Research Article

Approximate Solution of Tuberculosis Disease Population Dynamics Model

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We examine possible approximate solutions of both integer and noninteger systems of nonlinear differential equations describing tuberculosis disease population dynamics. The approximate solutions are obtained via the relatively new analytical technique, the homotopy decomposition method (HDM). The technique is described and illustrated with numerical example. The numerical simulations show that the approximate solutions are continuous functions of the noninteger-order derivative. The technique used for solving these problems is friendly, very easy, and less time consuming.

1. Introduction

Tuberculosis, MTB, or TB (short for tubercle bacillus) is a common and in many cases lethal, infectious disease caused by various strains of *Mycobacterium*, usually *Mycobacterium Tuberculosis* [1]. Tuberculosis typically attacks the lungs, but can also affect other parts of the body. It is spread through the air when people who have an active TB infection cough, sneeze, or otherwise transmit their saliva through the air [2]. Most infections are asymptomatic and latent, but about one in ten latent infections eventually progresses to active disease which, if left untreated, kills more than 50% of those so infected. Interested reader can find more about this model in [3–7].

Based on the standard SIRS model, the model population was compartmentalised into the susceptible (S) and the infected (I) which is further broken down into latently infected (I_L) and actively infected (I_A) while the recovered subpopulation is ploughed back into the susceptible group due to the possibility of reinfection after successful treatment of the earlier infection. The model monitors the temporary dynamics in the population of susceptible people (t), TB latently infected people $I_L(t)$, and TB actively infected people

$I_A(t)$ as captured in the model system of ordinary differential equations that follows.

$$\frac{dS(t)}{dt} = \nu f N - \alpha I_A S(t) + \delta S(t) + T_A I_A(t) + T_L I_L(t),$$

$$\frac{dI_L(t)}{dt} = (1-P)\alpha I_A S(t) - \beta_A I_L(t) - T_L I_L(t) - \delta I_L(t),$$

$$\frac{dI_A(t)}{dt} = P\alpha I_A S(t) + \beta_A I_L(t) - T_A I_A(t) - \delta I_A(t) - \varepsilon I_A(t) \tag{1}$$

subject to the initial conditions

$$S(0) = N, \qquad I_L(0) \geq 0, \qquad I_A(0) \geq 0, \tag{2}$$

where N is the total number of new people in the location of interest; S is the number of susceptible people in the location; I_L is the number of TB latently infected people; I_A is the number of TB actively infected people; ν is the probability that a susceptible person is not vaccinated; f is the efficient rate of vaccines; T_L is the success rate of latent T_B therapy; T_A is the active TB treatment cure rate; α is the TB instantaneous

incidence rate per susceptible; δ is humans natural death rate; P is the proportion of infection instantaneously degenerating into active TB; ε is the TB-induced death rate; and β_A is the breakdown rate from latent to active TB. The equilibrium analysis of the model was studied in [8]. Equation (1) together with (2) does not have an exact solution and is usually solved numerically.

The purpose of this paper is to derive approximate analytical solutions for the standard form as well as the fractional version of (1) together with (2) using the relatively new analytical technique, the homotopy decomposition method (HDM).

The paper is structured as follows. In Section 2, we present the basic ideal of the homotopy decomposition method for solving partial differential equations. We present the application of the HDM for system Tuberculosis disease population dynamics model in Section 3. In Section 4, we present the application of the HDM for system of fractional Tuberculosis disease population dynamics model. The conclusions are then given finally in Section 5.

2. Fundamental Information about Homotopy Decomposition Method

To demonstrate the elementary notion of this technique, we consider a universal nonlinear nonhomogeneous partial differential equation with initial conditions of the following form [9–13].

$$\frac{\partial^m U(x,t)}{\partial t^m} = L(U(x,t)) + N(U(x,t)) + f(x,t),$$

$$m = 1, 2, 3 \ldots,$$

(3)

focused on the primary condition

$$\frac{\partial^i U(x,0)}{\partial t^i} = y_i(x), \qquad \frac{\partial^{m-1} U(x,0)}{\partial t^{m-1}} = 0,$$

$$i = 0, 1, 2, \ldots, m-2,$$

(4)

where m is the order of the derivative, where f is an identified function, N is the common nonlinear differential operator, L denotes a linear differential operator, and m is the order of the derivative. The procedures first stage here is to apply the inverse operator $\partial^m/\partial t^m$ on both sides of (3) to obtain

$$U(x,t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} \frac{d^k u(x,0)}{dt^k}$$

$$+ \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} L(U(x,\tau))$$

$$+ N(U(x,\tau)) + f(x,\tau) \, d\tau \cdots dt.$$

(5)

