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Research Article

Nonlinear Instability for a Leslie-Gower Predator-Prey Model with Cross Diffusion

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A rigorous mathematical characterization for early-stage spatial and temporal patterns formation in a Leslie-Gower predator-prey model with cross diffusion is investigated. Given any general perturbation near an unstable constant equilibrium, we prove that its nonlinear evolution is dominated by the corresponding linear dynamics along a fixed finite number of the fastest growing modes.

1. Introduction

Since Turing proposed the striking idea of "diffusion-driven instability" in 1952 [1], reaction-diffusion systems are often employed to investigate chemical and biological pattern formations and have received much attention from the scientists [2–7]. However, most of the works concentrate on pattern formation in the case of linear instability, and there is a little discussion about the nonlinear effect of a reaction-diffusion system on the evolution of a nonuniform pattern.

In general, nonlinear instability is treated with great delicacy and difficulty. At first, nonlinear instability was established for nondissipative systems [8–11]. In 2004, Guo et al. [12] established nonlinear instability for an unstable Kirchhoff ellipse. Based upon a precise linear analysis, they found that the dynamics of general perturbation can be characterized by the linear dynamics of the fastest growing modes. This marks a beginning of a quantitative description of instability. Subsequently, Guo and Hwang dealt with nonlinear stability for a Keller-Segel model in [13] and described the early-stage pattern formation in that model.

Recently, Guo and Hwang considered the following reaction-diffusion system [14]

$$\frac{\partial U}{\partial t} = \nabla \cdot \left(D_1(U,V)\nabla U\right) + f(U,V),$$
$$\frac{\partial V}{\partial t} = \nabla \cdot \left(D_2(U,V)\nabla V\right) + g(U,V), \quad (1)$$

in a box $\mathbb{T}^N = (0,\pi)^N \subset \mathbb{R}^N (N \le 3)$ with the homogeneous Neumann boundary conditions. In system (1), $U(x,t)$, $V(x,t)$ denote the densities of two interactive species at time t, the functions D_1, D_2 are their diffusion rates, and f, g are the reaction functions. The classical Turing instability and Turing patterns were studied under some suitable conditions. Their result shows that the nonlinear evolution of patterns is dominated by the corresponding linear dynamics along a fixed finite number of the fastest growing modes over a time period.

In this paper, we consider the following Leslie-Gower predator-prey model with cross diffusion:

$$u_t - d_1\Delta u = \lambda u - u^2 - \beta uv, \quad x \in \mathbb{T}^N,\ t > 0,$$
$$v_t - d_2\Delta\left[(1+d_3u)v\right] = v\left(\mu - \frac{v}{m+u}\right), \quad x \in \mathbb{T}^N,\ t > 0,$$
$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial x_i} = 0, \quad x_i = 0,\pi,\ i = 1,\dots,N,\ t > 0,$$
$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{T}^N, \quad (2)$$

where $u(x,t)$ and $v(x,t)$ represent the densities of the species prey and predator, respectively. The parameters $\lambda, \beta, \mu, m, d_1, d_2$, and d_3 are all positive constants, where λ and μ are the intrinsic growth rates of the prey and predator, β is the predation rate, and the term $v/(m+u)$ is a modified Leslie-Gower term [15]. The constants d_1, d_2, called diffusion coefficients, represent the natural tendency of each species to diffuse to areas of smaller population concentration, and d_3, called a cross-diffusion coefficient, expresses the population flux of the predator resulting from the presence of the prey species. For more ecological backgrounds about this model, one can refer to [15–17].

System (2) and its variants were studied widely for pattern formation by applying the bifurcation theory and the degree theory [6, 18–20] in the case of linear instability. Inspired by the works [13, 14], in this paper, we attempt to study the nonlinear instability for this system and give a rigorous mathematical characterization for the nonlinear evolution of pattern by using a bootstrap technique. The mathematical approach in this paper is similar in spirit to that of [13, 14]. However, our problem (2) is much more complex. Notice that the diffusion term of the predator equation in the model (2) is

$$d_2\Delta\left[(1+d_3u)v\right]=\nabla\cdot\left[d_2d_3v\nabla u+d_2(1+d_3u)\nabla v\right]. \quad (3)$$

In some sense, the coupled degree in (2) is stronger than that in (1). As a result, our analysis here, especially in establishing H^2 estimates for nonlinear terms $d_2d_3\nabla(v\nabla u)$ and $d_2\nabla[(1+d_3u)\nabla v]$, is much more difficult and requires some techniques beyond those of [13, 14].

It is obvious that (2) has a unique positive equilibrium $(\tilde{u},\tilde{v})$ if and only if $\lambda>\beta\mu m$, where

$$\tilde{u}=\frac{\lambda-\beta\mu m}{1+\beta\mu},\qquad \tilde{v}=\frac{\mu(m+\lambda)}{1+\beta\mu}. \quad (4)$$

Let $\hat{u}=u(x,t)-\tilde{u}$, $\hat{v}=v(x,t)-\tilde{v}$ be the perturbation around $(\tilde{u},\tilde{v})$ and still denote it by (u,v). Then, the perturbation (u,v) satisfies the following strongly coupled equations:

$$\begin{aligned}
&u_t-d_1\Delta u=g_1(u+\tilde{u},v+\tilde{v}),\quad x\in\mathbb{T}^N,\ t>0,\\
&v_t-d_2\Delta\left[(1+d_3(u+\tilde{u}))(v+\tilde{v})\right]\\
&\quad=g_2(u+\tilde{u},v+\tilde{v}),\quad x\in\mathbb{T}^N,\ t>0,\\
&\frac{\partial u}{\partial x_i}=\frac{\partial v}{\partial x_i}=0,\quad x_i=0,\pi,\ i=1,\dots,N,\ t>0,\\
&u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad x\in\mathbb{T}^N,
\end{aligned} \quad (5)$$

where

$$g_1(u,v)=\lambda u-u^2-\beta uv,\qquad g_2(u,v)=v\left(\mu-\frac{v}{m+u}\right). \quad (6)$$

This paper is organized as follows. In Section 2, the growing modes in the linearized system are studied, which are important for our later discussions. Section 3 gives some estimates for the perturbation. The key is to control the nonlinear growth of high-order energy. In Section 4, the nonlinear instability is obtained.

2. Growing Modes in the Linearized System

The corresponding linearized system of (5) takes the form of

$$\begin{aligned}
&u_t-d_1\Delta u=-\tilde{u}u-\beta\tilde{u}v,\\
&v_t-d_2d_3\tilde{v}\Delta u-d_2(1+d_3\tilde{u})\Delta v=\mu^2u-\mu v.
\end{aligned} \quad (7)$$

We use $[\cdot,\cdot]$ to denote a column vector and let $\mathbf{w}(x,t)=[u(x,t),v(x,t)]$, $\mathbf{q}=(q_1,\dots,q_N)\in\mathbb{N}^N$. Then, $\mathbf{q}^2=\sum_{i=1}^N {q_i}^2$ are eigenvalues of $-\Delta$ on $\mathbb{T}^N$ under the homogeneous Neumann boundary condition, and the corresponding normalized eigenfunctions are given by

$$e_{\mathbf{q}}(x)=\begin{cases}\left(\dfrac{1}{\pi}\right)^{N/2}, & \mathbf{q}=\mathbf{0},\\[2mm] \left(\dfrac{2}{\pi}\right)^{N/2}\displaystyle\prod_{i=1}^N\cos(q_ix_i), & \mathbf{q}\neq\mathbf{0}.\end{cases} \quad (8)$$

This set of eigenfunctions forms an orthonormal basis in $L^2(\mathbb{T}^N)$.

We look for a normal mode to be the linear system (7) of the following form:

$$\mathbf{w}(x,t)=\mathbf{r}_{\mathbf{q}}e^{\lambda_{\mathbf{q}}t}e_{\mathbf{q}}(x), \quad (9)$$

where $\lambda_{\mathbf{q}}$ is a complex number and $\mathbf{r}_{\mathbf{q}}$ is a vector; they depend on $\mathbf{q}$. Substituting (9) into (7), we have

$$\lambda_{\mathbf{q}}\mathbf{r}_{\mathbf{q}}=\begin{pmatrix}-\tilde{u}-d_1\mathbf{q}^2 & -\beta\tilde{u}\\ \mu^2-d_2d_3\tilde{v}\mathbf{q}^2 & -\mu-d_2(1+d_3\tilde{u})\mathbf{q}^2\end{pmatrix}\mathbf{r}_{\mathbf{q}}:=L\mathbf{r}_{\mathbf{q}}. \quad (10)$$

System (7) possesses a nontrivial normal mode if and only if

$$\det\begin{pmatrix}\lambda_{\mathbf{q}}+\tilde{u}+d_1\mathbf{q}^2 & \beta\tilde{u}\\ -\mu^2+d_2d_3\tilde{v}\mathbf{q}^2 & \lambda_{\mathbf{q}}+\mu+d_2(1+d_3\tilde{u})\mathbf{q}^2\end{pmatrix}=0, \quad (11)$$

which is equivalent to

$$\begin{aligned}
&\lambda_{\mathbf{q}}^2+\left\{\tilde{u}+\mu+\left[d_1+d_2(1+d_3\tilde{u})\right]\mathbf{q}^2\right\}\lambda_{\mathbf{q}}+d_1d_2(1+d_3\tilde{u})\mathbf{q}^4\\
&\quad+\left[d_1\mu+d_2(1+d_3\tilde{u})\tilde{u}-\beta d_2d_3\tilde{u}\tilde{v}\right]\mathbf{q}^2+\mu\tilde{u}+\beta\mu^2\tilde{u}=0.
\end{aligned} \quad (12)$$

Thus, we deduce the following well-known aggregation (i.e., linear instability) criterion by requiring that there exists a $\mathbf{q}$, such that the constant term in (12) is

$$\begin{aligned}
&d_1d_2(1+d_3\tilde{u})\mathbf{q}^4+\left[d_1\mu+d_2(1+d_3\tilde{u})\tilde{u}-\beta d_2d_3\tilde{u}\tilde{v}\right]\mathbf{q}^2\\
&\quad+\mu\tilde{u}+\beta\mu^2\tilde{u}<0.
\end{aligned} \quad (13)$$

In this paper, we always assume that there exists a $\mathbf{q}$, such that (13) holds. Then, the discriminant of (12) is

$$\begin{aligned}
\Delta=\ &[d_1-d_2(1+d_3\tilde{u})]^2\mathbf{q}^4\\
&+\{2(\tilde{u}+\mu)\left[d_1+d_2(1+d_3\tilde{u})\right]\\
&\quad-4\left[d_1\mu+d_2(1+d_3\tilde{u})\tilde{u}\right]+4\beta d_2d_3\tilde{u}\tilde{v}\}\mathbf{q}^2\\
&+(\tilde{u}-\mu)^2-4\beta\mu^2\tilde{u},
\end{aligned} \quad (14)$$

and the coefficient of $\mathbf{q}^2$ is positive, since (13) implies

$$d_1\mu + d_2\left(1 + d_3\tilde{u}\right)\tilde{u} - \beta d_2 d_3 \tilde{u}\tilde{v} < 0. \quad (15)$$

For given $\mathbf{q}$, we denote the corresponding eigenvalues by $\lambda_{\mathbf{q}}^{\pm}$ and eigenvectors by $\mathbf{r}_{\pm}(\mathbf{q})$. We split it into three cases for the linear analysis.

