Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 498902, 11 pages
http://dx.doi.org/10.1155/2013/498902

Research Article

Optimal Homotopy Asymptotic Method for Solving Delay Differential Equations

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Received 4 May 2013; Accepted 13 September 2013

Academic Editor: Sotiris Ntouyas

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We extend for the first time the applicability of the optimal homotopy asymptotic method (OHAM) to find the algorithm of approximate analytic solution of delay differential equations (DDEs). The analytical solutions for various examples of linear and nonlinear and system of initial value problems of DDEs are obtained successfully by this method. However, this approach does not depend on small or large parameters in comparison to other perturbation methods. This method provides us with a convenient way to control the convergence of approximation series. The results which are obtained revealed that the proposed method is explicit, effective, and easy to use.

1. Introduction

Delay differential equation (DDE) is a form of differential equations in which derivative of the unknown function in a given time t is specified in terms of the values at an earlier point in time.

DDEs have the general form

$$u_i'(x) = f\left(x, u_i(x), u_i\left(\xi_j(x)\right)\right), \quad i = 1, 2, \dots, M, \quad j = 1, 2, \dots, N, \tag{1}$$

where $\xi_j(x) = a_j x + b_j$ is the delay function.

Many problems of physics, biological models, control system, and medical and biochemical fields are modelled by DDEs. Recent studies in such diverse fields have shown that DDEs play an important role in explaining many different phenomena. Patel et al. [1] introduced an iterative scheme for the optimal control systems described by DDEs with a quadratic cost functional. In physiology, Glass and Mackey [2] applied time delays to many physiological models. Busenberg and Tang [3] created a model for cell cycle by delay equations. In recent years, DDEs are used to design models as HIV-1 therapy for fighting a virus with another virus [4].

In the last years, a great deal of attention has been devoted to study DDEs. Hence, they are solved by numerical method and approximation approach, such as Adomian decomposition method [5, 6], homotopy perturbation method (HPM) [7, 8], multiquadric approximation scheme [9, 10], variational iteration method (VIM) [8, 11, 12], spline methods [13], homotopy analysis method (HAM) [14], Chebyshev polynomials [15], Galerkin method [16], Legendre wavelet method [17], differential transform method [18], and Runge-Kutta method [19]. Recently, a new approach of homotopy which is called optimal homotopy asymptotic method (OHAM) was proposed and developed by Marinca et al. [20–24] for the approximate solution nonlinear problems of thin film flow of a fourth-grade fluid and for the study of the behavior of nonlinear mechanical vibration of electrical machines. In OHAM, the control and adjustment of the convergence region are provided in a convenient way. Furthermore, the OHAM has been built in convergence criteria similar to those of HAM but with greater degree of flexibility. Islam et al. [25] have applied this method successfully to nonlinear problems and have also shown its effectiveness and accuracy. Idrees et al. [26] used OHAM to study the squeezing flow between two infinite planar plates slowly approaching each other.

The aim of this paper is to apply OHAM to get an approximate analytic solution of DDEs. The capability of this approach is tested upon several examples which offer an approximate solution in a series form that converges to exact solution in few terms. The rest of this paper is organized as follows. In Section 2, we describe the basic idea of OHAM. In Section 3, we provide the convergent theorem for this type of equations. Section 4 presents several examples to demonstrate the efficiency of the framework. The conclusion of this study is presented in Section 5.

2. Description of the Method

In this section, framework of the proposed method is given and represented in the following differential equation:

$$L_i(u_i(x)) + g_i(x) + N_i\left(u_i(x), u_i\left(\xi_j(x)\right)\right) = 0, \quad i = 1, 2, \dots, N, \quad j = 1, 2, \dots, M, \tag{2}$$

where L_i are the linear operators and N_i are the nonlinear operators contain delay function, $u_i(x)$ is an unknown function, x denotes an independent variable, $g_i(x)$ is a known function, and $\xi_j(x)$ are the delay functions.

According to OHAM, we construct a homotopy $h_i(v_i(x, p), p) : R \times [0, 1] \to R$ which satisfies

$$(1 - p) L_i\left[(v_i(x, p) - u_{i,0}(x)\right] = H_i(p)\left[L_i v(x, p) + g_i(x) + N_i\left(v_i(x, p), v_i\left(\xi_j(x), p\right)\right)\right], \tag{3}$$

where $x \in R$, $p \in [0, 1]$ is an embedding parameter, $H_i(p)$ is a nonzero auxiliary function for $p \neq 0$, $H(0) = 0$, and $v(x, p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$ it holds that $v_i(x, 0) = u_{i,0}(x)$ and $v_i(x, 1) = v_i(x)$, respectively. Thus, as p varies from 0 to 1, the solution $v_i(x, p)$ approach from $u_{i,0}$ to $v_i(x)$, where $u_{i,0}(x)$ is the initial guess that satisfies the linear operator and the initial conditions

$$L_i(u_{i,0}(x)) + g(x) = 0. \tag{4}$$

Next, we choose the auxiliary function $H_i(p)$ in the form

$$H_i(p) = pC_1 + p^2 C_2 + p^3 C_3 + \cdots, \tag{5}$$

where C_1, C_2, $C_3, \dots$ are convergence control parameters which can be determined later. $H(p)$ can be expressed in another form as reported by Herişanu and Marinca [24].

To get an approximate solution, we expand $v_i(x, p, C_k)$ in Taylor's series about p in the following manner:

$$v_i(x, p, C_k) = u_{i,0}(x) + \sum_{k=1}^{\infty} u_{i,k}(x, C_1, C_2, \dots, C_k) p^k. \tag{6}$$

By substituting (6) into (3) and equating the coefficient of like powers of p, we obtain the following linear equations. Define the vectors

$$\vec{C}_i = \{C_1, C_2, \dots, C_i\},$$
$$\vec{u}_{i,s} = \left\{u_{i,0}(x), u_{i,1}(x, C_1), \dots, u_{i,s}\left(x, \vec{C}_s\right), u_{i,0}\left(\xi_j(x)\right), u_{i,1}\left(\xi_j(x), C_1\right), \dots, u_{i,s}\left(\xi_j(x), \vec{C}_s\right)\right\}, \tag{7}$$

where $s = 1, 2, 3, \dots$ and $j = 1, 2, \dots, M$. The zeroth-order problem is given by (4), and the first- and second-order problems are given as

$$L_i(u_{i,1}(x)) = C_1 N_{i,0}(\vec{u}_{i,0}) + g(x), \tag{8}$$

$$L_i(u_{i,2}(x)) - L_i(u_{i,1}(x)) = C_2 N_{i,0}(\vec{u}_{i,0}) + C_1\left[L_i(u_{i,1}(x)) + N_{i,1}(\vec{u}_{i,1})\right]. \tag{9}$$