The multi-integral in (3) can be transformed to

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} L(U(x,\tau))$$

$$+ N(U(x,\tau)) + f(x,\tau) \, d\tau \cdots dt$$

$$= \frac{1}{(m-1)!} \int_0^t (t-\tau)^{m-1} L(U(x,\tau))$$

$$+ N(U(x,\tau)) + f(x,\tau) \, d\tau$$

(6)

so that (3) can be reformulated as

$$U(x,t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} y_i(x)$$

$$+ \frac{1}{(m-1)!} \int_0^t (t-\tau)^{m-1} L(U(x,\tau))$$

$$+ N(U(x,\tau)) + f(x,\tau) \, d\tau.$$

(7)

Using the homotopy scheme, the solution of the aforementioned integral equation is given in series form as

$$U(x,t,p) = \sum_{n=0}^{\infty} p^n U_n(x,t),$$

$$U(x,t) = \lim_{p \to 1} U(x,t,p),$$

(8)

and the nonlinear term can be decomposed as

$$NU(r,t) = \sum_{n=1}^{\infty} p^n \mathscr{H}_n(U),$$

(9)

where $p \in (0,1]$ is an implanting parameter. $\mathscr{H}_n(U)$ is the polynomials that can be engendered by

$$\mathscr{H}_n(U_0, \ldots, U_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{j=0}^{n} p^j U_j(x,t) \right) \right],$$

$$n = 0, 1, 2 \ldots.$$

(10)

The homotopy decomposition method is obtained by the combination of decomposition method with Abel integral and is given by

$$\sum_{n=0}^{\infty} p^n U_n(x,t)$$

$$= T(x,t) + p \frac{1}{(m-1)!}$$

$$\times \int_0^t (t-\tau)^{m-1} \left[f(x,\tau) + L \left(\sum_{n=0}^{\infty} p^n U_n(x,\tau) \right) \right.$$

$$\left. + \sum_{n=0}^{\infty} p^n \mathscr{H}_n(U) \right] d\tau$$

(11)

with

$$T(x,t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} y_i(x).$$

(12)

Relating the terms of same powers of p, this gives solutions of various orders. The initial guess of the approximation is $T(x,t)$ that is actually the Taylor series of the exact solution of order m. Note that this initial guess insures the uniqueness of the series decompositions [9].

3. Application of the HDM to the Model with Integer-Order Derivative

In this section, we employ this method for deriving the set of the mathematical equations describing the tuberculosis disease population dynamics model.

Resulting from the steps involved in the HDM method, we reach at the following integral equations that are very simple to solve:

$$p^0: \ S_0(t) = S(0),$$

$$p^0: \ I_{L0}(t) = I_L(0),$$

$$p^0: \ I_{A0}(t) = I_A(0),$$

$$p^1: S_1(t)$$
$$= \int_0^t \left(\nu fN - \alpha I_{A0}S_0(\tau) + \delta S_0(\tau) \right.$$
$$\left. + T_A I_{A0}(\tau) + T_L I_{L0}(\tau) \right) d\tau, \quad S_1(0) = 0,$$

$$p^1: I_{L1}(t)$$
$$= \int_0^t \left((1-P)\alpha I_{A0}S_0(\tau) - \beta_A I_{L0}(\tau) \right.$$
$$\left. - T_L I_L 0(\tau) - \delta I_{L0}(\tau) \right) d\tau, \quad I_{L1}(0) = 0,$$

$$p^1: I_{A1}(t)$$
$$= \int_0^t \left(P\alpha I_{A0}S_0(\tau) + \beta_A I_{L0}(\tau) \right.$$
$$\left. - T_A I_{A0}(\tau) - \delta I_{A0}(\tau) - \varepsilon I_{A0}(\tau) \right) d\tau, \quad I_{A1}(0) = 0,$$

$$\vdots$$

$$p^n: S_n(t)$$
$$= \int_0^t \left(\nu fN - \alpha \sum_{j=0}^{n-1} I_{Aj}S_{n-j-1}(\tau) + \delta S_{n-1}(\tau) \right.$$
$$\left. + T_A I_{A(n-1)}(\tau) + T_L I_{L(n-1)}(\tau) \right) d\tau,$$
$$S_{n-1}(0) = 0,$$

$$p^n: I_{Ln}(t)$$
$$= \int_0^t \left((1-P)\alpha \sum_{j=0}^{n-1} I_{Aj}S_{n-j-1}(\tau) - \beta_A I_{L(n-1)}(\tau) \right.$$