(1) $\Delta > 0$. Let $\Lambda_1 = \{\mathbf{q} \mid \Delta > 0\}$, and let $\lambda_{\mathbf{q}}^{\pm}$ be two distinct real roots with $\lambda_{\mathbf{q}}^{+} > \lambda_{\mathbf{q}}^{-}$, $\lambda_{\mathbf{q}}^{\pm}$ being the corresponding (linearly independent) real eigenvectors. It is easy to see that

$$\mathbf{r}_{\pm}(\mathbf{q}) = \left[1, -\frac{\lambda_{\mathbf{q}}^{\pm} + \tilde{u} + d_1\mathbf{q}^2}{\beta\tilde{u}}\right]. \quad (16)$$

Denote

$$\Lambda_* = \left\{\mathbf{q} \mid d_1 d_2\left(1 + d_3\tilde{u}\right)\mathbf{q}^4 + \left[d_1\mu + d_2\left(1 + d_3\tilde{u}\right)\tilde{u} - \beta d_2 d_3\tilde{u}\tilde{v}\right]\mathbf{q}^2 + \mu\tilde{u} + \beta\mu^2\tilde{u} < 0\right\}. \quad (17)$$

Clearly, $\lambda_{\mathbf{q}}^{+} > 0$ for $\mathbf{q} \in \Lambda_*$. Note that there are only finitely many $\mathbf{q}$ in Λ_* and $\Lambda_* \subset \Lambda_1$. Therefore, there are only finitely many linear growing modes, such that the constant equilibrium $(0,0)$ of (5) is unstable. Furthermore, we define

$$\lambda_{\max} = \max_{\lambda_{\mathbf{q}}^{+} > 0} \lambda_{\mathbf{q}}^{+}, \qquad \Lambda_{\max} = \left\{\mathbf{q} \mid \lambda_{\mathbf{q}}^{+} = \lambda_{\max}\right\}. \quad (18)$$

Then, $\Lambda_{\max} \subset \Lambda_* \subset \Lambda_1$.

(2) $\Delta = 0$. Let $\Lambda_2 = \{\mathbf{q} \mid \Delta = 0\}$. In this case, (12) possesses repeated real eigenvalues. Consider

$$\lambda_{\mathbf{q}} = \lambda_{\mathbf{q}}^{+} = \lambda_{\mathbf{q}}^{-} = -\frac{1}{2}\left\{\tilde{u} + \mu + \left[d_1 + d_2\left(1 + d_3\tilde{u}\right)\right]\mathbf{q}^2\right\} < 0. \quad (19)$$

The corresponding eigenvectors are

$$\mathbf{r}(\mathbf{q}) = \mathbf{r}_{+}(\mathbf{q}) = \mathbf{r}_{-}(\mathbf{q}) = \left[1, -\frac{\lambda_{\mathbf{q}} + \tilde{u} + d_1\mathbf{q}^2}{\beta\tilde{u}}\right], \quad (20)$$

and we can find another independent vector $\mathbf{r}'(\mathbf{q}) = [0, -1/(\beta\tilde{u})]$, satisfying

$$\left(L - \lambda_{\mathbf{q}} I\right)\mathbf{r}'(\mathbf{q}) = \mathbf{r}(\mathbf{q}). \quad (21)$$

(3) $\Delta < 0$. The complex case is where (12) possesses a pair of complex eigenvalues with a negative real part. Denote $\Lambda_3 = \{\mathbf{q}\Delta < 0\}$, and for any $\mathbf{q} \in \Lambda_3$, denote

$$\lambda_{\mathbf{q}}^{+} = \operatorname{Re}\lambda_{\mathbf{q}} + i\operatorname{Im}\lambda_{\mathbf{q}}, \qquad \mathbf{r}_{+}(\mathbf{q}) = \operatorname{Re}\mathbf{r}(\mathbf{q}) + i\operatorname{Im}\mathbf{r}(\mathbf{q}). \quad (22)$$

Then,

$$\lambda_{\mathbf{q}}^{-} = \operatorname{Re}\lambda_{\mathbf{q}} - i\operatorname{Im}\lambda_{\mathbf{q}}, \qquad \mathbf{r}_{-}(\mathbf{q}) = \operatorname{Re}\mathbf{r}(\mathbf{q}) - i\operatorname{Im}\mathbf{r}(\mathbf{q}), \quad (23)$$

where $\operatorname{Re}\mathbf{r}(\mathbf{q})$ and $\operatorname{Im}\mathbf{r}(\mathbf{q})$ are linearly independent vectors.

Given any initial perturbation $\mathbf{w}(x,0)$, we can expand it as follows:

$$\begin{aligned}\mathbf{w}(x,0) = {} & \sum_{\mathbf{q}\in\Lambda_1}\left\{w_{\mathbf{q}}^{-}\mathbf{r}_{-}(\mathbf{q}) + w_{\mathbf{q}}^{+}\mathbf{r}_{+}(\mathbf{q})\right\}e_{\mathbf{q}}(x) \\ & + \sum_{\mathbf{q}\in\Lambda_2}\left\{w_{\mathbf{q}}\mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}'\mathbf{r}'(\mathbf{q})\right\}e_{\mathbf{q}}(x) \\ & + \sum_{\mathbf{q}\in\Lambda_3}\left\{w_{\mathbf{q}}^{\mathrm{Re}}\operatorname{Re}\mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}^{\mathrm{Im}}\operatorname{Im}\mathbf{r}(\mathbf{q})\right\}e_{\mathbf{q}}(x) \\ := {} & \sum_{\mathbf{q}\in\mathbb{N}^N}\mathbf{w}_{\mathbf{q}}e_{\mathbf{q}}(x),\end{aligned} \quad (24)$$

where $w_{\mathbf{q}}^{-}$, $w_{\mathbf{q}}^{+}$, $w_{\mathbf{q}}$, $w_{\mathbf{q}}'$, $w_{\mathbf{q}}^{\mathrm{Re}}$, and $w_{\mathbf{q}}^{\mathrm{Im}}$ are constants, and

$$\begin{aligned}\mathbf{w}_{\mathbf{q}} &= w_{\mathbf{q}}^{-}\mathbf{r}_{-}(\mathbf{q}) + w_{\mathbf{q}}^{+}\mathbf{r}_{+}(\mathbf{q}), \quad \mathbf{q}\in\Lambda_1, \\ \mathbf{w}_{\mathbf{q}} &= w_{\mathbf{q}}\mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}'\mathbf{r}'(\mathbf{q}), \quad \mathbf{q}\in\Lambda_2, \\ \mathbf{w}_{\mathbf{q}} &= w_{\mathbf{q}}^{\mathrm{Re}}\operatorname{Re}\mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}^{\mathrm{Im}}\operatorname{Im}\mathbf{r}(\mathbf{q}), \quad \mathbf{q}\in\Lambda_3.\end{aligned} \quad (25)$$

The unique solution $\mathbf{w}(x,t) = [u(x,t), v(x,t)]$ to (7) is given by

$$\begin{aligned}\mathbf{w}(x,t) = {} & \sum_{\mathbf{q}\in\Lambda_1}\left\{w_{\mathbf{q}}^{-}\mathbf{r}_{-}(\mathbf{q})e^{\lambda_{\mathbf{q}}^{-}t} + w_{\mathbf{q}}^{+}\mathbf{r}_{+}(\mathbf{q})e^{\lambda_{\mathbf{q}}^{+}t}\right\}e_{\mathbf{q}}(x) \\ & + \sum_{\mathbf{q}\in\Lambda_2}\left\{w_{\mathbf{q}}\mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}'\mathbf{r}'(\mathbf{q}) + w_{\mathbf{q}}'\mathbf{r}(\mathbf{q})t\right\}e^{\lambda_{\mathbf{q}}t}e_{\mathbf{q}}(x) \\ & + \sum_{\mathbf{q}\in\Lambda_3}\left\{w_{\mathbf{q}}^{\mathrm{Re}}\left(\operatorname{Re}\mathbf{r}(\mathbf{q})\cos\left[\left(\operatorname{Im}\lambda_{\mathbf{q}}\right)t\right] - \operatorname{Im}\mathbf{r}(\mathbf{q})\sin\left[\left(\operatorname{Im}\lambda_{\mathbf{q}}\right)t\right]\right)\right. \\ & \left. + w_{\mathbf{q}}^{\mathrm{Im}}\left(\operatorname{Re}\mathbf{r}(\mathbf{q})\sin\left[\left(\operatorname{Im}\lambda_{\mathbf{q}}\right)t\right] + \operatorname{Im}\mathbf{r}(\mathbf{q})\cos\left[\left(\operatorname{Im}\lambda_{\mathbf{q}}\right)t\right]\right)\right\} \\ & \times e^{(\operatorname{Re}\lambda_{\mathbf{q}})t}e_{\mathbf{q}}(x) \\ := {} & e^{Lt}\mathbf{w}(x,0).\end{aligned} \quad (26)$$

In the sequel, the constant C_0 will only depend on the domain $\mathbb{T}^N$ and the dimension N, and the generic constants C_1, C_2, $\widehat{C}_1$, $\widehat{C}_2$, and so forth will depend on $\mathbb{T}^N$, N, and the parameters λ, β, m, μ, d_1, d_2, and d_3. Our main result of this section is the following lemma.

Lemma 1. *Assume that the instability criterion* (13) *is valid. Suppose that*

$$\mathbf{w}(x,t) = [u(x,t), v(x,t)] := e^{Lt}\mathbf{w}(x,0) \quad (27)$$

is a solution to the linearized system (7) *with the initial condition* $\mathbf{w}(x,0)$. *Then, there exists a constant* $\widehat{C}_1 \geq 1$, *such that*

$$\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{T}^N)} \leq \widehat{C}_1 e^{\lambda_{\max}t}\|\mathbf{w}(\cdot,0)\|_{L^2(\mathbb{T}^N)}, \quad (28)$$

for all $t \geq 0$.

Proof. We first consider the case for $t > 1$. For any $\mathbf{q} \in \Lambda_1$,

$$\left|\det\left[\mathbf{r}_-(\mathbf{q}),\mathbf{r}_+(\mathbf{q})\right]\right| = \frac{\lambda_{\mathbf{q}}^+ - \lambda_{\mathbf{q}}^-}{\beta\widetilde{u}} = \frac{\sqrt{\Delta}}{\beta\widetilde{u}}, \tag{29}$$

where Δ is given by (14). Applying Cramer's rule to (25), we have

$$\begin{aligned}\left|w_{\mathbf{q}}^{\pm}\right| &= \left|\frac{\det\left[\mathbf{r}_{\mp}(\mathbf{q}),\mathbf{w}_{\mathbf{q}}\right]}{\det\left[\mathbf{r}_-(\mathbf{q}),\mathbf{r}_+(\mathbf{q})\right]}\right| \le \frac{\left|\mathbf{r}_{\mp}(\mathbf{q})\right| \times \left|\mathbf{w}_{\mathbf{q}}\right|}{\left|\det\left[\mathbf{r}_-(\mathbf{q}),\mathbf{r}_+(\mathbf{q})\right]\right|} \\ &= \frac{\beta\widetilde{u}\left|\mathbf{r}_{\mp}(\mathbf{q})\right| \times \left|\mathbf{w}_{\mathbf{q}}\right|}{\sqrt{\Delta}},\end{aligned} \tag{30}$$

where

$$\left|\mathbf{r}_{\pm}(\mathbf{q})\right| = \sqrt{1 + \frac{\left(\lambda_{\mathbf{q}}^{\pm} + \widetilde{u} + d_1\mathbf{q}^2\right)^2}{(\beta\widetilde{u})^2}}. \tag{31}$$

It follows from (14) that there exist positive constants M_1 and

$$\begin{aligned}C_1 = {} & (\widetilde{u}+\mu)\left[d_1 + d_2\left(1+d_3\widetilde{u}\right)\right] \\ & - 2\left[d_1\mu + d_2\left(1+d_3\widetilde{u}\right)\widetilde{u}\right] + 2\beta d_2 d_3 \widetilde{u}\widetilde{v},\end{aligned} \tag{32}$$

such that $\Delta > C_1\mathbf{q}^2$ for all $|\mathbf{q}| > M_1$. Hence, for any $|\mathbf{q}| > M_1$,

$$\left|w_{\mathbf{q}}^{\pm}\right| \le \frac{\beta\widetilde{u}\left|\mathbf{r}_{\mp}(\mathbf{q})\right| \times \left|\mathbf{w}_{\mathbf{q}}\right|}{\sqrt{C_1}\,|\mathbf{q}|}, \tag{33}$$

and by (12),

$$\lim_{|\mathbf{q}|\to\infty} \frac{\lambda_{\mathbf{q}}^{\pm}}{\mathbf{q}^2} = -d_1, -d_2\left(1+d_3\widetilde{u}\right). \tag{34}$$