The general governing equations for $u_k(x)$ are

$$L(u_k(x)) - L(u_{k-1}(x)) = C_k N_0(u_0(x)) + \sum_{m=1}^{k-1} C_m\left[L(u_{k-m}(x)) + N_{k-m}(\vec{u}_{i,k-1})\right], \tag{10}$$

where $k = 2, 3, \dots$ and $N_{i,m}(u_{i,0}(x), u_{i,1}(x), \dots, u_{i,m}(x))$ is the coefficient of p^m in the expansion of $N(v(x, p))$ about the embedding parameter p:

$$N_i(v(x, p, C_i)) = N_{i,0}(u_0(x)) + \sum_{m=1}^{\infty} N_{i,m}(\vec{u}_{i,m}) p^m. \tag{11}$$

It has been observed that the convergence of the series (6) depends upon the auxiliary constants C_1, C_2, $C_3, \dots$. If it is convergent at $p = 1$, one has

$$v_i(x, C_k) = u_0(x) + \sum_{k=1}^{\infty} u_{i,k}(x, C_1, C_2, \dots, C_k). \tag{12}$$

The result of the mth-order approximation is given as

$$\tilde{v}_i(x, C_k) = u_0(x) + \sum_{k=1}^{m} u_{i,k}(x, C_1, C_2, \dots, C_k). \tag{13}$$

Substituting (12) into (2) yields the following residual:

$$R_i(x, C_1, C_2, C_3, \dots, C_m) = L_i(\tilde{v}_i(x, C_1, C_2, C_3, \dots, C_m)) + g_i(x) + N_i(\tilde{v}_i(x, C_1, C_2, C_3, \dots, C_m)). \tag{14}$$

If $R_i = 0$, then $\tilde{u}$ will be the exact solution. Generally such a case will not arise for nonlinear problems, but we can minimize the functional

$$J_i(C_1, C_2, C_3, \dots, C_m) = \int_a^b R_i^2(x, C_1, C_2, C_3, \dots, C_m)\, dx, \tag{15}$$

where a and b are the endpoints of the given problem. The unknown convergence control parameters C_i ($i = 1, 2, 3, \ldots, m$) can be calculated from the system of equations

$$\frac{\partial J_i}{\partial C_i} = 0, \quad i = 1, 2, \ldots, m. \tag{16}$$

It should be noted that our process included the auxiliary function $H_i(p)$ which provides us an easy way to set and optimally control the convergent area and the rate of the solution series.

3. Convergence Theorem

In this section, we introduce the convergence of the solution for DDEs.

Theorem 1. *If the series* (12) *converges to* $u(x)$, *where* $u_k(x) \in L(R^+)$ *is produced by* (8) *and the k-order deformation* (10), *then* $u(x)$ *is the exact solution of* (2).

Proof. Since the series

$$\sum_{k=1}^{\infty} u_{i,k}(x, C_1, C_2, \ldots, C_k) \tag{17}$$

converges, it can be written as

$$S_i(x) = \sum_{k=1}^{\infty} u_{i,k}(x, C_1, C_2, \ldots, C_k), \tag{18}$$

and it holds that

$$\lim_{k\to\infty} u_{i,k}(x, C_1, C_2, \ldots, C_k) = 0. \tag{19}$$

The left hand-side of (10) satisfies

$$\begin{aligned} u_{i,1}(x, C_1) &+ \sum_{k=2}^{n} u_{i,k}\left(x, \vec{C}_k\right) - \sum_{k=2}^{n} u_{i,k-1}\left(x, \vec{C}_{k-1}\right) \\ &= u_{i,2}\left(x, \vec{C}_2\right) - u_{i,1}(x, C_1) + \cdots + u_{i,n}\left(x, \vec{C}_n\right) \\ &\quad - u_{i,n-1}\left(x, \vec{C}_{n-1}\right) \\ &= u_{i,n}\left(x, \vec{C}_n\right). \end{aligned} \tag{20}$$

According to (18) we have

$$\begin{aligned} u_{i,1}(x, C_1) &+ \sum_{k=2}^{n} u_{i,k}\left(x, \vec{C}_k\right) - \sum_{k=2}^{n} u_{i,k-1}\left(x, \vec{C}_{k-1}\right) \\ &= \lim_{n\to\infty} u_{i,n}\left(x, \vec{C}_n\right) = 0. \end{aligned} \tag{21}$$

Using the linear operator L_i,

$$\begin{aligned} L_i(u_{i,1}(x, C_1)) &+ \sum_{k=2}^{\infty} L_i\left(u_{i,k}\left(x, \vec{C}_k\right)\right) \\ &- \sum_{k=2}^{\infty} L_i\left(u_{i,k-1}\left(x, \vec{C}_{k-1}\right)\right) \\ &= L_i(u_{i,1}(x, C_1)) + L_i \sum_{k=2}^{\infty} u_{i,k}\left(x, \vec{C}_k\right) \\ &- L_i \sum_{k-1}^{\infty} u_{i,k-1}\left(x, \vec{C}_{k-1}\right) = 0 \end{aligned} \tag{22}$$

which satisfies

$$\begin{aligned} L_i(u_{i,1}(x, C_1)) &+ L_i \sum_{k=2}^{\infty} u_{i,k}\left(x, \vec{C}_k\right) - L_i \sum_{k-1}^{\infty} u_{i,k-1}\left(x, \vec{C}_{k-1}\right) \\ = \sum_{k=2}^{\infty} \Bigg[& C_k N_{i,0}(u_{i,0}(x)) \\ &+ \sum_{m=1}^{k-1} C_m \left[L_i\left(u_{i,k-m}\left(x, \vec{C}_{k-m}\right)\right)\right. \\ &\left. + N_{i,k-1-m}(\vec{u}_{i,k-1}) \right] \Bigg] + g_i(x) = 0. \end{aligned} \tag{23}$$

Also the right hand side can be written as

$$\begin{aligned} \sum_{k=1}^{\infty} \Bigg[& \sum_{m=1}^{k} C_{m-k} \\ & \times \left[L_i\left(u_{i,m-1}\left(x, \vec{C}_{m-1}\right)\right) + N_{i,m-1}(\vec{u}_{i,k-1}) \right] \Bigg] \\ & + g_i(x) = 0. \end{aligned} \tag{24}$$

Now, if the C_m, $m = 1, 2, \ldots$, is properly chosen, then (24) leads to

$$L_i(u_i(x)) + N_i\left(u_i(x), u_i\left(\xi_j(x)\right)\right) + g_i(x) = 0, \tag{25}$$

which is the exact solution. □

4. Applications

In this section, we will present a few examples with a known analytic solution in order to demonstrate the effectiveness and high precision of this algorithm.