$$\left. - T_L I_{L(n-1)}(\tau) - \delta I_{L(n-1)}(\tau) \right) d\tau,$$

$$I_{Ln}(0) = 0$$

$$p^n: I_{An}(t)$$
$$= \int_0^t \left(P\alpha \sum_{j=0}^{n-1} I_{Aj}S_{n-j-1}(\tau) + \beta_A I_{L(n-1)}(\tau) - T_A I_{A(n-1)}(\tau) \right.$$

$$\left. - \delta I_{A(n-1)}(\tau) - \varepsilon I_{A(n-1)}(\tau) \right) d\tau, \quad I_{An}(0) = 0.$$

$$(13)$$

Integrating the previous, we obtain the following components:

$$S_0(t) = S(0); \qquad I_{L0}(t) = I_L(0);$$

$$I_{A0}(t) = I_A(0),$$

$$S_1(t) = \left(\nu fN - \alpha I_{A0}S_0 + \delta S_0 + T_A I_{A0} + T_L I_{L0} \right) t,$$

$$I_{L1}(t) = \left((1-P)\alpha I_{A0}S_0 - \beta_A I_{L0} - T_L I_L 0 - \delta I_{L0} \right) t,$$

$$I_{A1}(t) = \left(P\alpha I_{A0}S_0 + \beta_A I_{L0} - T_A I_{A0} - \delta I_{A0} - \varepsilon I_{A0} \right) t.$$

$$(14)$$

For simplicity, let us put

$$a = \left(\nu fN - \alpha I_{A0}S_0 + \delta S_0 + T_A I_{A0} + T_L I_{L0} \right),$$

$$b = \left((1-P)\alpha I_{A0}S_0 - \beta_A I_{L0} - T_L I_L 0 - \delta I_{L0} \right),$$

$$c = \left(P\alpha I_{A0}S_0 + \beta_A I_{L0} - T_A I_{A0} - \delta I_{A0} - \varepsilon I_{A0} \right),$$

$$S_2(t) = \frac{1}{2}t^2 \left(bT_A + cT_L - aI_{A0}\alpha - bS_0\alpha + a\delta \right)$$
$$= \frac{t^2}{2}a_1,$$

$$(15)$$

$$I_{L2}(t) = \frac{1}{2}t^2 \left(-cT_L + aI_{A0}\alpha - aI_{A0}P\alpha \right.$$
$$\left. + bS_0\alpha - c\beta_A - c\delta \right) = \frac{t^2}{2}b_1,$$

$$I_{A2}(t) = \frac{1}{2}t^2 \left(aI_{A0}P\alpha + bP\alpha S_0 - bT_A \right.$$
$$\left. + c\beta_A - b\delta - b\varepsilon \right) = c_1 \frac{t^2}{2}.$$

In general, we obtain the following recursive formulas:

$$S_n(t) = \frac{t^n}{n!}a_n,$$

$$I_{Ln}(t) = \frac{t^n}{n!}b_n,$$

$$I_{An}(t) = c_n \frac{t^n}{n!},$$

$$(16)$$

where a_n, b_n, and c_n depend on the fixed set of empirical parameters. It therefore follows that the approximate solution of the system (1) is given as

$$S_N(t) = \sum_{n=0}^{N} \frac{t^n}{n!} a_n,$$

$$I_{LN}(t) = \sum_{n=0}^{N} \frac{t^n}{n!} b_n, \qquad (17)$$

$$I_{AN}(t) = \sum_{n=0}^{N} \frac{t^n}{n!} c_n.$$

If for instance one supposes that the total number of new people in the location of interest is $N = 100$; the initial number of susceptible people in the location is $S(0) = 96$; the initial number of TB latently infected people is $I_L(0) = 3$; the initial number of TB actively infected people is $I_A(0) = 1$; the probability that a susceptible person is not vaccinated is $v = 0.5$; the efficient rate of vaccines is $f = 0.5$; the success rate of latent TB therapy is $T_L = 0.8$; the active TB treatment cure rate is $T_A = 0.74$; the TB instantaneous incidence rate per susceptible is $\alpha = 0.41$; humans natural death rate is $\delta = 1/(366 \times 70)$; the proportion of infection instantaneously degenerating into active TB is $P = 0.0197$; the TB-induced death rate is $\varepsilon = 0.0735$; and the breakdown rate from latent to active TB is $\beta_A = 0.01$, then the following approximate solution is obtained as a result of the first 8 terms of the series decomposition:

$$S(t) = 96 - 11.2162t + 62.1069t^2 - 29.5924t^3 - 149.2t^4$$
$$+ 48.3455t^5 - 20.6378t^6 + 15.5857t^7 + \cdots$$

$$I_L(t) = 3 + 36.8527t - 62.9161t^2 - 797.302t^3 + 151.174t^4$$
$$- 48.8926t^5 + 20.7629t^6 - 15.6036t^7 + \cdots$$

$$I_A(t) = 1 - 0.706394t + 0.252053t^2 - 0.252832t^3$$
$$- 1.96203t^4 + 0.573666t^5 - 0.131459t^6$$
$$+ 0.0190148t^7 + \cdots.$$

$$(18)$$

If in addition we assume that no new person migrates or is born in this area, we obtain the following figures. The approximate solutions of the main problem are depicted in Figures 1, 2, and 3, respectively.