Thus,

$$\begin{aligned}\lim_{|\mathbf{q}|\to\infty} \frac{\left|\mathbf{r}_{\pm}(\mathbf{q})\right|}{\mathbf{q}^2} &= \lim_{|\mathbf{q}|\to\infty} \frac{\left|\lambda_{\mathbf{q}}^{\pm} + \widetilde{u} + d_1\mathbf{q}^2\right|}{\beta\widetilde{u}\mathbf{q}^2} \\ &= \frac{1}{\beta\widetilde{u}}\left(\lim_{|\mathbf{q}|\to\infty}\left|\frac{\lambda_{\mathbf{q}}^{\pm}}{\mathbf{q}^2} + d_1\right|\right) \\ &\le \frac{d_1 + d_2\left(1+d_3\widetilde{u}\right)}{\beta\widetilde{u}}.\end{aligned} \tag{35}$$

Consequently, there exists a positive constant $M_2 > M_1$, such that

$$\left|\mathbf{r}_{\pm}(\mathbf{q})\right| \le \left(\frac{d_1 + d_2\left(1+d_3\widetilde{u}\right)}{\beta\widetilde{u}} + 1\right)\mathbf{q}^2 \tag{36}$$

for any $|\mathbf{q}| > M_2$. Substituting this into (33) yields

$$\left|w_{\mathbf{q}}^{\pm}\right| \le C_2\,|\mathbf{q}|\left|\mathbf{w}_{\mathbf{q}}\right| \tag{37}$$

for any $|\mathbf{q}| > M_2$, where $C_2 = (d_1 + d_2(1+d_3\widetilde{u}) + \beta\widetilde{u})/\sqrt{C_1}$. We thus obtain

$$\left|w_{\mathbf{q}}^{\pm}\mathbf{r}_{\pm}(\mathbf{q})\,e^{\lambda_{\mathbf{q}}^{\pm}t}\right| \le C_2\,|\mathbf{q}|\left|\mathbf{w}_{\mathbf{q}}\right|\left|\mathbf{r}_{\pm}(\mathbf{q})\right|e^{\lambda_{\mathbf{q}}^{\pm}t}. \tag{38}$$

Since

$$\begin{aligned}&\lim_{|\mathbf{q}|\to\infty} |\mathbf{q}|\left|\mathbf{r}_{\pm}(\mathbf{q})\right|e^{\lambda_{\mathbf{q}}^{\pm}t} \\ &\quad\le \lim_{|\mathbf{q}|\to\infty} |\mathbf{q}|\sqrt{1 + \frac{\left(\lambda_{\mathbf{q}}^{\pm} + \widetilde{u} + d_1\mathbf{q}^2\right)^2}{(\beta\widetilde{u})^2}} \\ &\qquad\times e^{-\min\{d_1/2,\, d_2(1+d_3\widetilde{u})/2\}\mathbf{q}^2 t} \\ &\quad= 0,\end{aligned} \tag{39}$$

there exists a constant $M_3 > M_2$, such that

$$\left|w_{\mathbf{q}}^{\pm}\mathbf{r}_{\pm}(\mathbf{q})\,e^{\lambda_{\mathbf{q}}^{\pm}t}\right| \le C_2\left|\mathbf{w}_{\mathbf{q}}\right| \tag{40}$$

for any $|\mathbf{q}| > M_3$.

For any $\mathbf{q} \in \Lambda_1$ and $|\mathbf{q}| \le M_3$, as Δ is an increasing function of $|\mathbf{q}|^2$, we denote

$$M_* = \min\left\{|\mathbf{q}|\,\Delta(\mathbf{q}) > 0\right\}. \tag{41}$$

With the help of (30) and (31), we get

$$\left|w_{\mathbf{q}}^{\pm}\mathbf{r}_{\pm}(\mathbf{q})\,e^{\lambda_{\mathbf{q}}^{\pm}t}\right| \le C_3\left|\mathbf{w}_{\mathbf{q}}\right|e^{\lambda_{\max}t}, \tag{42}$$

where C_3 only depends on $\lambda, \beta, \mu, m, M_*$, and M_3. Hence, we conclude that, for any $\mathbf{q} \in \Lambda_1$, there exists a positive constant $C_4 = \max\{C_2, C_3\}$, such that

$$\left|w_{\mathbf{q}}^{\pm}\mathbf{r}_{\pm}(\mathbf{q})\,e^{\lambda_{\mathbf{q}}^{\pm}t}\right| \le C_4\left|\mathbf{w}_{\mathbf{q}}\right|e^{\lambda_{\max}t}. \tag{43}$$

For all $\mathbf{q} \in \Lambda_2$ and $\mathbf{q} \in \Lambda_3$, by some similar arguments as above we can show that there exist positive constants C_5 and C_6, such that

$$\begin{aligned}&\left|w_{\mathbf{q}}\mathbf{r}(\mathbf{q})\right|, \left|w_{\mathbf{q}}'\mathbf{r}'(\mathbf{q})\right|, \left|w_{\mathbf{q}}'\mathbf{r}(\mathbf{q})\right| \le C_5\left|\mathbf{w}_{\mathbf{q}}\right|, \quad te^{\lambda_{\mathbf{q}}t} \le C_5, \\ &\left|w_{\mathbf{q}}^{\mathrm{Re}}\operatorname{Re}\mathbf{r}(\mathbf{q})\right|, \left|w_{\mathbf{q}}^{\mathrm{Re}}\operatorname{Im}\mathbf{r}(\mathbf{q})\right|, \left|w_{\mathbf{q}}^{\mathrm{Im}}\operatorname{Re}\mathbf{r}(\mathbf{q})\right|, \left|w_{\mathbf{q}}^{\mathrm{Im}}\operatorname{Im}\mathbf{r}(\mathbf{q})\right| \\ &\qquad \le C_6\left|\mathbf{w}_{\mathbf{q}}\right|.\end{aligned} \tag{44}$$

Next, we derive the energy estimate in L^2 for $\mathbf{w}(x,t)$. Recall that $\{e_{\mathbf{q}}(x)\}_{\mathbf{q}\in\mathbb{N}^N}$ is an orthonormal basis in $L^2(\mathbb{T}^N)$. Then,

$$\|\,\mathbf{w}(\cdot,t)\,\|_{L^2(\mathbb{T}^N)}^2 = A_1 + A_2 + A_3, \tag{45}$$

where

$$A_1 = \sum_{\mathbf{q}\in\Lambda_1} \{w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) e^{\lambda_{\mathbf{q}}^- t} + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e^{\lambda_{\mathbf{q}}^+ t}\}^2,$$
$$A_2 = \sum_{\mathbf{q}\in\Lambda_2} \{w_{\mathbf{q}} \mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}' \mathbf{r}'(\mathbf{q}) + w_{\mathbf{q}}' \mathbf{r}(\mathbf{q}) t\}^2 e^{2\lambda_{\mathbf{q}} t},$$
$$A_3 = \sum_{\mathbf{q}\in\Lambda_3} \Big\{ w_{\mathbf{q}}^{\mathrm{Re}} \big(\operatorname{Re} \mathbf{r}(\mathbf{q}) \cos\left[\left(\operatorname{Im}\lambda_{\mathbf{q}}\right) t\right] - \operatorname{Im} \mathbf{r}(\mathbf{q}) \sin\left[\left(\operatorname{Im}\lambda_{\mathbf{q}}\right) t\right] \big) + w_{\mathbf{q}}^{\mathrm{Im}} \big(\operatorname{Re} \mathbf{r}(\mathbf{q}) \sin\left[\left(\operatorname{Im}\lambda_{\mathbf{q}}\right) t\right] + \operatorname{Im} \mathbf{r}(\mathbf{q}) \cos\left[\left(\operatorname{Im}\lambda_{\mathbf{q}}\right) t\right] \big) \Big\}^2 e^{2(\operatorname{Re}\lambda_{\mathbf{q}}) t}. \quad (46)$$

From (43) and (44), we obtain

$$\begin{aligned}
A_1 &\le \sum_{\mathbf{q}\in\Lambda_1} \{|w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) e^{\lambda_{\mathbf{q}}^- t}| + |w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e^{\lambda_{\mathbf{q}}^+ t}|\}^2 \\
&\le 4C_4^2 \sum_{\mathbf{q}\in\Lambda_1} |\mathbf{w}_{\mathbf{q}}|^2 e^{2\lambda_{\max} t}, \\
A_2 &\le \sum_{\mathbf{q}\in\Lambda_2} \left\{ \left| w_{\mathbf{q}} \mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}' \mathbf{r}'(\mathbf{q}) \right| + \left| w_{\mathbf{q}}' \mathbf{r}(\mathbf{q}) \right| t \right\}^2 e^{2\lambda_{\mathbf{q}} t} \\
&\le 8C_5^2 \sum_{\mathbf{q}\in\Lambda_2} \left\{ \left| \mathbf{w}_{\mathbf{q}} \right|^2 + \left| w_{\mathbf{q}}' \mathbf{r}(\mathbf{q}) \right|^2 t^2 \right\} e^{2\lambda_{\mathbf{q}} t} \\
&\le 8C_5^2 \sum_{\mathbf{q}\in\Lambda_2} \left\{ \left| \mathbf{w}_{\mathbf{q}} \right|^2 e^{2\lambda_{\mathbf{q}} t} + C_5^2 \left| w_{\mathbf{q}}' \mathbf{r}(\mathbf{q}) \right|^2 \right\} \\
&\le 8C_5^2 \sum_{\mathbf{q}\in\Lambda_2} \left\{ \left| \mathbf{w}_{\mathbf{q}} \right|^2 e^{2\lambda_{\mathbf{q}} t} + C_5^4 \left| \mathbf{w}_{\mathbf{q}} \right|^2 \right\} \\
&\le 8C_5^2 \left(1 + C_5^4\right) \sum_{\mathbf{q}\in\Lambda_2} |\mathbf{w}_{\mathbf{q}}|^2 e^{2\lambda_{\max} t}, \\
A_3 &\le \sum_{\mathbf{q}\in\Lambda_3} \left\{ \left(|\operatorname{Re} \mathbf{r}(\mathbf{q})| + |\operatorname{Im} \mathbf{r}(\mathbf{q})| \right) \left(\left| w_{\mathbf{q}}^{\mathrm{Re}} \right| + \left| w_{\mathbf{q}}^{\mathrm{Im}} \right| \right) \right\}^2 e^{2(\operatorname{Re}\lambda_{\mathbf{q}}) t} \\
&\le 16C_6^2 \sum_{\mathbf{q}\in\Lambda_3} |\mathbf{w}_{\mathbf{q}}|^2 e^{2(\operatorname{Re}\lambda_{\mathbf{q}}) t} \\
&\le 16C_6^2 \sum_{\mathbf{q}\in\Lambda_3} |\mathbf{w}_{\mathbf{q}}|^2 e^{2\lambda_{\max} t}.
\end{aligned} \quad (47)$$

Thus,

$$\begin{aligned}
\|\mathbf{w}(\cdot, t)\|_{L^2(\mathbb{T}^N)}^2 &\le C_7^2 \left(\sum_{\mathbf{q}\in\Lambda_1} |\mathbf{w}_{\mathbf{q}}|^2 + \sum_{\mathbf{q}\in\Lambda_2} |\mathbf{w}_{\mathbf{q}}|^2 + \sum_{\mathbf{q}\in\Lambda_3} |\mathbf{w}_{\mathbf{q}}|^2 \right) e^{2\lambda_{\max} t} \quad (48) \\
&= C_7^2 \|\mathbf{w}(\cdot, 0)\|_{L^2(\mathbb{T}^N)}^2 e^{2\lambda_{\max} t},
\end{aligned}$$

where $C_7^2 = \max\{4C_4^2, 8C_5^2(1 + C_5^4), 16C_6^2\}$. Finally, for any $t > 1$, we have

$$\|\mathbf{w}(\cdot, t)\|_{L^2(\mathbb{T}^N)} \le C_7 \|\mathbf{w}(\cdot, 0)\|_{L^2(\mathbb{T}^N)} e^{\lambda_{\max} t}. \quad (49)$$

