c} = \frac{w_{02}-v_{02}}{v_{03}+v_{01}} \frac{xyz}{r_a r_b r_c},$$
$$\frac{\sinh c}{\sin\delta} = \frac{w_{03}-v_{03}}{r_c yx} : \frac{z(w_{01}+v_{02})}{r_b r_a} = \frac{w_{03}-v_{03}}{w_{01}+v_{02}} \frac{xyz}{r_a r_b r_c}. \quad (98)$$

It is seen that these equations contain a common factor which is symmetric with respect to a, b, c and x, y, z. Multiply all equations of (97) by this factor. We arrive to (78). □

Theorem 6. *The sides and the angles of triangle $\widetilde{\triangle}ABC$ satisfy the following equation:*

$$\cos\delta = \sin\alpha\sin\beta\cosh c - \cos\alpha\cos\beta. \quad (99)$$

Proof. Evaluate the first term of the right-hand side of (99) by making use of (43a), (43b), and (43c):

$$\begin{aligned}\sin\alpha\sin\beta\cosh c &= \frac{1}{r_a r_b} O_{ca} O_{cb} (1 - \cos c_1 \cos c_2) \\ &= \frac{1}{r_a r_b} O_{ca} O_{cb} \times \left(1 - \frac{O_{cb}^2 - r_b^2 + r_c^2}{2r_c O_{cb}} \frac{O_{ca}^2 - r_a^2 + r_c^2}{2r_c O_{ac}}\right) \\ &= \frac{1}{r_a r_b} O_{ac} O_{cb} - \frac{1}{r_a r_b} O_{ac} O_{cb} \times \left(\frac{O_{cb}^2 - r_b^2 + r_c^2}{2r_c O_{cb}} \frac{O_{ac}^2 - r_a^2 + r_c^2}{2r_c O_{ac}}\right) \\ &= \frac{1}{4r_a r_b r_c^2} \left(4O_{ac}O_{cb}r_c^2 - O_{cb}^2 O_{ca}^2 - (O_{cb}^2 + O_{ca}^2) r_c^2 + O_{cb}^2 r_a^2 + O_{ac}^2 r_b^2\right).\end{aligned} \quad (100)$$

Now calculate the product $\cos\alpha\cos\beta$ by using the following formulae:

$$\cos\alpha = \frac{1}{2r_b r_c}\left(r_c^2 + r_b^2 - O_{cb}^2\right), \quad \cos\beta = -\frac{1}{2r_a r_c}\left(r_c^2 + r_a^2 - O_{ac}^2\right). \quad (101)$$

We obtain

$$\begin{aligned}\cos\alpha\cos\beta &= \frac{1}{2r_b r_c}\left(r_c^2 + r_b^2 - O_{cb}^2\right) \frac{1}{2r_a r_c}\left(r_c^2 + r_a^2 - O_{ac}^2\right) \\ &= \frac{1}{4r_b r_a r_c^2} \times \left(O_{cb}^2 O_{ac}^2 - (O_{cb}^2 + O_{ac}^2) r_c^2 - O_{cb}^2 r_a^2 - O_{ac}^2 r_b^2\right).\end{aligned} \quad (102)$$

By using (100) and (102) calculate the difference

$$\begin{aligned}&\sin\alpha\sin\beta\cosh c - \cos\alpha\cos\beta \\ &= \frac{1}{4r_a r_b r_c^2}\left(4O_{ac}O_{cb}r_c^2 - O_{cb}^2 O_{ac}^2 - (O_{cb}^2 + O_{ac}^2) r_c^2 + O_{cb}^2 r_a^2 + O_{ac}^2 r_b^2\right) - \frac{1}{4r_b r_a r_c^2} \\ &\quad \times \left(+O_{cb}^2 O_{ac}^2 - (O_{cb}^2 + O_{ac}^2) r_c^2 - O_{cb}^2 r_a^2 - O_{ac}^2 r_b^2\right) \\ &= \frac{1}{2r_b r_a}\left(r_a^2 + r_b^2 - O_{ab}^2\right) = \cos\delta.\end{aligned} \quad (103)$$

Thus, we got (99). □

The other two equations, (80) and (81), are proved analogously.

6. Concluding Remarks

We have proved three hyperbolic cosine-sine theorems for the triangular formed by arcs of three intersecting semicircles by using only elements of the Euclidean geometry. This geometrical figure is one of the basic figures of the Poincaré model of hyperbolic geometry. Conventionally, in the textbooks on the Poincaré model of hyperbolic geometry these theorems are proved by employing invariance properties of the cross-ratio with respect to Möbius group transformations. In that case one remains with the impression that the hyperbolic cosine-sine theorems are consequences of a special structure of the H half-plane. We have shown that the hyperbolic trigonometry as well as periodic trigonometry arises on Euclidean plane in a natural way.