Figure 1 shows that, if there is migration or newborn in the location of interest, the number of susceptible people will vanish as time goes, because of the natural death rate and due to TB. Note that any person that is latently infected is removed from the set of susceptible. Figure 2 indicates that the number of people that are latently infected will increase up to a certain time and then vanish as time goes. The number of susceptible people, will become latently infected since some are not vaccinated against the TB and finally will vanish due to. Figure 3 indicates that the number of TB actively infected
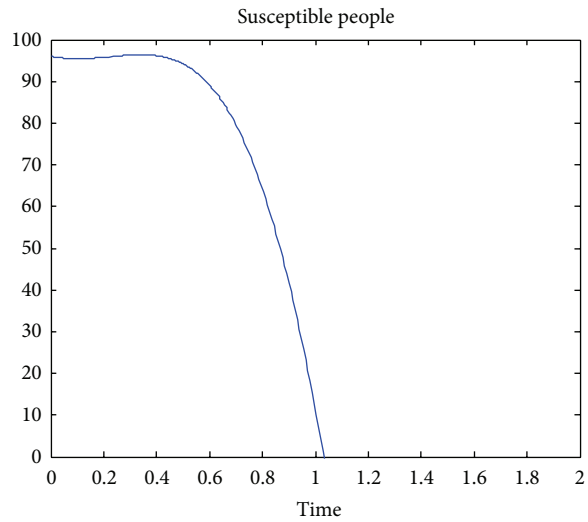

FIGURE 1: Approximate solution for the number of susceptible people in the location.
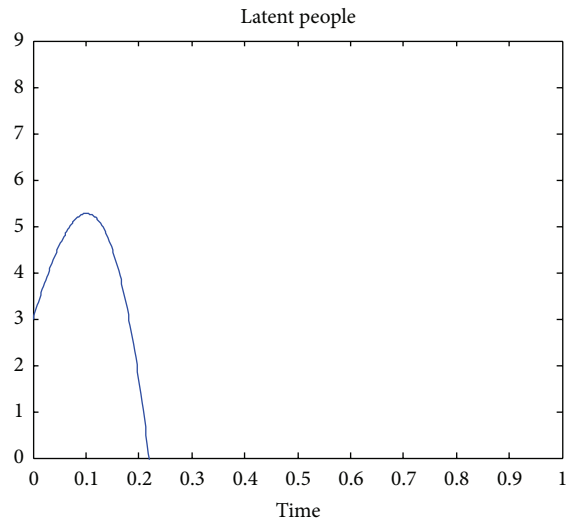

FIGURE 2: Approximate solution for the number of TB latently infected people.

people will also vanish because of the natural death rate and the death due to TB.

4. Application of the HDM to the Model with Noninteger-Order Derivative

Fractional calculus has been used to model physical and engineering processes, which are found to be best described by fractional differential equations. It is worth noting that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases. In the recent years, fractional calculus has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory, and signal and image processing. Major topics
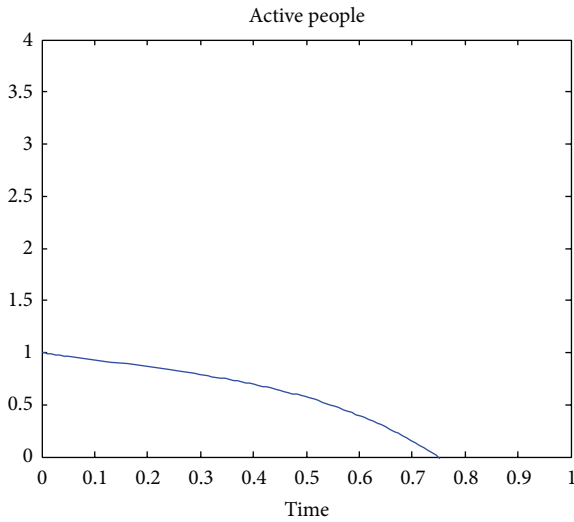
Active people

FIGURE 3: Approximate solution for the number of TB actively infected people.

include anomalous diffusion; vibration and control; continuous time random walk; Levy statistics, fractional Brownian motion; fractional neutron point kinetic model; power law; Riesz potential; fractional derivative and fractals; computational fractional derivative equations; nonlocal phenomena; history-dependent process; porous media; fractional filters; biomedical engineering; fractional phase-locked loops, and groundwater problem (see [14–21]).