For finite time $t \le 1$, we multiply the first and second equations of (7) by u and Av, respectively, then add them and use the integration by parts to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_{\mathbb{T}^N} \left\{ |u|^2 + A|v|^2 \right\} dx \\
&\quad + \int_{\mathbb{T}^N} \left\{ d_1 |\nabla u|^2 + A d_2 \left(1 + d_3 \tilde{u}\right) |\nabla v|^2 + A d_2 d_3 \tilde{v} \nabla u \nabla v \right\} dx \\
&\quad + \int_{\mathbb{T}^N} \left\{ \tilde{u} u^2 + A\mu v^2 \right\} dx \\
&= \int_{\mathbb{T}^N} \left(A\mu^2 - \beta\tilde{u} \right) uv \, dx.
\end{aligned} \quad (50)$$

Firstly, we claim that the integrand of the second integral in (50) satisfies

$$\begin{aligned}
&d_1 |\nabla u|^2 + A d_2 \left(1 + d_3 \tilde{u}\right) |\nabla v|^2 + A d_2 d_3 \tilde{v} \nabla u \nabla v \\
&\quad \ge \frac{1}{2} \left(d_1 |\nabla u|^2 + A d_2 \left(1 + d_3 \tilde{u}\right) |\nabla v|^2 \right)
\end{aligned} \quad (51)$$

for some positive constant A. Obviously, it suffices to require that

$$\left(A d_2 d_3 \tilde{v}\right)^2 - A d_1 d_2 \left(1 + d_3 \tilde{u}\right) < 0. \quad (52)$$

This is equivalent to

$$A < \frac{d_1 \left(1 + d_3 \tilde{u}\right)}{d_2 d_3^2 \tilde{v}^2}. \quad (53)$$

Denote

$$A_0 = \frac{d_1 \left(1 + d_3 \tilde{u}\right)}{2 d_2 d_3^2 \tilde{v}^2}. \quad (54)$$

On the other hand, the term on the right of (50) is

$$\begin{aligned}
\int_{\mathbb{T}^N} \left(A\mu^2 - \beta\tilde{u} \right) uv \, dx &\le \int_{\mathbb{T}^N} \left(A\mu^2 + \beta\tilde{u} \right) |u| \cdot |v| \, dx \\
&\le \int_{\mathbb{T}^N} \left(A\mu^2 + \beta\tilde{u} \right) \left(\frac{u^2 + v^2}{2} \right) dx \\
&= \frac{A\mu^2 + \beta\tilde{u}}{2} \int_{\mathbb{T}^N} \left(u^2 + v^2 \right) dx.
\end{aligned} \quad (55)$$

Taking $A = A_0$, and substituting (51) and (55) into (50), we get

$$\frac{d}{dt} \int_{\mathbb{T}^N} \left\{ |u|^2 + A_0 |v|^2 \right\} dx \le \left(A_0 \mu^2 + \beta\tilde{u} \right) \int_{\mathbb{T}^N} \left(u^2 + v^2 \right) dx. \quad (56)$$

Integrating (56) from 0 to t leads to

$$\int_{\mathbb{T}^N} \left\{|u(x,t)|^2 + A_0|v(x,t)|^2\right\} dx \leq \int_{\mathbb{T}^N} \left\{|u(x,0)|^2 + A_0|v(x,0)|^2\right\} dx + \left(A_0\mu^2 + \beta\tilde{u}\right)\int_0^t \int_{\mathbb{T}^N} \left(u^2 + v^2\right) dx\, dt. \tag{57}$$

If $A_0 \geq 1$, then it follows from (57) that

$$\int_{\mathbb{T}^N} \left\{|u(x,t)|^2 + |v(x,t)|^2\right\} dx \leq A_0 \int_{\mathbb{T}^N} \left\{|u(x,0)|^2 + |v(x,0)|^2\right\} dx + \left(A_0\mu^2 + \beta\tilde{u}\right)\int_0^t \int_{\mathbb{T}^N} \left(u^2 + v^2\right) dx\, dt; \tag{58}$$

thus, the Gronwall inequality implies

$$\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{T}^N)} \leq \sqrt{A_0}\|\mathbf{w}(\cdot,0)\|_{L^2(\mathbb{T}^N)} \cdot e^{((A_0\mu^2+\beta\tilde{u})/2)t}. \tag{59}$$

Consequently, there exists a positive constant C_8, such that

$$\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{T}^N)} \leq C_8\sqrt{A_0}\|\mathbf{w}(\cdot,0)\|_{L^2(\mathbb{T}^N)} e^{\lambda_{\max} t} \tag{60}$$

for all $t \in [0,1]$ due to the boundedness of

$$e^{((A_0\mu^2+\beta\tilde{u})/2-\lambda_{\max})t}. \tag{61}$$

If $0 < A_0 < 1$, in the same way as above, there exists a positive constant C_9, such that

$$\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{T}^N)} \leq C_9\|\mathbf{w}(\cdot,0)\|_{L^2(\mathbb{T}^N)} e^{\lambda_{\max} t}. \tag{62}$$

The proof is completed by taking $\widehat{C}_1 = \max\{C_7, C_8\sqrt{A_0}, C_9\}$. □

3. The Estimates for the Solutions of the Full System (5)

The general theory in [21] guarantees that (5) has a unique nonnegative local solution. The results can be summarized as follows.

Lemma 2. *Suppose that* $\mathbf{w}(x,t) = [u,v]$ *is a solution of the full system* (5)*. For* $s \geq 1$ $(N = 1)$ *and* $s \geq 2$ $(N = 2,3)$*, there exist a* $T > 0$ *and a constant C, such that*

$$\|\mathbf{w}(\cdot,t)\|_{H^s(\mathbb{T}^N)} \leq C\|\mathbf{w}(\cdot,0)\|_{H^s(\mathbb{T}^N)}, \quad 0 \leq t < T \tag{63}$$

if $u_0(x), v_0(x) \in H^s(\mathbb{T}^N)$.

Denote

$$\partial_{ij}u := \frac{\partial^2 u}{\partial x_i \partial x_j}, \qquad \partial_i u := \frac{\partial u}{\partial x_i}, \qquad i,j = 1,\ldots,N. \tag{64}$$

In order to derive the H^2 estimate for the solution of (5), we first prove the following energy estimates.

Lemma 3. *Suppose that* $\mathbf{w}(x,t) = [u,v]$ *is a solution of the full system* (5)*. Then,*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{i,j=1}^N \int_{\mathbb{T}^N} \left\{\left|\partial_{ij}u\right|^2 + \left|\partial_{ij}v\right|^2\right\} dx &+ \sum_{i,j=1}^N \int_{\mathbb{T}^N} \left\{\frac{d_1}{4}\left|\nabla\partial_{ij}u\right|^2 dx + \frac{A_0}{2}d_2\left(1 + d_3\tilde{u}\right)\left|\nabla\partial_{ij}v\right|^2\right\} dx \\ &+ \tilde{u}\sum_{i,j=1}^N \int_{\mathbb{T}^N} \left|\partial_{ij}u\right|^2 dx + \frac{A_0\mu}{2}\sum_{i,j=1}^N \int_{\mathbb{T}^N} \left|\partial_{ij}v\right|^2 dx \\ &\leq \widehat{C}_2(1+\eta)^2\|\mathbf{w}(\cdot,t)\|_{H^2(\mathbb{T}^N)}\left\|\nabla^3\mathbf{w}(\cdot,t)\right\|^2_{L^2(\mathbb{T}^N)} + \widehat{C}_2\| u(\cdot,t)\|^2_{L^2(\mathbb{T}^N)}\end{aligned} \tag{65}$$

for $\|\mathbf{w}\|_{H^2(\mathbb{T}^N)} \leq \eta$.

Proof. We first notice that system (5) preserves the evenness of the solution; that is, if $\mathbf{w}(x_1,x_2,x_3,t)$ is a solution to (5), then $\mathbf{w}(-x_1,x_2,x_3,t)$, $\mathbf{w}(x_1,-x_2,x_3,t)$, and $\mathbf{w}(x_1,x_2,-x_3,t)$ are also solutions of (5). We can regard system (5) as a special case with the evenness of the periodic problem by a reflective and an even extension. For this reason, we may assume periodicity at the boundary of the extended $2\mathbb{T}^N = (-\pi,\pi)^N$. Taking the second order partial derivative of the first equation of (5), multiplying $\partial_{ij}u$, and integrating over the domain $2\mathbb{T}^N$ to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{2\mathbb{T}^N}\left|\partial_{ij}u\right|^2 dx &= \int_{2\mathbb{T}^N} \partial_{ij}\left\{d_1\Delta u + g_1\left(u+\tilde{u}, v+\tilde{v}\right)\right\}\partial_{ij}u\, dx \\ &= \int_{2\mathbb{T}^N} \partial_{ij}\left\{L_1(u,v) + N_1(u,v)\right\}\partial_{ij}u\, dx,\end{aligned} \tag{66}$$

where

$$L_1(u,v) = d_1\Delta u - \tilde{u}u - \beta\tilde{u}v, \tag{67}$$

$$N_1(u,v) = g_1\left(u+\tilde{u}, v+\tilde{v}\right) + \tilde{u}u + \beta\tilde{u}v = -u^2 - \beta uv \tag{68}$$

are the linear and nonlinear terms, respectively, then, we have

$$\begin{aligned}\int_{2\mathbb{T}^N}\left(\partial_{ij}L_1(u,v)\right)\partial_{ij}u\, dx &= -d_1\int_{2\mathbb{T}^N}\left|\nabla\partial_{ij}u\right|^2 dx - \tilde{u}\int_{2\mathbb{T}^N}\left|\partial_{ij}u\right|^2 dx \\ &\quad - \beta\tilde{u}\int_{2\mathbb{T}^N}\partial_{ij}u\partial_{ij}v\, dx,\end{aligned}$$

$$\begin{aligned}
&\int_{2\mathbb{T}^N} \left(\partial_{ij} N_1(u,v)\right) \partial_{ij} u \, dx \\
&= -\int_{2\mathbb{T}^N} \partial_{ij} \left\{u^2 + \beta u v\right\} \partial_{ij} u \, dx, \\
&= -\int_{2\mathbb{T}^N} \left\{2\partial_i u \partial_j u + 2u\partial_{ij} u + \beta v \partial_{ij} u + \beta \partial_i u \partial_j v \right. \\
&\qquad \left. + \beta \partial_j u \partial_i v + \beta u \partial_{ij} v\right\} \partial_{ij} u \, dx \\
&= -\int_{2\mathbb{T}^N} \left\{(2u + \beta v) \partial_{ij} u + \beta u \partial_{ij} v \right. \\
&\qquad \left. + \left(2\partial_i u \partial_j u + \beta \partial_i u \partial_j v + \beta \partial_j u \partial_i v\right)\right\} \partial_{ij} u \, dx \\
&\leq (2 + 2\beta) \|\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)} \left\|\partial_{ij} \mathbf{w}\right\|^2_{L^2(2\mathbb{T}^N)} \\
&\quad + (2 + 2\beta) \|\nabla \mathbf{w}\|_{L^\infty(2\mathbb{T}^N)} \|\nabla \mathbf{w}\|_{L^2(2\mathbb{T}^N)} \left\|\partial_{ij} \mathbf{w}\right\|_{L^2(2\mathbb{T}^N)}.
\end{aligned} \tag{69}$$