The method of parametrization of the mass-shell equation via the hyperbolic angle, *the rapidity*, is widely used in the relativistic physics [4]. The quadratic equation (3) is closely related to the mass-shell equation

$$p_0^2 - p^2 = m^2 c^2, \quad (104)$$

where mc plays the role of radius of the semicircle $2mc = x_2 - x_1$. Within the framework of the present method we come to the following parametrization of energy momentum of the relativistic particle [11]:

$$p_0 = mc\coth(mc\phi), \qquad p = \frac{mc}{\sinh(mc\phi)}, \qquad 0 \le \phi < \infty. \quad (105)$$

In this way, the method developed in this paper may be used in order to give a new geometrical interpretation of rapidity widely used in relativistic physics (see, e.g., [12] and references therein).

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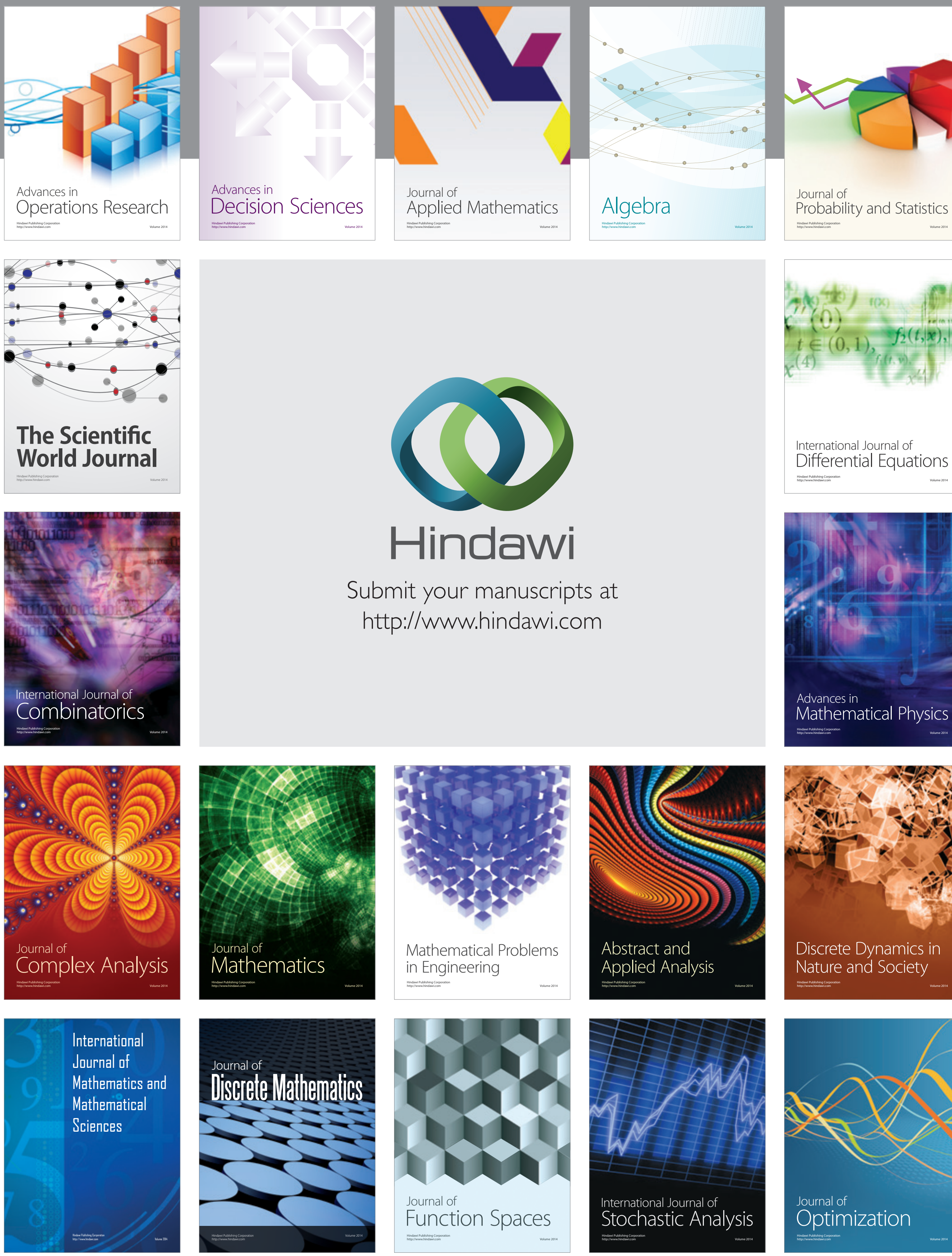
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