4.1. Properties and Definitions

Definition 1. A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where $h(x) \in C[0, \infty)$, and it is said to be in space C_μ^m if $f^{(m)} \in C_\mu, m \in \mathbb{N}$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt, \quad \alpha > 0, \ x > 0$$

$$J^0 f(x) = f(x). \tag{19}$$

Properties of the operator can be found in [14–16]. We mention only the following: for $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$, and $\gamma > -1$,

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x) \quad J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \tag{20}$$

Lemma 3. *If $m - 1 < \alpha \leq m, m \in \mathbb{N}$, and $f \in C_\mu^m, \mu \geq -1$, then*

$$D^\alpha J^\alpha f(x) = f(x),$$

$$J^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \tag{21}$$

Definition 4 (partial derivatives of fractional order). Assume now that $f(\mathbf{x})$ is a function of n variables $x_i \, i = 1, \dots, n$ also of class C on $D \in \mathbb{R}_n$. We define partial derivative of order α for f respect to x_i the function

$$a\partial_{\underline{x}}^\alpha f = \frac{1}{\Gamma(m-\alpha)} \int_a^{x_i} (x_i - t)^{m-\alpha-1} \partial_{x_i}^m f(x_j)\big|_{x_j=t} dt. \tag{22}$$

where $\partial_{x_i}^m$ is the usual partial derivative of integer-order m.

4.2. Approximate Solution of Fractional Version. The system of equations under investigation here is given as

$$\frac{d^\mu S(t)}{dt^\mu} = vfN - \alpha I_A S(t) + \delta S(t)$$

$$+ T_A I_A(t) + T_L I_L(t), \quad 0 < \mu \leq 1,$$

$$\frac{d^\eta I_L(t)}{dt^\eta} = (1-P)\alpha I_A S(t) - \beta_A I_L(t)$$

$$- T_L I_L(t) - \delta I_L(t), \quad 0 < \eta \leq 1, \tag{23}$$

$$\frac{d^v I_A(t)}{dt^v} = P\alpha I_A S(t) + \beta_A I_L(t) - T_A I_A(t)$$

$$- \delta I_A(t) - \varepsilon I_A(t), \quad 0 < v \leq 1.$$

Following the discussion presented earlier, we arrive at the following equations:

$$p^0: \ S_0(t) = S(0),$$

$$p^0: \ I_{L0}(t) = I_L(0),$$

$$p^0: \ I_{A0}(t) = I_A(0),$$

$$p^1: S_1(t)$$

$$= \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1}$$

$$\times (vfN - \alpha I_{A0} S_0(\tau) + \delta S_0(\tau)$$

$$+ T_A I_{A0}(\tau) + T_L I_{L0}(\tau)) \, d\tau,$$

$$S_1(0) = 0,$$

$$p^1: I_{L1}(t)$$

$$= \frac{1}{\Gamma(\eta)} \int_0^t (t-\tau)^{\eta-1}$$

$$\times ((1-P)\alpha I_{A0} S_0(\tau) - \beta_A I_{L0}(\tau)$$

$$- T_L I_L 0(\tau) - \delta I_{L0}(\tau)) \, d\tau,$$

$$I_{L1}(0) = 0,$$

$$p^1: I_{A1}(t)$$

$$= \frac{1}{\Gamma(v)} \int_0^t (t-\tau)^{v-1}$$

$$\times (P\alpha I_{A0} S_0(\tau) + \beta_A I_{L0}(\tau)$$

$$- T_A I_{A0}(\tau) - \delta I_{A0}(\tau) - \varepsilon I_{A0}(\tau)) \, d\tau,$$

$$I_{A1}(0) = 0,$$

$$p^n : S_n(t)$$

$$= \frac{1}{\Gamma(\mu)} \int_0^t (t - \tau)^{\mu-1}$$

$$\times \left(\nu f N - \alpha \sum_{j=0}^{n-1} I_{Aj} S_{n-j-1}(\tau) + \delta S_{n-1}(\tau) \right.$$

$$\left. + T_A I_{A(n-1)}(\tau) + T_L I_{L(n-1)}(\tau) \right) d\tau,$$

$$S_{n-1}(0) = 0,$$

$$p^n : I_{Ln}(t)$$

$$= \frac{1}{\Gamma(\eta)} \int_0^t (t - \tau)^{\eta-1}$$

$$\times \left((1 - P) \alpha \sum_{j=0}^{n-1} I_{Aj} S_{n-j-1}(\tau) - \beta_A I_{L(n-1)}(\tau) \right.$$