Example 1. Consider the following linear delay differential equation [5]:

$$u'(x) = \frac{1}{2} e^{x/2} u\left(\frac{x}{2}\right) + \frac{1}{2} u(x), \quad 0 \le x \le 1, \; u(0) = 1, \tag{26}$$

with the exact solution

$$u(x) = e^{x}. \tag{27}$$

Applying the procedure which is described in Section 2, the linear and nonlinear operators are

$$\begin{gathered} L[v(x,p)] = \frac{dv(x,p)}{dx}, \\ N[v(x,p)] \\ = \frac{dv(x,p)}{dx} - \frac{1}{2}\psi(x;p)\,v\left(\frac{x}{2},p\right) - \frac{1}{2}v(x,p), \\ v(0,p) = 1, \end{gathered} \tag{28}$$

where $\psi(x;p)$ is the expansion Taylor series of $e^{x/2}$ with respect to p, which can be written as

$$\psi(x;p) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{x}{2}\right)^{k}p^{k}. \tag{29}$$

Now, apply (4) to $p = 0$ to give the zeroth-order problem as

$$u_0'(x) = 0, \quad u_0(0) = 1. \tag{30}$$

The solution of the zeroth-order deformation is

$$u_0(x) = 0. \tag{31}$$

The first-order deformation which is obtained from (8) is given as

$$u_1'(x,C_1) = -\frac{1}{4}(4+x)C_1, \quad u_0(0) = 0, \tag{32}$$

and has the solution

$$u_1(x,C_1) = -C_1x - \frac{1}{8}C_1x^2. \tag{33}$$

The second-order deformation is given by (9):

$$\begin{aligned} u_2'(x,C_1,C_2) = {} & u_1'(x,C_1) - C_2 - \frac{1}{4}xC_2 \\ & - \frac{1}{16}x^2C_2 - \frac{1}{2}C_1u_1\left(\frac{x}{2},C_1\right) \\ & - \frac{1}{4}xC_1u_1\left(\frac{x}{2},C_1\right) - \frac{1}{16}x^2C_1u_1\left(\frac{x}{2},C_1\right) \\ & - \frac{1}{2}C_1u_1(x,C_1) + C_1u_1'(x,C_1), \end{aligned} \tag{34}$$

with initial condition

$$u_2(0) = 0. \tag{35}$$

The solution of (34) is given by

$$\begin{aligned} u_2(x,C_1,C_2) = {} & -C_1x - \frac{1}{8}C_1x^2 - C_1^2x + \frac{1}{4}C_1^2x^2 \\ & + \frac{13}{192}C_1^2x^3 + \frac{5}{512}C_1^2x^4 + \frac{1}{2560}C_1^2x^5 \\ & - C_2x - \frac{1}{8}C_2x^2 - \frac{1}{48}C_2x^3. \end{aligned} \tag{36}$$

According to (10), the third-order deformation is defined as

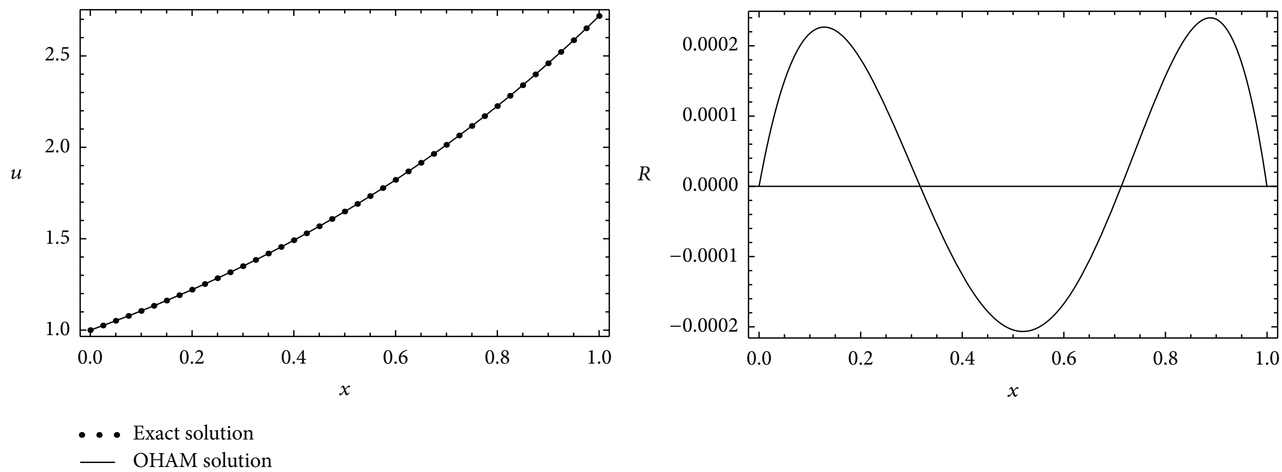
$$\begin{aligned} & u_3'(x,C_1,C_2,C_3) \\ & = u_2'(x,C_1,C_2) - C_3 - \frac{1}{4}C_3x - \frac{1}{16}C_3x^2 - \frac{1}{96}C_3x^3 \\ & - \frac{1}{2}C_2u_1\left(\frac{x}{2},C_1\right) - \frac{1}{4}C_2xu_1\left(\frac{x}{2},C_1\right) \\ & - \frac{1}{16}C_2x^2u_1\left(\frac{x}{2},C_1\right) - \frac{1}{96}C_2x^3u_1\left(\frac{x}{2},C_1\right) \\ & - \frac{1}{2}C_2u_1(x,C_1) - \frac{1}{2}C_1u_2\left(\frac{x}{2},C_1,C_2\right) \\ & - \frac{1}{4}C_1xu_2\left(\frac{x}{2},C_1,C_2\right) - \frac{1}{16}C_1x^2u_2\left(\frac{x}{2},C_1,C_2\right) \\ & - \frac{1}{96}C_1x^3u_2\left(\frac{x}{2},C_1,C_2\right) - \frac{1}{2}C_1u_2(x,C_1,C_2) \\ & + C_2u_1'(x,C_1) + C_1u_2'(x,C_1,C_2), \end{aligned} \tag{37}$$