Similarly, taking the second order partial derivative of the second equation of (5), multiplying $\partial_{ij} v$, and integrating over the domain $2\mathbb{T}^N$ to get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{2\mathbb{T}^N} \left|\partial_{ij} v\right|^2 dx \\
&= \int_{2\mathbb{T}^N} \partial_{ij} \left\{d_2 \Delta \left[(1 + d_3 (u + \tilde{u}))(v + \tilde{v})\right] \right. \\
&\qquad \left. + g_2 (u + \tilde{u}, v + \tilde{v})\right\} \partial_{ij} v \, dx \\
&= \int_{2\mathbb{T}^N} \partial_{ij} \left\{L_2(u,v) + N_2(u,v)\right\} \partial_{ij} v \, dx,
\end{aligned} \tag{70}$$

where

$$L_2(u,v) = d_2 d_3 \tilde{v} \Delta u + d_2 (1 + d_3 \tilde{u}) \Delta v + \mu^2 u - \mu v, \tag{71}$$

$$\begin{aligned}
N_2(u,v) &= d_2 \Delta \left[(1 + d_3 (u + \tilde{u}))(v + \tilde{v})\right] \\
&\quad + g_2 (u + \tilde{u}, v + \tilde{v}) - L_2(u,v) \\
&= d_2 d_3 (v \Delta u + 2\nabla u \nabla v + u \Delta v) - \frac{(\mu u - v)^2}{m + u + \tilde{u}} \\
&:= N_2^{(1)}(u,v) + N_2^{(2)}(u,v),
\end{aligned} \tag{72}$$

thus,

$$\begin{aligned}
&\int_{2\mathbb{T}^N} \left(\partial_{ij} L_2(u,v)\right) \partial_{ij} v \, dx \\
&= -d_2 d_3 \tilde{v} \int_{2\mathbb{T}^N} \nabla \partial_{ij} u \nabla \partial_{ij} v \, dx \\
&\quad - d_2 (1 + d_3 \tilde{u}) \int_{2\mathbb{T}^N} \left|\nabla \partial_{ij} v\right|^2 dx \\
&\quad + \mu^2 \int_{2\mathbb{T}^N} \partial_{ij} u \partial_{ij} v \, dx - \mu \int_{2\mathbb{T}^N} |\partial_{ij} v|^2 dx,
\end{aligned}$$

$$\begin{aligned}
&\int_{2\mathbb{T}^N} \left(\partial_{ij} N_2^{(1)}(u,v)\right) \partial_{ij} v \, dx \\
&= d_2 d_3 \int_{2\mathbb{T}^N} \partial_{ij} (v \Delta u + 2\nabla u \nabla v + u \Delta v) \partial_{ij} v \, dx \\
&= d_2 d_3 \int_{2\mathbb{T}^N} \left\{\left(\partial_{ij} v\right) \Delta u + v \left(\Delta \partial_{ij} u\right) + \left(\partial_i v\right) \left(\Delta \partial_j u\right) \right. \\
&\qquad \left. + \left(\partial_j v\right) \left(\Delta \partial_i u\right)\right\} \partial_{ij} v \, dx \\
&\quad + 2 d_2 d_3 \int_{2\mathbb{T}^N} \left\{\left(\nabla \partial_{ij} u\right) \nabla v + \nabla u \left(\nabla \partial_{ij} v\right) \right. \\
&\qquad + \left(\nabla \partial_i u\right) \left(\nabla \partial_j v\right) \\
&\qquad \left. + \left(\nabla \partial_j u\right) \left(\nabla \partial_i v\right)\right\} \partial_{ij} v \, dx \\
&\quad + d_2 d_3 \int_{2\mathbb{T}^N} \left\{\left(\partial_{ij} u\right) \Delta v + u \left(\Delta \partial_{ij} v\right) \right. \\
&\qquad + \left(\partial_i u\right) \left(\Delta \partial_j v\right) \\
&\qquad \left. + \left(\partial_j u\right) \left(\Delta \partial_i v\right)\right\} \partial_{ij} v \, dx \\
&= -d_2 d_3 \int_{2\mathbb{T}^N} \left(\partial_{ij} v\right) \left(\nabla \partial_{ij} v\right) \nabla u \, dx \\
&\quad - d_2 d_3 \int_{2\mathbb{T}^N} \nabla v \left(\partial_{ij} u\right) \left(\nabla \partial_{ij} v\right) dx \\
&\quad - d_2 d_3 \int_{2\mathbb{T}^N} v \left(\nabla \partial_{ij} v\right) \left(\nabla \partial_{ij} u\right) dx + d_2 d_3 \\
&\quad \times \int_{2\mathbb{T}^N} \left\{\left(\partial_i v\right) \left(\Delta \partial_j u\right) + \left(\partial_j v\right) \left(\Delta \partial_i u\right)\right\} \partial_{ij} v \, dx \\
&\quad - 2 d_2 d_3 \int_{2\mathbb{T}^N} \left\{\left(\Delta \partial_i u\right) \partial_{ij} v \partial_j v \right. \\
&\qquad + \left(\nabla \partial_i u\right) \left(\nabla \partial_{ij} v\right) \partial_j v \\
&\qquad + \left(\Delta \partial_j u\right) \partial_{ij} v \partial_i v \\
&\qquad \left. + \left(\nabla \partial_j u\right) \left(\nabla \partial_{ij} v\right) \partial_i v\right\} dx \\
&\quad - d_2 d_3 \int_{2\mathbb{T}^N} u |\nabla \partial_{ij} v|^2 dx + d_2 d_3 \\
&\quad \times \int_{2\mathbb{T}^N} \left\{\left(\partial_i u\right) \left(\Delta \partial_j v\right) + \left(\partial_j u\right) \left(\Delta \partial_i v\right)\right\} \partial_{ij} v \, dx \\
&\leq 14 d_2 d_3 \|\nabla \mathbf{w}\|_{L^\infty(2\mathbb{T}^N)} \left\|\partial_{ij} \mathbf{w}\right\|_{L^2(2\mathbb{T}^N)} \left\|\nabla^3 \mathbf{w}\right\|_{L^2(2\mathbb{T}^N)} \\
&\quad + 2 d_2 d_3 \|\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)} \left\|\nabla^3 \mathbf{w}\right\|^2_{L^2(2\mathbb{T}^N)},
\end{aligned}$$

$$\begin{aligned}
&\int_{2\mathbb{T}^N} \left(\partial_{ij} N_2^{(2)}(u,v)\right) \partial_{ij} v \, dx \\
&= -\int_{2\mathbb{T}^N} \partial_{ij} \left(\frac{(\mu u - v)^2}{m + u + \tilde{u}}\right) \partial_{ij} v \, dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{2\mathbb{T}^N}\Big\{-\Big(2(\mu u - v)\left(\mu\partial_{ij}u - \partial_{ij}v\right) \\
&\qquad +2\left(\mu\partial_i u - \partial_i v\right)\left(\mu\partial_j u - \partial_j v\right)\Big)\times(m+u+\tilde{u})^{-1} \\
&\qquad -(\mu u - v)^2\left[\frac{2\partial_i u\partial_j u}{(m+u+\tilde{u})^3} - \frac{\partial_{ij}u}{(m+u+\tilde{u})^2}\right] \\
&\qquad +\Big(2(\mu u - v)\big[\left(\mu\partial_i u - \partial_i v\right)\partial_j u \\
&\qquad\qquad +\left(\mu\partial_j u - \partial_j v\right)\partial_i u\big]\Big) \\
&\qquad \times(m+u+\tilde{u})^{-2}\Big\}\,\partial_{ij}v\,dx \\
&\le \frac{2(\mu+1)^2}{m}\|\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)}\left\|\partial_{ij}\mathbf{w}\right\|^2_{L^2(2\mathbb{T}^N)} \\
&\quad + \frac{2(\mu+1)^2}{m}\|\nabla\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)}\|\nabla\mathbf{w}\|_{L^2(2\mathbb{T}^N)}\left\|\partial_{ij}\mathbf{w}\right\|_{L^2(2\mathbb{T}^N)} \\
&\quad + \frac{2(\mu+1)^2}{m^3}\|\mathbf{w}\|^2_{L^\infty(2\mathbb{T}^N)}\|\nabla\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)} \\
&\quad \times\|\nabla\mathbf{w}\|_{L^2(2\mathbb{T}^N)}\left\|\partial_{ij}\mathbf{w}\right\|_{L^2(2\mathbb{T}^N)} \\
&\quad + \frac{(\mu+1)^2}{m^2}\|\mathbf{w}\|^2_{L^\infty(2\mathbb{T}^N)}\left\|\partial_{ij}\mathbf{w}\right\|^2_{L^2(2\mathbb{T}^N)} \\
&\quad + \frac{4(\mu+1)^2}{m^2}\|\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)}\|\nabla\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)} \\
&\quad \times\|\nabla\mathbf{w}\|_{L^2(2\mathbb{T}^N)}\left\|\partial_{ij}\mathbf{w}\right\|_{L^2(2\mathbb{T}^N)} \\
&\le \frac{(\mu+1)^2}{m}\left(2+\frac{\|\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)}}{m}\right)\|\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)}\left\|\partial_{ij}\mathbf{w}\right\|^2_{L^2(2\mathbb{T}^N)} \\
&\quad + \frac{2(\mu+1)^2}{m}\left(1+\frac{\|\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)}}{m}\right)^2\|\nabla\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)} \\
&\quad \times\|\nabla\mathbf{w}\|_{L^2(2\mathbb{T}^N)}\left\|\partial_{ij}\mathbf{w}\right\|_{L^2(2\mathbb{T}^N)}.
\end{aligned}
\tag{73}
$$

Substituting (70) $\times A_0$+ (66) into (69) and (73) to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{2\mathbb{T}^N}\left\{\left|\partial_{ij}u\right|^2 + A_0\left|\partial_{ij}v\right|^2\right\}dx \\
&\qquad + \int_{2\mathbb{T}^N}\Big\{d_1\left|\nabla\partial_{ij}u\right|^2 + A_0 d_2 d_3\tilde{v}\nabla\partial_{ij}u\nabla\partial_{ij}v \\
&\qquad\qquad + A_0 d_2\left(1+d_3\tilde{u}\right)\left|\nabla\partial_{ij}v\right|^2\Big\}dx \\
&\qquad + \tilde{u}\int_{2\mathbb{T}^N}\left|\partial_{ij}u\right|^2 dx + A_0\mu\int_{2\mathbb{T}^N}\left|\partial_{ij}v\right|^2 dx \\
&\quad \le I_1 + I_2,
\end{aligned}
\tag{74}
$$

where

$$
\begin{aligned}
I_1 &= C_{10}(1+\eta)\|\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)}\left\|\partial_{ij}\mathbf{w}\right\|^2_{L^2(2\mathbb{T}^N)} \\
&\quad + C_{11}(1+\eta)^2\|\nabla\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)}\|\nabla\mathbf{w}\|_{L^2(2\mathbb{T}^N)}\left\|\partial_{ij}\mathbf{w}\right\|_{L^2(2\mathbb{T}^N)} \\
&\quad + C_{12}\|\nabla\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)}\left\|\partial_{ij}\mathbf{w}\right\|_{L^2(2\mathbb{T}^N)}\left\|\nabla^3\mathbf{w}\right\|_{L^2(2\mathbb{T}^N)} \\
&\quad + C_{13}\|\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)}\left\|\nabla^3\mathbf{w}\right\|^2_{L^2(2\mathbb{T}^N)}, \\
I_2 &= \left(A_0\mu^2 - \beta\tilde{u}\right)\int_{2\mathbb{T}^N}\partial_{ij}u\,\partial_{ij}v\,dx,
\end{aligned}
\tag{75}
$$

we apply the Sobolev imbedding to control the L^∞ norm by

$$\|g\|_{L^\infty(2\mathbb{T}^N)} \le C_0\|g\|_{H^2(2\mathbb{T}^N)} \tag{76}$$

for $N \le 3$. From the Hölder inequality, the Poincaré inequality, and the Sobolev imbedding theorem, it follows that

$$\|g\|_{L^2(2\mathbb{T}^N)} \le (2\pi)^{N/4}\|g\|_{L^4(2\mathbb{T}^N)} \le C_0\|\nabla g\|_{L^2(2\mathbb{T}^N)} \tag{77}$$

for $\int_{2\mathbb{T}^N} g\,dx = 0$. Recall the even extension of (5), and the solution $[u, v]$ satisfies

$$
\begin{aligned}
&\int_{2\mathbb{T}^N}\nabla u\,dx = \int_{2\mathbb{T}^N}\nabla v\,dx = 0, \\
&\int_{2\mathbb{T}^N}\partial_{ij}u\,dx = \int_{2\mathbb{T}^N}\partial_{ij}v\,dx = 0, \quad i, j = 1, \ldots, N.
\end{aligned}
\tag{78}
$$