$$\left. - T_L I_{L(n-1)}(\tau) - \delta I_{L(n-1)}(\tau) \right) d\tau,$$

$$I_{Ln}(0) = 0,$$

$$p^n : I_{An}(t)$$

$$= \frac{1}{\Gamma(\upsilon)} \int_0^t (t - \tau)^{\upsilon-1}$$

$$\times \left(P\alpha \sum_{j=0}^{n-1} I_{Aj} S_{n-j-1}(\tau) + \beta_A I_{L(n-1)}(\tau) \right.$$

$$- T_A I_{A(n-1)}(\tau) - \delta I_{A(n-1)}(\tau)$$

$$\left. - \varepsilon I_{A(n-1)}(\tau) \right) d\tau,$$

$$I_{An}(0) = 0.$$

$$(24)$$

Integrating the previous, we obtain the following components:

$$S_0(t) = S(0); \qquad I_{L0}(t) = I_L(0);$$
$$I_{A0}(t) = I_A(0),$$

$$S_1(t) = -\frac{11.2162 t^\mu}{\Gamma(1 + \mu)};$$

$$I_{L1}(t) = \frac{36.8527 t^\eta}{\Gamma(1 + \eta)},$$

$$I_{A1}(t) = -\frac{0.706394 t^\upsilon}{\Gamma(1 + \upsilon)},$$

$$S_2(t) = t^\mu \left(\frac{29.4822 t^\eta}{\Gamma(1 + \eta + \mu)} + \frac{4.59822 t^\mu}{\Gamma(1 + 2\mu)} \right.$$

$$\left. + \frac{27.2809 t^\upsilon}{\Gamma(1 + \upsilon + \mu)} \right),$$

$$I_{L2}(t) = -t^\eta \left(\frac{29.8522 t^\eta}{\Gamma(1 + 2\eta)} + \frac{4.58965 t^\mu}{\Gamma(1 + \eta + \mu)} \right.$$

$$\left. + \frac{27.7492 t^\upsilon}{\Gamma(1 + \upsilon + \mu)} \right),$$

$$I_{A2}(t) = t^\upsilon \left(\frac{0.368527 t^\eta}{\Gamma(1 + \eta + \upsilon)} - \frac{0.00901337 t^\mu}{\Gamma(1 + \upsilon + \mu)} \right.$$

$$\left. + \frac{0.520184 t^\upsilon}{\Gamma(1 + 2\upsilon)} \right),$$

$$S_3(t) = t^\mu \left(-\frac{3.24846 t^{\mu+\upsilon} \Gamma(1 + \mu + \upsilon)}{\Gamma(1 + \mu)\Gamma(1 + \upsilon)\Gamma(1 + 2\mu + \upsilon)} \right.$$

$$- \frac{0.298522 t^{2\eta}}{\Gamma(1 + 2\eta + \upsilon)} - \frac{15.7583 t^{\eta+\mu}}{\Gamma(1 + \eta + 2\mu)}$$

$$- \frac{1.88509 t^{2\mu}}{\Gamma(1 + 3\mu)} - \frac{36.4319 t^{\eta+\upsilon}}{\Gamma(1 + \mu + \upsilon + \eta)}$$

$$\left. - \frac{10.836 t^{\mu+\upsilon}}{\Gamma(1 + 2\mu + \upsilon)} - \frac{20.0895 t^{2\upsilon}}{\Gamma(1 + \mu + 2\upsilon)} \right),$$

$$I_{L3}(t) = t^\eta \left(-\frac{1148.5 t^{2\eta}}{\Gamma(1 + 3\eta)} - \frac{164.513 t^{\eta+\mu}}{\Gamma(1 + 2\eta + \mu)} + \frac{1.88158 t^{2\mu}}{\Gamma(1 + \eta + 2\mu)} \right.$$

$$- \frac{1067.59 t^\eta}{\Gamma(1 + 2\eta + \upsilon)} + \frac{11.1633 t^{\mu+\upsilon}}{\Gamma(1 + \eta + \mu + \upsilon)}$$

$$\left. + \frac{3.2421 t^{\mu+\upsilon} \Gamma(1 + \mu + \upsilon)}{\Gamma(1 + \mu)\Gamma(1 + \upsilon)\Gamma(1 + \eta + \mu + \upsilon)} \right),$$

$$I_{A3}(t) = t^\upsilon \left(\frac{0.00636699 t^{\mu+\upsilon} \Gamma(1 + \mu + \upsilon)}{\Gamma(1 + \mu)\Gamma(1 + \upsilon)\Gamma(1 + 2\upsilon + \mu)} \right.$$