with initial condition

$$u_3(0) = 0, \tag{38}$$

and has the solution

$$\begin{aligned} & u_3(x,C_1,C_2,C_3) \\ & = -C_1x - \frac{1}{8}C_1x^2 - 2C_1^2x + \frac{1}{2}C_1^2x^2 + \frac{13}{96}C_1^2x^3 \\ & + \frac{5}{256}C_1^2x^4 + \frac{7}{3840}C_1^2x^5 + \frac{1}{18432}C_1^2x^6 \\ & - C_1^3x + \frac{5}{8}C_1^3x^2 + \frac{11}{192}C_1^3x^3 + \frac{17}{4096}C_1^3x^4 \\ & - \frac{199}{245760}C_1^3x^5 - \frac{377}{1474560}C_1^3x^6 \\ & - \frac{109}{5898240}C_1^3x^7 - \frac{7}{7864320}C_1^3x^8 \\ & - \frac{1}{70778880}C_1^3x^9 - C_2x - \frac{1}{8}C_2x^2 - \frac{1}{48}C_2x^3 \\ & - 2C_1C_2x + \frac{1}{2}C_1C_2x^2 + \frac{11}{96}C_1C_2x^3 \\ & + \frac{23}{1024}C_1C_2x^4 - \frac{23}{7680}C_1C_2x^5 \end{aligned}$$



(a) (b)

FIGURE 1: (a) Comparison between the three-term OHAM approximate solution and the exact solution for Example 1 and (b) residual error $R(x)$ given by (41) using the three-term OHAM approximate solution.

$$
\begin{aligned}
&+ \frac{5}{36864} C_1 C_2 x^6 + \frac{1}{258048} C_1 C_2 x^7 \\
&- C_3 x - \frac{1}{8} C_3 x^2 - \frac{1}{48} C_3 x^3 - \frac{1}{384} C_3 x^4.
\end{aligned} \tag{39}
$$

By using (31), (33), (36), and (39), the third-order approximate solution by OHAM for $p = 1$ is

$$
\begin{aligned}
\tilde{u}(x, C_1, C_2, C_3) = {} & u_0(x) + u_1(x, C_1) + u_2(x, C_1, C_2) \\
& + u_3(x, C_1, C_2, C_3).
\end{aligned} \tag{40}
$$

By using the proposed method of Section 2 on $[0, 1]$, we use the residual error:

$$
\begin{aligned}
R = {} & \tilde{u}'(x, C_1, C_2, C_3) - \frac{1}{2} e^{x/2} \tilde{u}\left(\frac{x}{2}, C_1, C_2, C_3\right) \\
& - \frac{1}{2} \tilde{u}(x, C_1, C_2, C_3).
\end{aligned} \tag{41}
$$

The Less Square error can be formed as

$$
\begin{gathered}
J(C_1, C_2, C_3) = \int_0^1 R^2 dx, \\
\frac{\partial J(C_1, C_2, C_3)}{\partial C_1} = \frac{\partial J(C_1, C_2, C_3)}{\partial C_2} = \frac{\partial J(C_1, C_2, C_3)}{\partial C_3} = 0.
\end{gathered} \tag{42}
$$

Thus, the following optimal values of C_i's are obtained:

$$
\begin{gathered}
C_1 = -1.1862449850, \qquad C_2 = 0.0255300261, \\
C_3 = -0.0070171914.
\end{gathered} \tag{43}
$$

In this case, our approximate solution is

$$
\begin{aligned}
& \tilde{u}(x, C_1, C_2, C_3) \\
& \quad = 1 + 1.00397x + 0.479341x^2 + 0.194878x^3 \\
& \qquad + 0.0349963x^4 + 0.00455719x^5 \\
& \qquad + 0.00050723x^6 + 0.0000309654x^7 \\
& \qquad + 1.4858 \times 10^{-6} x^8 + 2.35841 \times 10^{-8} x^9.
\end{aligned} \tag{44}
$$

Equations (44) and (41) are plotted in Figures 1(a) and 1(b), respectively. Figure 1(a) shows a comparison between the approximate solution which is obtained by using OHAM and exact solution (27). The residual error is plotted in Figure 1(b). We noted that the absolute maximum error for solving this example via HAM is 0.04 while the absolute maximum error via OHAM is 0.2×10^{-3}, which leads to conclude that OHAM is more accurate than HAM.

Example 2. Consider the linear delay differential equation of third order [5]

$$
\begin{gathered}
u'''(x) = -u(x) - u(x - 0.3) + e^{-x+0.3}, \quad 0 \le x \le 1, \\
u'(0) = -1, \qquad u''(0) = 1, \qquad u(x) = e^{-x}, \quad x \le 0,
\end{gathered} \tag{45}
$$

with exact solution

$$
u(x) = e^{-x}. \tag{46}
$$

According to the method which was described in the above section, we start with

$$L[v(x,p)] = \frac{d^3 v(x,p)}{dx^3},$$
$$N[v(x,p)] = \frac{d^3 v(x,p)}{dx^3} + v(x,p) + v(x-0.3,p) - e^{-x+0.3}. \quad (47)$$

By applying OHAM, we have the following zero-, first-, second-, and the third-order approximate solutions:

$$u_0(x) = 1 - x + 0.5x^2, \quad (48)$$

$$u_1(x,C_1) = 0.174167C_1x^3 - 0.0541667C_1x^4 + 0.0166667C_1x^5, \quad (49)$$

$$\begin{aligned} u_2(x,C_1,C_2) &= 0.174167C_1x^3 - 0.0541667C_1x^4 \\ &+ 0.0166667C_1x^5 + 0.173303C_1^2x^3 \\ &- 0.0519354C_1^2x^4 + 0.0134917C_1^2x^5 \\ &+ 0.00356944C_1^2x^6 - 0.000634921C_1^2x^7 \\ &+ 0.0000992063C_1^2x^8 + 0.166667C_2x^3 \\ &- 0.0416667C_2x^4 + 0.00833333C_2x^5, \end{aligned} \quad (50)$$