By (76) and (77), we find that

$$\|\nabla\mathbf{w}\|_{L^\infty(2\mathbb{T}^N)} \le C_0\|\nabla\mathbf{w}\|_{H^2(2\mathbb{T}^N)} \le C_0\left\|\nabla^3\mathbf{w}\right\|_{L^2(2\mathbb{T}^N)}, \tag{79}$$

where C_0 is a universal constant. Therefore, when $N \le 3$, it follows from (76) and (79) that

$$I_1 \le C_{14}(1+\eta)^2\|\mathbf{w}\|_{H^2(2\mathbb{T}^N)}\left\|\nabla^3\mathbf{w}\right\|^2_{L^2(2\mathbb{T}^N)}. \tag{80}$$

Applying the Young inequality to get

$$I_2 \le \frac{\left(A_0\mu^2 + \beta\tilde{u}\right)^2}{2A_0\mu}\int_{2\mathbb{T}^N}\left|\partial_{ij}u\right|^2 dx + \frac{A_0\mu}{2}\int_{2\mathbb{T}^N}\left|\partial_{ij}v\right|^2 dx, \tag{81}$$

which is combined with the interpolation inequality and the ε-Young inequality to imply

$$\int_{2\mathbb{T}^N}\left|\partial_{ij}u\right|^2 dx \le C_0\left(a\left\|\nabla\partial_{ij}u\right\|^2_{L^2(2\mathbb{T}^N)}\right) + \frac{1}{4a^2}\|u\|^2_{L^2(2\mathbb{T}^N)}, \tag{82}$$

where a is a positive constant, in the same way as above, we obtain that the second integral satisfies

$$
\begin{aligned}
&\int_{2\mathbb{T}^N}\Big\{d_1\left|\nabla\partial_{ij}u\right|^2 + A_0 d_2 d_3\tilde{v}\nabla\partial_{ij}u\nabla\partial_{ij}v \\
&\qquad + A_0 d_2\left(1+d_3\tilde{u}\right)\left|\nabla\partial_{ij}v\right|^2\Big\}dx \\
&\quad \ge \left\{\frac{d_1}{2}\left|\nabla\partial_{ij}u\right|^2 + \frac{A_0 d_2\left(1+d_3\tilde{u}\right)}{2}\left|\nabla\partial_{ij}v\right|^2\right\}dx.
\end{aligned}
\tag{83}
$$

Substituting (80)–(83) into (74), take $((A_0\mu^2+\beta\widetilde{u})^2/2A_0\mu)C_0a=d_1/4$ to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{2\mathbb{T}^N}\left\{\left|\partial_{ij}u\right|^2+A_0\left|\partial_{ij}v\right|^2\right\}dx\\&\quad+\int_{2\mathbb{T}^N}\left\{\frac{d_1}{4}\left|\nabla\partial_{ij}u\right|^2+\frac{A_0}{2}d_2\left(1+d_3\widetilde{u}\right)\left|\nabla\partial_{ij}v\right|^2\right\}dx\\&\quad+\widetilde{u}\int_{2\mathbb{T}^N}\left|\partial_{ij}u\right|^2dx+\frac{A_0\mu}{2}\int_{2\mathbb{T}^N}\left|\partial_{ij}v\right|^2dx\\&\le C_{14}(1+\eta)^2\|\mathbf{w}\|_{H^2(2\mathbb{T}^N)}\left\|\nabla^3\mathbf{w}\right\|^2_{L^2(2\mathbb{T}^N)}\\&\quad+C_{15}\|u\|^2_{L^2(2\mathbb{T}^N)}.\end{aligned}\tag{84}$$

Similar as the proof of Lemma 1, we proceed in the two cases: $A_0\ge 1$ and $0<A_0<1$. Then, we conclude

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{2\mathbb{T}^N}\left\{\left|\partial_{ij}u\right|^2+\left|\partial_{ij}v\right|^2\right\}dx\\&\quad+\int_{2\mathbb{T}^N}\left\{\frac{d_1}{4}\left|\nabla\partial_{ij}u\right|^2+\frac{A_0}{2}d_2\left(1+d_3\widetilde{u}\right)\left|\nabla\partial_{ij}v\right|^2\right\}dx\\&\quad+\widetilde{u}\int_{2\mathbb{T}^N}\left|\partial_{ij}u\right|^2dx+\frac{A_0\mu}{2}\int_{2\mathbb{T}^N}\left|\partial_{ij}v\right|^2dx\\&\le C_{16}(1+\eta)^2\|\mathbf{w}\|_{H^2(2\mathbb{T}^N)}\left\|\nabla^3\mathbf{w}\right\|^2_{L^2(2\mathbb{T}^N)}\\&\quad+C_{17}\|u\|^2_{L^2(2\mathbb{T}^N)}.\end{aligned}\tag{85}$$

So, the even extension implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i,j=1}^{N}\int_{\mathbb{T}^N}\left\{\left|\partial_{ij}u\right|^2+\left|\partial_{ij}v\right|^2\right\}dx\\&\quad+\sum_{i,j=1}^{N}\int_{\mathbb{T}^N}\left\{\frac{d_1}{4}\left|\nabla\partial_{ij}u\right|^2+\frac{A_0}{2}d_2\left(1+d_3\widetilde{u}\right)\left|\nabla\partial_{ij}v\right|^2\right\}dx\\&\quad+\widetilde{u}\sum_{i,j=1}^{N}\int_{\mathbb{T}^N}\left|\partial_{ij}u\right|^2dx+\frac{A_0\mu}{2}\sum_{i,j=1}^{N}\int_{\mathbb{T}^N}\left|\partial_{ij}v\right|^2dx\\&\le\widehat{C}_2(1+\eta)^2\|\mathbf{w}\|_{H^2(\mathbb{T}^N)}\left\|\nabla^3\mathbf{w}\right\|^2_{L^2(\mathbb{T}^N)}+\widehat{C}_2\|u\|^2_{L^2(\mathbb{T}^N)}.\end{aligned}\tag{86}$$

□

Next, we control the H^2 growth of $\mathbf{w}(x,t)$ in terms of its L^2 growth.

Lemma 4. *Suppose that* $\mathbf{w}(x,t)$ *is a solution of the full system* (5)*, such that*

$$\|\mathbf{w}(\cdot,t)\|_{H^2(\mathbb{T}^N)}\le\min\left\{\eta,\frac{d_1}{4\widehat{C}_2(1+\eta)^2},\frac{A_0d_2\left(1+d_3\widetilde{u}\right)}{2\widehat{C}_2(1+\eta)^2}\right\},\tag{87}$$

$$\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{T}^N)}\le 2\widehat{C}_1e^{\lambda_{\max}t}\|\mathbf{w}(\cdot,0)\|_{L^2(\mathbb{T}^N)}.\tag{88}$$

Then,

$$\begin{aligned}\|\mathbf{w}(\cdot,t)\|^2_{H^2(\mathbb{T}^N)}\le\widehat{C}_3&\left\{\|\mathbf{w}(\cdot,0)\|^2_{H^2(\mathbb{T}^N)}\right.\\&\left.+e^{2\lambda_{\max}t}\|\mathbf{w}(\cdot,0)\|^2_{L^2(\mathbb{T}^N)}\right\},\end{aligned}\tag{89}$$

where $\widehat{C}_3=2\max\{4\widehat{C}_1^2,4(C_0+1)\widehat{C}_1^2\widehat{C}_2/\lambda_{\max},(C_0+1)/2\}$.

Proof. We first consider the second-order derivatives of $\mathbf{w}(x,t)$. From Lemmas 3 and 4 and our assumption for $\mathbf{w}(x,t)$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{i,j=1}^{N}\int_{\mathbb{T}^N}\left\{\left|\partial_{ij}u\right|^2+\left|\partial_{ij}v\right|^2\right\}dx&\le\widehat{C}_2\|u\|^2_{L^2(\mathbb{T}^N)}\\&\le 4\widehat{C}_1^2\widehat{C}_2e^{2\lambda_{\max}t}\|\mathbf{w}(\cdot,0)\|^2_{L^2(\mathbb{T}^N)}.\end{aligned}\tag{90}$$

By an integration from 0 to t, we deduce that

$$\begin{aligned}&\sum_{i,j=1}^{N}\int_{\mathbb{T}^N}\left\{\left|\partial_{ij}u(x,t)\right|^2+\left|\partial_{ij}v(x,t)\right|^2\right\}dx\\&\quad\le\sum_{i,j=1}^{N}\int_{\mathbb{T}^N}\left\{\left|\partial_{ij}u(x,0)\right|^2+\left|\partial_{ij}v(x,0)\right|^2\right\}dx\\&\qquad+\frac{4\widehat{C}_1^2\widehat{C}_2}{\lambda_{\max}}e^{2\lambda_{\max}t}\|\mathbf{w}(\cdot,0)\|^2_{L^2(\mathbb{T}^N)}.\end{aligned}\tag{91}$$

For the first derivations of $\mathbf{w}(x,t)$, we apply the Poincaré inequality to get

$$\|\nabla\mathbf{w}\|_{L^2(2\mathbb{T}^N)}\le C_0\left\|\nabla^2\mathbf{w}\right\|_{L^2(2\mathbb{T}^N)}.\tag{92}$$

Applying the even extension, we have

$$\|\nabla\mathbf{w}\|_{L^2(\mathbb{T}^N)}\le C_0\left\|\nabla^2\mathbf{w}\right\|_{L^2(\mathbb{T}^N)}.\tag{93}$$

It follows from (88)–(93) that

$$\begin{aligned}&\|\mathbf{w}(\cdot,\mathbf{t})\|^2_{H^2(\mathbb{T}^N)}\\&\quad=\|\mathbf{w}(\cdot,\mathbf{t})\|^2_{L^2(\mathbb{T}^N)}+\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^N)}+\left\|\nabla^2\mathbf{w}\right\|^2_{L^2(\mathbb{T}^N)}\\&\quad\le\|\mathbf{w}(\cdot,\mathbf{t})\|^2_{L^2(\mathbb{T}^N)}+\left(C_0+1\right)\left\|\nabla^2\mathbf{w}\right\|^2_{L^2(\mathbb{T}^N)}\\&\quad\le 4\widehat{C}_1^2e^{2\lambda_{\max}t}\|\mathbf{w}(\cdot,\mathbf{0})\|^2_{L^2(\mathbb{T}^N)}\\&\qquad+\left(C_0+1\right)\|\mathbf{w}(\cdot,\mathbf{0})\|^2_{H^2(\mathbb{T}^N)}\\&\qquad+\frac{4\left(C_0+1\right)\widehat{C}_1^2\widehat{C}_2}{\lambda_{\max}}e^{2\lambda_{\max}t}\|\mathbf{w}(\cdot,0)\|^2_{L^2(\mathbb{T}^N)}\\&\quad\le\widehat{C}_3\left\{\|\mathbf{w}(\cdot,\mathbf{0})\|^2_{H^2(\mathbb{T}^N)}+e^{2\lambda_{\max}t}\|\mathbf{w}(\cdot,\mathbf{0})\|^2_{L^2(\mathbb{T}^N)}\right\},\end{aligned}\tag{94}$$

where $\widehat{C}_3=2\max\{4\widehat{C}_1^2,4(C_0+1)\widehat{C}_1^2\widehat{C}_2/\lambda_{\max},(C_0+1)/2\}$. □