$$- \frac{0.298522 t^{2\eta}}{\Gamma(1 + 2\eta + \upsilon)} - \frac{0.0222046 t^{\eta+\mu}}{\Gamma(1 + \eta + 2\mu)}$$

$$+ \frac{0.00369513 t^{2\mu}}{\Gamma(1 + 3\mu)} - \frac{0.548873 t^{\eta+\upsilon}}{\Gamma(1 + \mu + 2\upsilon)}$$

$$\left. + \frac{0.0285603 t^{\mu+\upsilon}}{\Gamma(1 + \mu + 2\upsilon)} - \frac{0.38306 t^{2\upsilon}}{\Gamma(1 + 3\upsilon)} \right).$$

$$(25)$$

The remaining terms can be obtained in the same manner. But here only few terms of the series solutions are considered, and the asymptotic solution is given as

$$S(t) = S_0(t) + S_1(t) + S_2(t) + S_3(t) + \cdots,$$

$$I_L(t) = I_{L0}(t) + I_{L1}(x, t) + I_{L2}(x, t) + I_{L3}(x, t) + \cdots,$$

$$I_A(t) = I_{A0}(t) + I_{A1}(x, t) + I_{A2}(x, t) + I_{A3}(x, t) + \cdots.$$

$$(26)$$

The following figures show the simulated solutions for different values of the fractional order derivatives. The approximate

FIGURE 4: Approximate for $\mu = 0.45$, $\eta = 0.7$, and $v = 0.85$.

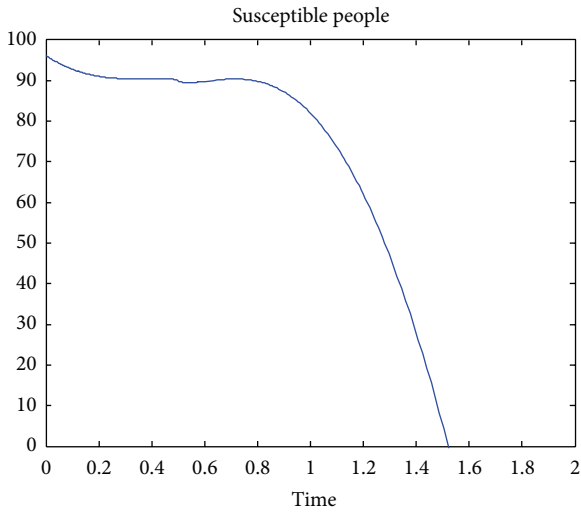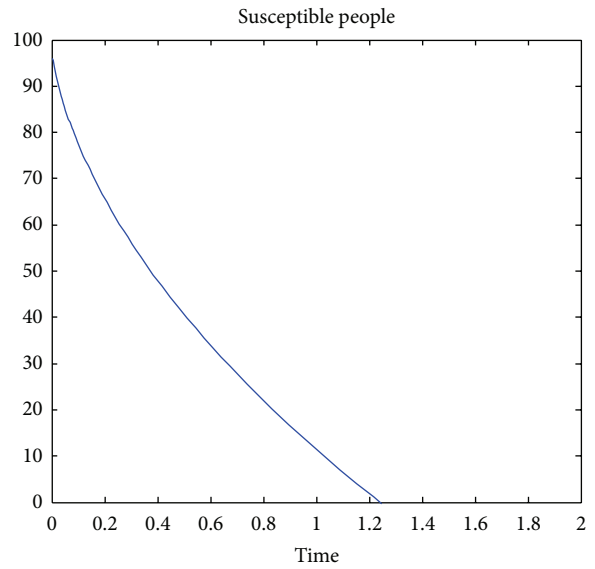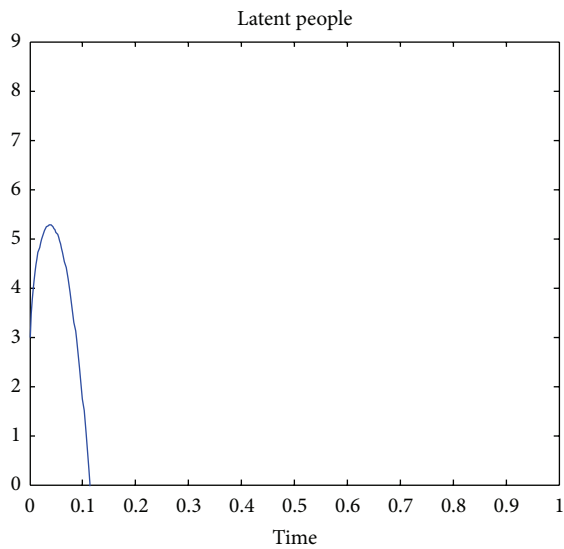FIGURE 5: Approximate for $\mu = 0.45$, $\eta = 0.7$, and $v = 0.85$.