$$\begin{aligned} u_3(x,C_1,C_2,C_3) &= 0.174167C_1x^3 - 0.0541667C_1x^4 \\ &+ 0.0166667C_1x^5 + 0.346606C_1^2x^3 \\ &- 0.103871C_1^2x^4 + 0.0269833C_1^2x^5 \\ &+ 0.00713889C_1^2x^6 - 0.00126984C_1^2x^7 \\ &+ 0.000198413C_1^2x^8 + 0.172448C_1^3x^3 \\ &- 0.0497316C_1^3x^4 + 0.0103718C_1^3x^5 \\ &+ 0.0070607C_1^3x^6 - 0.00119984C_1^3x^7 \\ &+ 0.000156374C_1^3x^8 + 0.000017306C_1^3x^9 \\ &- 2.09436 \times 10^{-6}C_1^3x^{10} \\ &+ 2.00417 \times 10^{-7}C_1^3x^{11} + 0.166667C_2x^3 \\ &- 0.0416667C_2x^4 + 0.00833333C_2x^5 \\ &+ 0.33916C_1C_2x^3 - 0.0915255C_1C_2x^4 \\ &+ 0.0189125C_1C_2x^5 + 0.00682639C_1C_2x^6 \\ &- 0.00109127C_1C_2x^7 + 0.00014881C_1C_2x^8 \\ &+ 0.165917C_3x^3 - 0.0397917C_3x^4 \\ &+ 0.00583333C_3x^5 + 0.00138889C_3x^6. \end{aligned} \quad (51)$$

By adding (48)–(50) and (2), we obtain

$$\tilde{u}(x,C_1,C_2,C_3) = u_0(x) + u_1(x,C_1) + u_2(x,C_1,C_2) + u_3(x,C_1,C_2,C_3). \quad (52)$$

Following the procedure described in Section 2 regarding the domain between $a = 0$ and $b = 1$, we use the residual

$$R = \tilde{u}'''(x,C_1,C_2,C_3) + \tilde{u}(x,C_1,C_2,C_3) + \tilde{u}(x-0.3,C_1,C_2,C_3) - e^{-x+0.3}. \quad (53)$$

The following optimal values of C_i's are obtained:

$$C_1 = 0.2237074205, \qquad C_2 = -0.5559972626, \qquad C_3 = -0.4753337145. \quad (54)$$

By substituting values in (52), we have

$$\begin{aligned} \tilde{u}(x,C_1,C_2,C_3) &= 1 - x + 0.5x^2 - 0.161547x^3 + 0.0319249x^4 \\ &- 0.00106469x^5 - 0.000894311x^6 + 0.0000269763x^7 \\ &- 1.86401 \times 10^{-6}x^8 + 1.93748 \times 10^{-7}x^9 \\ &- 2.34472 \times 10^{-8}x^{10} + 2.24376 \times 10^{-9}x^{11}. \end{aligned} \quad (55)$$

The comparison between the approximate solution and the exact solution is shown in Figures 2(a) and 2(b). We observe that the results agree very well with the exact solution.

Example 3. Consider the first order of nonlinear delay differential equation [14]

$$u'(x) = -2u^2\left(\frac{x}{2}\right) + 1, \quad 0 \le x \le 1, \; u(0) = 0, \quad (56)$$

which has the exact solution

$$u(x) = \sin(x). \quad (57)$$

By applying the same method as in Examples 1 and 2, we have the following:

$$L[v(x,p)] = \frac{dv(x,p)}{dx},$$
$$N[v(x,p)] = \frac{dv(x,p)}{dx} + 2v^2\left(\frac{x}{2},p\right) - 1. \quad (58)$$

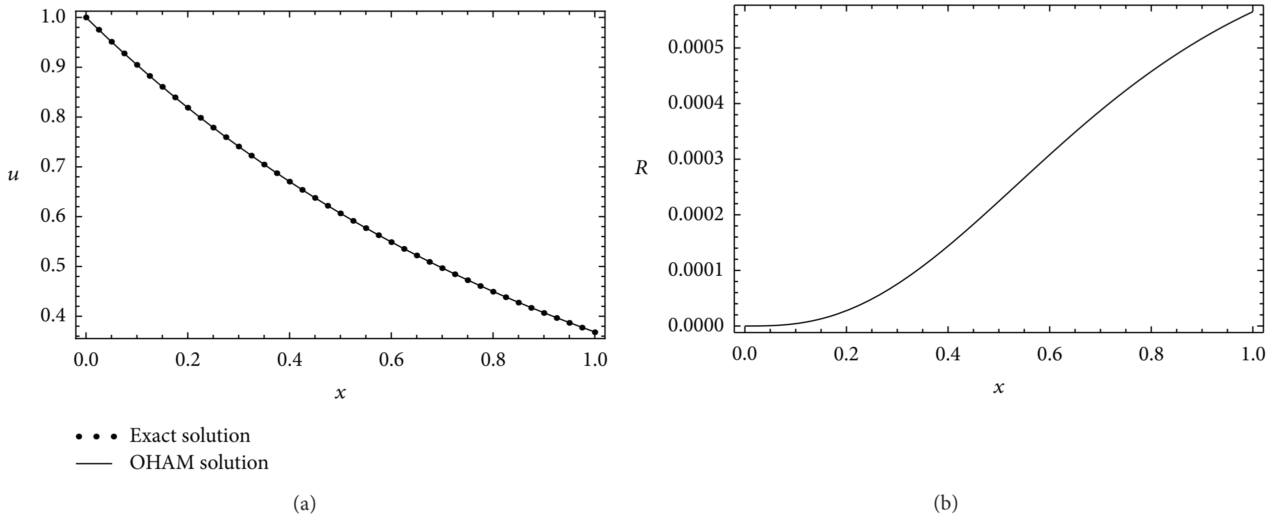


FIGURE 2: (a) Comparison between the three-term OHAM approximate solution and the exact solution for Example 2 and (b) residual error $R(x)$ given by (50) using three-term OHAM approximate solution.