4. Nonlinear Instability and Pattern Formation

Let θ be a fixed constant. For an arbitrary positive number δ, we define the escape time T^δ by

$$\theta = \delta e^{\lambda_{\max} T^\delta} \tag{95}$$

or equivalently

$$T^\delta = \frac{1}{\lambda_{\max}} \ln \frac{\theta}{\delta}. \tag{96}$$

Theorem 5. *Assume that there exists a* $\mathbf{q}$, *such that the instability criterion* (13) *holds. Let*

$$\begin{aligned}\mathbf{w}_0(x) = &\sum_{\mathbf{q}\in\Lambda_1} \left\{w_{\mathbf{q}}^{-}\mathbf{r}_{-}(\mathbf{q}) + w_{\mathbf{q}}^{+}\mathbf{r}_{+}(\mathbf{q})\right\} e_{\mathbf{q}}(x)\\ &+ \sum_{\mathbf{q}\in\Lambda_2} \left\{w_{\mathbf{q}}\mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}^{\prime}\mathbf{r}^{\prime}(\mathbf{q})\right\} e_{\mathbf{q}}(x)\\ &+ \sum_{\mathbf{q}\in\Lambda_3} \left\{w_{\mathbf{q}}^{\mathrm{Re}}\operatorname{Re}\mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}^{\mathrm{Im}}\operatorname{Im}\mathbf{r}(\mathbf{q})\right\} e_{\mathbf{q}}(x),\end{aligned} \tag{97}$$

such that $\mathbf{w}_0(x) \in H^2(\mathbb{T}^N)$, $\|\mathbf{w_0}\|_{L^2(\mathbb{T}^N)} = 1$. *Then, there exist* $\theta_0 > 0$, $\delta_0 > 0$, *and* $C > 0$, *such that for all* $\theta \in (0, \theta_0)$ *and* $\delta \in (0, \delta_0]$, *the nonlinear evolution* $\mathbf{w}^\delta(x,t)$ *of* (5) *with the initial perturbation* $\mathbf{w}^\delta(x,0) = \delta\mathbf{w}_0(x)$ *satisfies*

$$\begin{aligned}&\left\|\mathbf{w}^\delta(\cdot,t) - \delta e^{\lambda_{\max}t}\sum_{\mathbf{q}\in\Lambda_{\max}} w_{\mathbf{q}}^{+}\mathbf{r}_{+}(\mathbf{q})e_{\mathbf{q}}(x)\right\|_{L^2(\mathbb{T}^N)}\\ &\quad\le C\left\{e^{-\sigma t} + \delta\|\mathbf{w_0}\|_{H^2(\mathbb{T}^N)}^2 + \delta e^{\lambda_{\max}t}\right\}\delta e^{\lambda_{\max}t},\\ &\qquad \forall t\in\left[0,T^\delta\right],\end{aligned} \tag{98}$$

where the constant σ, *called the gap between* $\lambda_{\max}$ *and the rest of eigenvalues, is positive, and*

$$\begin{aligned}\delta_0 = \min\Bigg\{ &\frac{1}{2\sqrt{\widehat{C}_3}\|\mathbf{w_0}\|_{H^2(T^N)}},\\ &\min\left\{\eta, \frac{d_1}{4\widehat{C}^2(1+\eta)^2}, \frac{A_0 d_2(1+d_3\tilde{u})}{2\widehat{C}_1(1+\eta)^2}\right\},\\ &\frac{\lambda_{\max}}{4\widehat{C}_3\widehat{C}_4\|\mathbf{w_0}\|_{H^2(T^N)}^2}\Bigg\},\\ \theta_0 = \min\Bigg\{ &\frac{\lambda_{\max}}{4\widehat{C}_3\widehat{C}_4}, \frac{\eta}{2\sqrt{\widehat{C}_3}},\\ &\frac{d_1}{8\widehat{C}_2\sqrt{\widehat{C}_3}(1+\eta)^2}, \frac{A_0 d_2(1+d_3\tilde{u})}{4\widehat{C}_2\sqrt{\widehat{C}_3}(1+\eta)^2}\Bigg\}.\end{aligned} \tag{99}$$

Remark 6. First, we notice that the part of the fastest growing modes

$$\delta e^{\lambda_{\max}t}\sum_{\mathbf{q}\in\Lambda_{\max}} w_{\mathbf{q}}^{+}\mathbf{r}_{+}(\mathbf{q})e_{\mathbf{q}}(x) \tag{100}$$

in (98) is nontrivial. Recall that the initial profile $\mathbf{w}_0(x)$ is any H^2 function satisfying $\|\mathbf{w_0}\|_{L^2(\mathbb{T}^N)} = 1$. Thus, we can choose $\mathbf{w}_0(x)$, such that there is at least one $\mathbf{q_0} \in \Lambda_{\max}$ with $w_{\mathbf{q_0}}^{+} \neq 0$. Consequently,

$$\begin{aligned}&\left\|\delta e^{\lambda_{\max}t}\sum_{\mathbf{q}\in\Lambda_{\max}} w_{\mathbf{q}}^{+}\mathbf{r}_{+}(\mathbf{q})e_{\mathbf{q}}(x)\right\|_{L^2(\mathbb{T}^N)}\\ &\quad\ge \delta e^{\lambda_{\max}t}\left|w_{\mathbf{q_0}}^{+}\right|\left|\mathbf{r}_{+}(\mathbf{q_0})\right| > 0.\end{aligned} \tag{101}$$

Remark 7. Fix θ to be a positive small number. If δ is small sufficiently, then T^δ is large, and for $0 \le t \le T^\delta$, $\delta e^{\lambda_{\max}t} \le \theta$. Our estimate (98) implies that the dynamics of a general perturbation can be characterized by such linear dynamics over a long time period $(0, T^\delta]$, when the initial perturbation is small.

Remark 8. In particular, if we take

$$\mathbf{w}_0(x) = \frac{\mathbf{r}_{+}(\mathbf{q_0})}{\left|\mathbf{r}_{+}(\mathbf{q_0})\right|}e_{\mathbf{q_0}}(x) \tag{102}$$

in Remark 6, then, at the time $t = T^\delta$, the estimate (98) gives

$$\begin{aligned}&\left\|\mathbf{w}^\delta\left(\cdot,T^\delta\right) - \delta e^{\lambda_{\max}T^\delta}\frac{\mathbf{r}_{+}(\mathbf{q_0})}{\left|\mathbf{r}_{+}(\mathbf{q_0})\right|}e_{\mathbf{q_0}}(\cdot)\right\|_{L^2(\mathbb{T}^N)}\\ &\quad\le C\left\{e^{-\sigma T^\delta} + \delta\|\mathbf{w_0}\|_{H^2(\mathbb{T}^N)}^2 + \theta\right\}\theta,\\ &\quad= C\left\{\theta^{1-\sigma/\lambda_{\max}}\delta^{\sigma/\lambda_{\max}} + \delta\theta\|\mathbf{w_0}\|_{H^2(\mathbb{T}^N)}^2 + \theta^2\right\}.\end{aligned} \tag{103}$$

Therefore,

$$\begin{aligned}\left\|\mathbf{w}^\delta\left(\cdot,T^\delta\right)\right\| \ge \theta - C\Big\{&\theta^{1-\sigma/\lambda_{\max}}\delta^{\sigma/\lambda_{\max}}\\ &+ \delta\theta\|\mathbf{w_0}\|_{H^2(\mathbb{T}^N)}^2 + \theta^2\Big\}.\end{aligned} \tag{104}$$

For θ sufficiently small, $\|\mathbf{w}^\delta(.,T^\delta)\| \ge \theta/2$ as $\delta \to 0$, which implies that nonlinear instability occurs.

Remark 9. From a view of pattern formation, Theorem 5 shows that if the unique positive equilibrium $(\tilde{u},\tilde{v})$ of (2) is linear unstable, then a general small perturbation near $(\tilde{u},\tilde{v})$ can induce pattern formation. Furthermore, the patterns can be characterized by the fastest growing modes in the corresponding linear dynamics over a long time period $(0, T^\delta]$.

Proof of Theorem 5. Define

$$T^{*}=\sup\left\{t \mid \left\|\mathbf{w}^{\delta}(\cdot,t)-\delta e^{Lt}\mathbf{w}_0\right\|_{L^2(\mathbb{T}^N)}\leq\frac{\widehat{C}_1}{2}\delta e^{\lambda_{\max}t}\right\},\tag{105}$$

$$T^{**}=\sup\left\{t \mid \left\|\mathbf{w}^{\delta}(\cdot,t)\right\|_{H^2(\mathbb{T}^N)}\leq\min\left\{\eta,\frac{d_1}{4\widehat{C}_2(1+\eta)^2},\frac{A_0d_2(1+d_3\bar{u})}{2\widehat{C}_2(1+\eta)^2}\right\}\right\}.\tag{106}$$

Now, we proceed in the following four steps.

Step 1. We establish H^2 estimate for the solution $\mathbf{w}^{\delta}(x,t)$ of $0\leq t\leq\min\{T^{\delta},T^{*},T^{**}\}$.

From Lemma 1, for any $t\geq 0$, we have

$$\left\|\delta e^{Lt}\mathbf{w}_0\right\|_{L^2(\mathbb{T}^N)}\leq\delta\widehat{C}_1e^{\lambda_{\max}t}.\tag{107}$$

By the definition of T^{*}, for any $0\leq t\leq T^{*}$, it follows that

$$\left\|\mathbf{w}^{\delta}(\cdot,t)\right\|_{L^2(\mathbb{T}^N)}-\left\|\delta e^{Lt}\mathbf{w}_0\right\|_{L^2(\mathbb{T}^N)}\leq\frac{\widehat{C}_1}{2}\delta e^{\lambda_{\max}t}.\tag{108}$$

Substituting (107) into (108), we obtain

$$\left\|\mathbf{w}^{\delta}(\cdot,t)\right\|_{L^2(\mathbb{T}^N)}\leq\frac{3\widehat{C}_1}{2}\delta e^{\lambda_{\max}t}.\tag{109}$$

Furthermore, it follows from (109) and Lemma 4 that

$$\left\|\mathbf{w}^{\delta}(\cdot,t)\right\|_{H^2(\mathbb{T}^N)}\leq\sqrt{\widehat{C}_3}\cdot\sqrt{\delta^2\left\|\mathbf{w}_0\right\|^2_{H^2(\mathbb{T}^N)}+\delta^2e^{2\lambda_{\max}t}}\tag{110}$$

$$\leq\sqrt{\widehat{C}_3}\cdot\left(\delta\left\|\mathbf{w}_0\right\|_{H^2(\mathbb{T}^N)}+\delta e^{\lambda_{\max}t}\right)\tag{111}$$

for any $t\leq T^{**}$.

Step 2. We establish L^2 estimate for $\mathbf{w}^{\delta}(x,t)$ of $0\leq t\leq\min\{T^{\delta},T^{*},T^{**}\}$. Applying Duhamel's principle, we have

$$\mathbf{w}^{\delta}(x,t)=\delta e^{Lt}\mathbf{w}_0-\int_0^t e^{L(t-\tau)}N\left(\mathbf{w}^{\delta}(x,\tau)\right)d\tau,\tag{112}$$

where $N[\mathbf{w}^{\delta}(x,\tau)]=(N_1(\mathbf{w}^{\delta}(x,\tau)),N_2(\mathbf{w}^{\delta}(x,\tau)))$, $N_1(\mathbf{w}^{\delta}(x,\tau))$, and $N_2(\mathbf{w}^{\delta}(x,\tau))$ are given by (68) and (72). Using Lemma 1, it follows that

$$\begin{aligned}
&\left\|\mathbf{w}^{\delta}(\cdot,t)-\delta e^{Lt}\mathbf{w}_0\right\|_{L^2(\mathbb{T}^N)}\\
&\quad=\left\|\int_0^t e^{L(t-\tau)}N\left(\mathbf{w}^{\delta}(x,\tau)\right)d\tau\right\|_{L^2(\mathbb{T}^N)}\\
&\quad\leq\int_0^t\left\|e^{L(t-\tau)}N\left(\mathbf{w}^{\delta}(x,\tau)\right)\right\|_{L^2(\mathbb{T}^N)}d\tau\\
&\quad\leq\widehat{C}_1\int_0^t e^{\lambda_{\max}(t-\tau)}\left\|N\left(\mathbf{w}^{\delta}(x,\tau)\right)\right\|_{L^2(\mathbb{T}^N)}d\tau,\\
&\left\|N(\mathbf{w}^{\delta}(x,\tau))\right\|_{L^2(\mathbb{T}^N)}\\
&\leq\left\|N_1\left(\mathbf{w}^{\delta}(x,\tau)\right)\right\|_{L^2(\mathbb{T}^N)}+\left\|N_2(\mathbf{w}^{\delta}(x,\tau))\right\|_{L^2(\mathbb{T}^N)}\\
&=\left\|-(u^{\delta})^2-\beta u^{\delta}v^{\delta}\right\|_{L^2(\mathbb{T}^N)}\\
&\quad+\left\|d_2d_3v^{\delta}\Delta u^{\delta}+2d_2d_3\nabla u^{\delta}\nabla v^{\delta}+d_2d_3u^{\delta}\Delta v^{\delta}\right.\\
&\qquad\left.-\frac{\left(\mu u^{\delta}-v^{\delta}\right)^2}{m+u^{\delta}+\bar{u}}\right\|_{L^2(\mathbb{T}^N)}\\
&\leq(\beta+1)\left\|\mathbf{w}^{\delta}\right\|_{L^{\infty}(\mathbb{T}^N)}\left\|\mathbf{w}^{\delta}\right\|_{L^2(\mathbb{T}^N)}+2d_2d_3\left\|\mathbf{w}^{\delta}\right\|_{L^{\infty}(\mathbb{T}^N)}\\
&\quad\times\left\|\mathbf{w}^{\delta}\right\|_{H^2(\mathbb{T}^N)}+2d_2d_3\left\|\nabla\mathbf{w}^{\delta}\right\|^2_{L^4(\mathbb{T}^N)}\\
&\quad+\frac{(\mu+1)^2}{m}\left\|\mathbf{w}^{\delta}\right\|_{L^{\infty}(\mathbb{T}^N)}\left\|\mathbf{w}^{\delta}\right\|_{L^2(\mathbb{T}^N)}\\
&\leq\widehat{C}_4\left\|\mathbf{w}^{\delta}\right\|^2_{H^2(\mathbb{T}^N)},
\end{aligned}\tag{113}$$