FIGURE 6: Approximate for $\mu = 0.45$, $\eta = 0.7$, and $v = 0.85$.

FIGURE 7: Approximate for $\mu = 0.045$, $\eta = 0.5$, and $v = 0.085$.

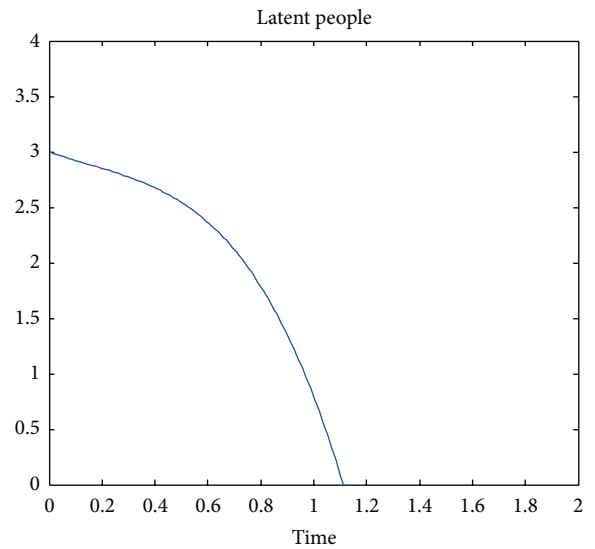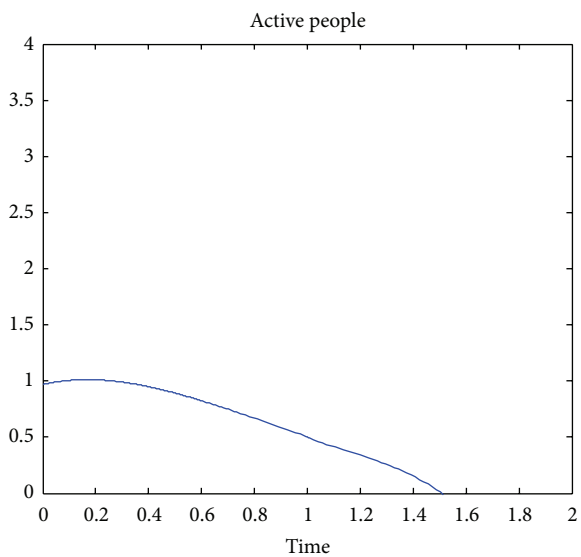FIGURE 8: Approximate for $\mu = 0.045$, $\eta = 0.5$, and $v = 0.085$.

solutions of the main problem are depicted in Figures 4, 5, 6, 7, 8, and 9, respectively.

The numerical simulations show that the approximate solutions are continuous functions of the noninteger-order derivative. It is worth noting that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases. It is therefore advisable to use the fractional model for describing this problem.

5. Conclusion

The tuberculosis model was examined for the case of integer- and noninteger-order derivatives. Both systems of nonlinear equations were solved with an iterative analytical model
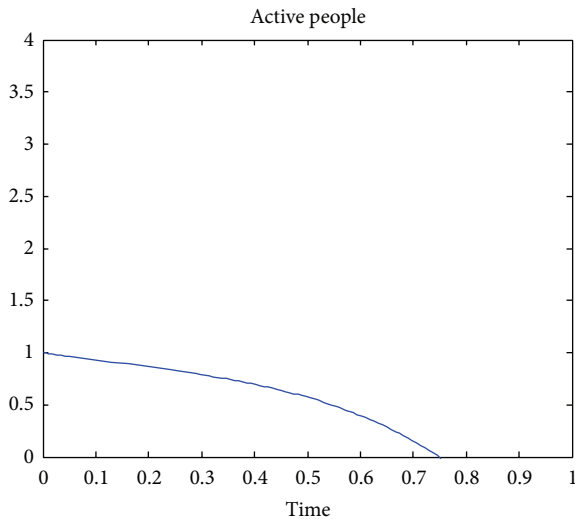
FIGURE 9: Approximate for $\mu = 0.045$, $\eta = 0.5$, and $\upsilon = 0.085$.

called the homotopy decomposition model method. The basic characters of the relatively new technique are presented in detail. The approximate solutions of the noninteger case are increasing continuous functions of the fractional order derivative. The technique used for solving these problems is friendly, very easy, and less time consuming. The numerical solutions in both cases display the biological behaviour of the real world situation.

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