According to OHAM, we have the following zero-, first-, second- and the third-order approximate solutions:

$$
\begin{aligned}
u_0(x) &= x, \\
u_1(x, C_1) &= \frac{1}{6}x^3, \\
u_2(x, C_1, C_2) &= \frac{1}{6}C_1x^3 + \frac{1}{6}C_1^2x^3 + \frac{1}{120}C_1^2x^5 + \frac{1}{6}C_2x^3, \\
u_3(x, C_1, C_2, C_3) &= \frac{1}{6}C_1x^3 + \frac{1}{3}C_1^2x^3 + \frac{1}{60}C_1^2x^5 + \frac{1}{6}C_1^3x^3 \\
&\quad + \frac{1}{60}C_1^3x^5 + \frac{1}{5040}C_1^3x^7 + \frac{1}{6}C_2x^3 + \frac{1}{3}C_1C_2x^3 \\
&\quad + \frac{1}{60}C_1C_2x^5 + \frac{1}{6}C_3x^3.
\end{aligned}
\tag{59}
$$

From (59), the third-order approximate solution by OHAM is given as

$$
\begin{aligned}
\tilde{u}(x, C_1, C_2, C_3) &= u_0(x) + u_1(x, C_1) \\
&\quad + u_2(x, C_1, C_2) + u_3(x, C_1, C_2, C_3).
\end{aligned}
\tag{60}
$$

By using (60) in (14) and applying the method as discussed in (15) and (16), we obtain the following values of C_i's:

$$
C_1 = -0.9892887781, \qquad C_2 = 0.0001159690, \qquad C_3 = 3.3939542758.
\tag{61}
$$

The approximate solution now becomes

$$
\begin{aligned}
\tilde{u}(x, C_1, C_2, C_3) &= x - 0.166665x^3 \\
&\quad + 0.00832857x^5 - 0.000192105x^7.
\end{aligned}
\tag{62}
$$

From Figures 3(a) and 3(b), we observe that the results agree very well with the exact solution; as we increase the order of the problem the accuracy increases and the residual error will decrease as shown in Figure 3(b). We observed that the absolute maximum error for solving this example via HAM is 1.2×10^{-6} while the absolute maximum error by using OHAM is 4×10^{-8}, which revealed that the proposed method is more accurate than HAM.

Example 4. Consider the third-order nonlinear delay differential equation [14]

$$
\begin{gathered}
u'''(x) = 2u^2\left(\frac{x}{2}\right) - 1, \\
0 \le x \le 1, \; u(0) = 0, \; u'(0) = 1, \; u''(0) = 0.
\end{gathered}
\tag{63}
$$

The exact solution of the above problem is given as

$$
u(x) = \sin(x).
\tag{64}
$$

By applying the present method, the linear and nonlinear operators are defined as

$$
\begin{aligned}
L[v(x, p)] &= \frac{d^3v(x, p)}{dx^3}, \\
N[v(x, p)] &= \frac{d^3v(x, p)}{dx^3} - 2v^2\left(\frac{x}{2}, p\right) + 1.
\end{aligned}
\tag{65}
$$

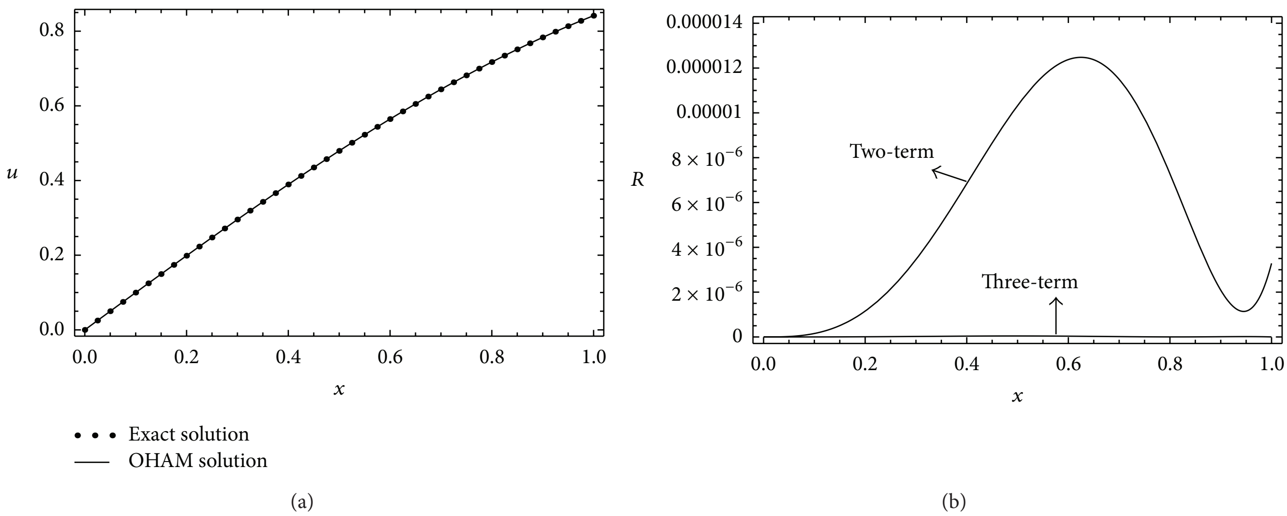


Figure 3: (a) Comparison between three-term OHAM approximate solution and the exact solution for Example 3 and (b) the residual error for Example 3 using two-term and three-term OHAM approximate solutions.

According to OHAM, we have the following zero-, first-, second- and third-order approximate solutions:

$$
\begin{aligned}
u_0(x) &= x - \frac{x^3}{6}, \\
u_1(x, C_1) &= -\frac{1}{120} C_1 x^5 + \frac{1}{5040} C_1 x^7 - \frac{1}{580608} C_1 x^9, \\
u_2(x, C_1, C_2) &= -\frac{1}{120} C_1 x^5 + \frac{1}{5040} C_1 x^7 - \frac{1}{580608} C_1 x^9 \\
&\quad - \frac{1}{120} C_1^2 x^5 + \frac{1}{5040} C_1^2 x^7 - \frac{1}{1451520} C_1^2 x^9 \\
&\quad - \frac{1}{39916800} C_1^2 x^{11} + \frac{101}{1275293859840} C_1^2 x^{13} \\
&\quad - \frac{1}{9738607656960} C_1^2 x^{15} - \frac{1}{120} C_2 x^5 \\
&\quad + \frac{1}{5040} C_2 x^7 - \frac{1}{580608} C_2 x^9, \\
u_3(x, C_1, C_2, C_3) &= -\frac{1}{120} C_1 x^5 + \frac{1}{5040} C_1 x^7 - \frac{1}{580608} C_1 x^9 \\
&\quad - \frac{1}{60} C_1^2 x^5 + \frac{1}{2520} C_1^2 x^7 - \frac{1}{725760} C_1^2 x^9 \\
&\quad - \frac{1}{19958400} C_1^2 x^{11} + \frac{101}{637646929920} C_1^2 x^{13} \\
&\quad - \frac{1}{4869303829480} C_1^2 x^{15} - \frac{1}{120} C_1^3 x^5 \\
&\quad + \frac{1}{5040} C_1^3 x^7 + \frac{1}{2903040} C_1^3 x^9 \\
&\quad - \frac{1}{19958400} C_1^3 x^{11} + \frac{491}{6376469299200} C_1^3 x^{13} \\
&\quad + \frac{1223}{2678117105664000} C_1^3 x^{15} \\
&\quad - \frac{341827}{149186120241276518400} C_1^3 x^{17} \\
&\quad + \frac{1}{5040} C_3 x^7 - \frac{1}{580608} C_3 x^9 \\
&\quad - \frac{263}{91675269781634049638400} C_1^3 x^{21} \\
&\quad - \frac{1}{120} C_2 x^5 + \frac{1}{5040} C_2 x^7 - \frac{1}{580608} C_2 x^9 \\
&\quad - \frac{1}{60} C_1 C_2 x^5 + \frac{1}{2520} C_1 C_2 x^7 \\
&\quad - \frac{1}{725760} C_1 C_2 x^9 - \frac{1}{19958400} C_1 C_2 x^{11} \\
&\quad + \frac{101}{637646929920} C_1 C_2 x^{13} - \frac{1}{120} C_3 x^5 \\
&\quad - \frac{1}{4869303828480} C_1 C_2 x^{15} \\
&\quad + \frac{38033}{102043306245033138585 60} C_1^3 x^{19}.
\end{aligned} \tag{66}
$$