where $\widehat{C}_4=\beta+1+4d_2d_3+(\mu+1)^2/m$. Therefore,

$$\begin{aligned}
&\left\|\mathbf{w}^{\delta}(\cdot,t)-\delta e^{Lt}\mathbf{w}_0\right\|_{L^2(\mathbb{T}^N)}\\
&\quad\leq\widehat{C}_1\widehat{C}_4\int_0^t e^{\lambda_{\max}(t-\tau)}\left\|\mathbf{w}^{\delta}(\tau)\right\|^2_{H^2(\mathbb{T}^N)}d\tau.
\end{aligned}\tag{114}$$

Substituting (110) into (114), we obtain

$$\begin{aligned}
&\left\|\mathbf{w}^{\delta}(\cdot,t)-\delta e^{Lt}\mathbf{w}_0\right\|_{L^2(\mathbb{T}^N)}\\
&\quad\leq\widehat{C}_1\widehat{C}_3\widehat{C}_4\int_0^t e^{\lambda_{\max}(t-\tau)}\\
&\qquad\times\left(\delta^2\left\|\mathbf{w}_0\right\|^2_{H^2(\mathbb{T}^N)}+\delta^2e^{2\lambda_{\max}\tau}\right)d\tau\\
&\quad\leq\widehat{C}_1\widehat{C}_3\widehat{C}_4\left\{\frac{\delta\left\|\mathbf{w}_0\right\|^2_{H^2(\mathbb{T}^N)}}{\lambda_{\max}}+\frac{\delta e^{\lambda_{\max}t}}{\lambda_{\max}}\right\}\delta e^{\lambda_{\max}t}.
\end{aligned}\tag{115}$$

Step 3. We prove by contradiction that

$$T^{\delta} = \min\left\{T^{\delta}, T^{*}, T^{**}\right\} \tag{116}$$

for δ sufficiently small and

$$\theta < \min\left\{\frac{\lambda_{\max}}{4\widehat{C}_3\widehat{C}_4}, \frac{\eta}{2\sqrt{\widehat{C}_3}}, \frac{d_1}{8\widehat{C}_2\sqrt{\widehat{C}_3}(1+\eta)^2}, \frac{A_0 d_2\left(1+d_3\widetilde{u}\right)}{4\widehat{C}_2\sqrt{\widehat{C}_3}(1+\eta)^2}\right\}. \tag{117}$$

If $T^{**} = \min\{T^{\delta}, T^{*}, T^{**}\}$, we can let $t = T^{**} \leq T^{\delta}$ in (111) to obtain

$$\begin{aligned}\left\|\mathbf{w}^{\delta}\left(\cdot, T^{**}\right)\right\|_{H^2(\mathbb{T}^N)} &\leq \sqrt{\widehat{C}_3}\left(\delta\left\|\mathbf{w}_0\right\|_{H^2(\mathbb{T}^N)} + \delta e^{\lambda_{\max}T^{**}}\right) \\ &\leq \sqrt{\widehat{C}_3}\left(\delta\left\|\mathbf{w}_0\right\|_{H^2(\mathbb{T}^N)} + \theta\right).\end{aligned} \tag{118}$$

By choosing δ sufficiently small, such that

$$\begin{aligned}&\sqrt{\widehat{C}_3}\delta\left\|\mathbf{w}_0\right\|_{H^2(\mathbb{T}^N)} \\ &\quad\leq \frac{1}{2}\min\left\{\eta, \frac{d_1}{4\widehat{C}_2(1+\eta)^2}, \frac{A_0 d_2\left(1+d_3\widetilde{u}\right)}{2\widehat{C}_2(1+\eta)^2}\right\},\end{aligned} \tag{119}$$

and the choice of θ in (117), we have

$$\begin{aligned}&\left\|\mathbf{w}^{\delta}\left(\cdot, T^{**}\right)\right\|_{H^2(\mathbb{T}^N)} \\ &\quad< \min\left\{\eta, \frac{d_1}{4\widehat{C}_2(1+\eta)^2}, \frac{A_0 d_2\left(1+d_3\widetilde{u}\right)}{2\widehat{C}_2(1+\eta)^2}\right\}.\end{aligned} \tag{120}$$

This is a contradiction to the definition of T^{**}.

On the other hand, if $T^{*} = \min\{T^{\delta}, T^{*}, T^{**}\}$, we can let $t = T^{*}$ in (115) to get

$$\begin{aligned}&\left\|\mathbf{w}^{\delta}\left(\cdot, T^{*}\right) - \delta e^{LT^{*}}\mathbf{w}_0\right\|_{L^2(\mathbb{T}^N)} \\ &\quad\leq \widehat{C}_1\widehat{C}_3\widehat{C}_4\left\{\frac{\delta\left\|\mathbf{w}_0\right\|^2_{H^2(\mathbb{T}^N)}}{\lambda_{\max}} + \frac{\theta}{\lambda_{\max}}\right\}\delta e^{\lambda_{\max}T^{*}}.\end{aligned} \tag{121}$$

By choosing δ sufficiently small, such that

$$\widehat{C}_3\widehat{C}_4\frac{\delta\left\|\mathbf{w}_0\right\|^2_{H^2(\mathbb{T}^N)}}{\lambda_{\max}} \leq \frac{1}{4}, \tag{122}$$

and the choice of θ in (117), we have

$$\left\|\mathbf{w}^{\delta}\left(\cdot, T^{*}\right) - \delta e^{LT^{*}}\mathbf{w}_0\right\|_{L^2(\mathbb{T}^N)} < \frac{\widehat{C}_1}{2}\delta e^{\lambda_{\max}T^{*}}. \tag{123}$$

This again contradicts the definition of T^{*}. From these arguments, (116) holds.

Step 4. Rewriting the left-hand term in (115) as the form of (26), and separating $\mathbf{q} \in \Lambda_{\max}$ and moving $\mathbf{q} \notin \Lambda_{\max}$ to the right-hand side, it follows that

$$\begin{aligned}&\left\|\mathbf{w}^{\delta}(\cdot, t) - \delta e^{\lambda_{\max}t}\sum_{\mathbf{q}\in\Lambda_{\max}} w^{+}_{\mathbf{q}}\mathbf{r}_{+}(\mathbf{q})\, e_{\mathbf{q}}(x)\right\|_{L^2(\mathbb{T}^N)} \\ &\quad\leq \delta\left\|\sum_{\mathbf{q}\in\Lambda_{\max}} w^{-}_{\mathbf{q}}\mathbf{r}_{-}(\mathbf{q})\, e^{\lambda^{-}_{\mathbf{q}}t}e_{\mathbf{q}}(x)\right\|_{L^2(\mathbb{T}^N)} \\ &\qquad+ \delta\left\|\sum_{\mathbf{q}\in\Lambda_1\setminus\Lambda_{\max}}\left\{w^{-}_{\mathbf{q}}\mathbf{r}_{-}(\mathbf{q})\, e^{\lambda^{-}_{\mathbf{q}}t} + w^{+}_{\mathbf{q}}\mathbf{r}_{+}(\mathbf{q})\, e^{\lambda^{+}_{\mathbf{q}}t}\right\}e_{\mathbf{q}}(x)\right\|_{L^2(\mathbb{T}^N)} \\ &\qquad+ \delta\left\|\sum_{\mathbf{q}\in\Lambda_2}\left\{w_{\mathbf{q}}\mathbf{r}(\mathbf{q}) + w'_{\mathbf{q}}\mathbf{r}'(\mathbf{q}) + w'_{\mathbf{q}}\mathbf{r}(\mathbf{q})\, t\right\}e^{\lambda_{\mathbf{q}}t}e_{\mathbf{q}}(x)\right\|_{L^2(\mathbb{T}^N)} \\ &\qquad+ \delta\left\|\sum_{\mathbf{q}\in\Lambda_3}\left\{w^{\mathrm{Re}}_{\mathbf{q}}\left(\mathrm{Re}\,\mathbf{r}(\mathbf{q})\cos\left[\left(\mathrm{Im}\,\lambda_{\mathbf{q}}\right)t\right] - \mathrm{Im}\,\mathbf{r}(\mathbf{q})\sin\left[\left(\mathrm{Im}\,\lambda_{\mathbf{q}}\right)t\right]\right)\right.\right. \\ &\qquad\qquad\left.\left.+ w^{\mathrm{Im}}_{\mathbf{q}}\left(\mathrm{Re}\,\mathbf{r}(\mathbf{q})\sin\left[\left(\mathrm{Im}\,\lambda_{\mathbf{q}}\right)t\right] + \mathrm{Im}\,\mathbf{r}(\mathbf{q})\cos\left[\left(\mathrm{Im}\,\lambda_{\mathbf{q}}\right)t\right]\right)\right\} \times e^{(\mathrm{Re}\,\lambda_{\mathbf{q}})t}e_{\mathbf{q}}(x)\right\|_{L^2(\mathbb{T}^N)} \\ &\qquad+ \widehat{C}_1\widehat{C}_3\widehat{C}_4\left\{\frac{\delta\left\|\mathbf{w}_0\right\|^2_{H^2(\mathbb{T}^N)}}{\lambda_{\max}} + \frac{\delta e^{\lambda_{\max}t}}{\lambda_{\max}}\right\}\delta e^{\lambda_{\max}t}.\end{aligned} \tag{124}$$

Next, we process the first term on the right side of (124) to get

$$\begin{aligned}&\delta\left\|\sum_{\mathbf{q}\in\Lambda_{\max}} w^{-}_{\mathbf{q}}\mathbf{r}_{-}(\mathbf{q})e^{\lambda^{-}_{\mathbf{q}}t}e_{\mathbf{q}}(x)\right\|_{L^2(\mathbb{T}^N)} \\ &\quad= \delta\left(\sum_{\mathbf{q}\in\Lambda_{\max}}\left(w^{-}_{\mathbf{q}}\right)^2\left|\mathbf{r}_{-}(\mathbf{q})\right|^2 e^{2\lambda^{-}_{\mathbf{q}}t}dx\right)^{1/2}.\end{aligned} \tag{125}$$

Recall that there are only finite elements in $\Lambda_{\max}$ and for any $\mathbf{q} \in \Lambda_{\max}$, there exists a constant M_0, such that $0 < |\mathbf{q}| < M_0$. Therefore,

$$\delta \left\| \sum_{\mathbf{q} \in \Lambda_{\max}} w_{\mathbf{q}}^{-} \mathbf{r}_{-}(\mathbf{q})\, e^{\lambda_{\mathbf{q}}^{-} t} e_{\mathbf{q}}(x) \right\|_{L^2(\mathbb{T}^N)} \leq C\delta e^{(\lambda_{\max} - \sigma)t}. \tag{126}$$

Similar to the arguments in the proof of Lemma 1, we can treat the second, third, and fourth terms to obtain some similar estimates as (126), and our theorem follows. □

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