From (66), the third-order approximate solution by OHAM is given as

$$
\begin{aligned}
\tilde{u}(x, C_1, C_2, C_3) &= u_0(x) + u_1(x, C_1) \\
&\quad + u_2(x, C_1, C_2) + u_3(x, C_1, C_2, C_3).
\end{aligned} \tag{67}
$$

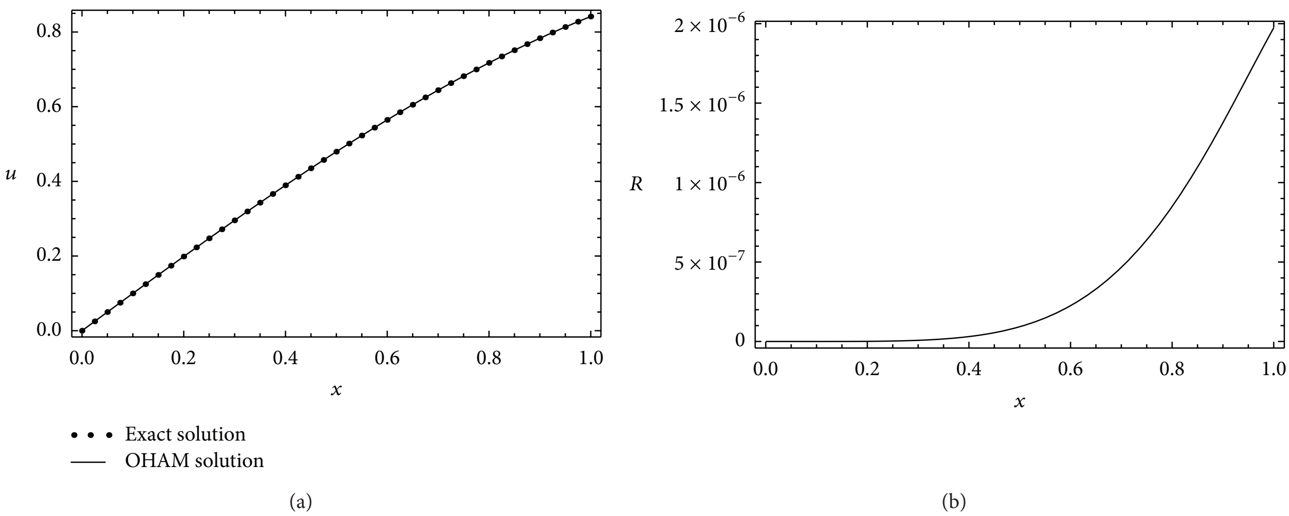


FIGURE 4: (a) Comparison between three-term OHAM approximate solution and the exact solution for Example 4 and (b) the residual error for Example 4 using three-term OHAM approximate solution.

By using (67) in (14) and applying the method as discussed in (15) and (16), we obtain the following values of C_i's:

$$C_1 = 2.7354549148, \qquad C_2 = -1.3589185618, \qquad C_3 = -1.0003665216. \tag{68}$$

The approximate solution now becomes

$$\begin{aligned}\tilde{u}(x, C_1, C_2, C_3) &= x - \frac{1}{6}x^3 + 0.00833639x^5 - 0.000198485x^7 \\ &\quad + 1.72296 \times 10^{-6}x^9 + 1.80627 \times 10^{-29}x^{11} \\ &\quad - 5.71017 \times 10^{-32}x^{13} + 7.40356 \times 10^{-35}x^{15} \\ &\quad - 4.68993 \times -50x^{17} + 7.62895 \times 10^{53}x^{19} \\ &\quad - 5.87208 \times 10^{-56}x^{21}.\end{aligned} \tag{69}$$

Numerical results of the solution are displayed in Figures 4(a) and 4(b).

Example 5. Consider the system of delay differential equation [14]

$$\begin{aligned} u_1'(x) &= u_1(x-1), \\ u_2'(x) &= u_1(x-1) + u_2(x-0.2), \\ u_3'(x) &= u_2(x-1), \end{aligned} \tag{70}$$

with initial conditions

$$u_1(0) = 1, \qquad u_2(0) = 1, \qquad u_3(0) = 1. \tag{71}$$

Following the same procedure, we have

$$\begin{aligned} L_i[v_i(x,p)] &= \frac{dv_i(x,p)}{dx}, \quad i = 1, 2, 3, \\ N_1[v_1(x,p)] &= \frac{dv_1(x,p)}{dx} - v_1(x-1,p), \\ N_2[v_2(x,p)] &= \frac{dv_2(x,p)}{dx} - v_1(x-1,p) - v_2(x-0.2,p), \\ N_3[v_3(x,p)] &= \frac{dv_3(x,p)}{dx} - v_2(x-1,p). \end{aligned} \tag{72}$$

According to OHAM formulation, we have the following:

zeroth-order solution:

$$\begin{aligned} u_{1,0}(x) &= 1, \\ u_{2,0}(x) &= 1, \\ u_{3,0}(x) &= 1, \end{aligned} \tag{73}$$

first-order solution:

$$\begin{aligned} u_{1,1}(x) &= -C_1 x, \\ u_{2,1}(x) &= -2K_1 x, \\ u_{3,1}(x) &= -A_1 x, \end{aligned} \tag{74}$$

second-order solution:

$$\begin{aligned} u_{1,2}(x) &= -C_1 x - 2C_1^2 x + \frac{1}{2}C_1^2 x^2 - C_2 x, \\ u_{2,2}(x) &= -2K_1 x - 3.4K_1^2 x + \frac{3}{2}K_1^2 x^2 - 2K_2 x, \\ u_{3,2}(x) &= -A_1 x - 3A_1^2 x + A_1^2 x^2 - A_2 x. \end{aligned} \tag{75}$$

Making use of (73)–(75) and extending the solutions up to a fifth order, the approximate solutions by OHAM for $p = 1$ are

$$\begin{aligned}
&\tilde{u}_1(x, C_1, C_2, C_3, C_4, C_5)\\
&\quad= u_{1,0}(x) + u_{1,1}(x, C_1) + u_{1,2}(x, C_1, C_2)\\
&\qquad+ u_{1,3}(x, C_1, C_2, C_3) + u_{1,4}(x, C_1, C_2, C_3, C_4)\\
&\qquad+ u_{1,5}(x, C_1, C_2, C_3, C_4, C_5),\\
&\tilde{u}_2(x, K_1, K_2, K_3, K_4, K_5)\\
&\quad= u_{2,0}(x) + u_{2,1}(x, K_1) + u_{2,2}(x, K_1, K_2)\\
&\qquad+ u_{2,3}(x, K_1, K_2, K_3) + u_{2,4}(x, K_1, K_2, K_3, K_4)\\
&\qquad+ u_{2,5}(x, K_1, K_2, K_3, K_4, K_5),\\
&\tilde{u}_3(x, A_1, A_2, A_3, A_4, A_5)\\
&\quad= u_{2,0}(x) + u_{2,1}(x, A_1) + u_{2,2}(x, A_1, A_2)\\
&\qquad+ u_{2,3}(x, A_1, A_2, A_3) + u_{2,4}(x, A_1, A_2, A_3, A_4)\\
&\qquad+ u_{2,5}(x, A_1, A_2, A_3, A_4, A_5).
\end{aligned} \tag{76}$$

By using the proposed procedure which is described in Section 2 on [0, 1], we use the residual error

$$\begin{aligned}
R_1 &= \widetilde{u_1}'(x, C_1, C_2, C_3, C_4, C_5)\\
&\quad- u_1(x-1, C_1, C_2, C_3, C_4, C_5),\\
R_2 &= \widetilde{u_2}'(x, K_1, K_2, K_3, K_4, K_5)\\
&\quad- u_1(x-1, C_1, C_2, C_3, C_4, C_5)\\
&\quad- u_2(x-0.2, K_1, K_2, K_3, K_4, K_5),\\
R_3 &= \widetilde{u_3}'(x, A_1, A_2, A_3, A_4, A_5)\\
&\quad- u_2(x-1, K_1, K_2, K_3, K_4, K_5).
\end{aligned} \tag{77}$$

The following values of C_i's, K_i's and A_i's are obtained:

$$\begin{gathered}
C_1 = -0.58668069, \qquad C_2 = -0.08423352,\\
C_3 = 0.00786513, \qquad C_4 = 0.00069702,\\
C_5 = -0.00034937,\\
K_1 = -0.75856373, \qquad K_2 = -0.14305522,\\
K_3 = 0.07154350, \qquad K_4 = -0.02323472,\\
K_5 = -0.00280683,\\
A_1 = -0.67739965, \qquad A_2 = -0.08965635,\\
A_3 = 0.03646196, \qquad A_4 = -0.01330410,\\
A_5 = 0.00453433.
\end{gathered} \tag{78}$$

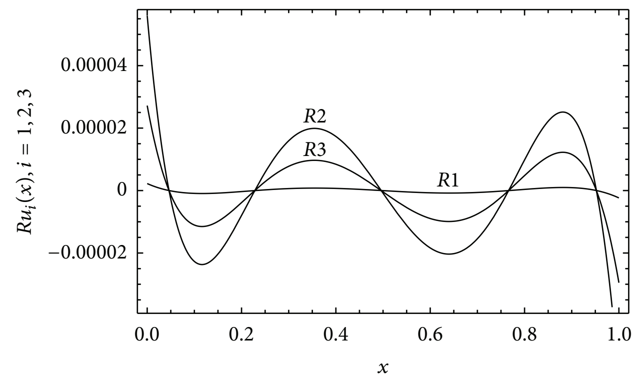


Figure 5: The residual errors for Example 5 using the five-term OHAM approximate solution.

By using the above values, the approximate solutions are

$$\begin{aligned}
&\tilde{u}_1(x, C_1, C_2, C_3, C_4, C_5)\\
&\quad= 1 + 0.567178x + 0.160857x^2 + 0.0302719x^3\\
&\qquad+ 0.00434813x^4 + 0.000579199x^5,\\
&\tilde{u}_2(x, K_1, K_2, K_3, K_4, K_5)\\
&\quad= 1 + 1.3299x + 0.714785x^2 + 0.205164x^3\\
&\qquad+ 0.0536786x^4 + 0.0125583x^5,\\
&\tilde{u}_2(x, A_1, A_2, A_3, A_4, A_5)\\
&\quad= 1 + 0.182376x + 0.230372x^2 + 0.0785302x^3\\
&\qquad+ 0.0228859x^4 + 0.00594311x^5.
\end{aligned} \tag{79}$$

From Figure 5, we can observe the accuracy of the solution obtained by the five-term approximate solution using OHAM which is quite good.

5. Conclusions

In this work, OHAM is employed for the first time to propose a new analytic approximate solution of delay differential equations (DDEs). This method has been tested in various examples of linear and nonlinear and system of initial value problems of DDEs and was seen to yield satisfactory results. The OHAM provides us with a simple way to optimally control and adjust the convergence solution series and it gives a good approximation in few terms which is converged to the exact solution and proved the efficiency and reliability of the method. This fact is obvious from the use of the auxiliary function $H(p)$. In OHAM, it is important to solve a set of nonlinear algebraic equations with m unknown convergence control parameters, $C_1, C_2, \ldots, C_m$, and this makes it time consuming, especially for large m.

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