Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 346824, 15 pages
doi:10.1155/2012/346824

Research Article

A Class of PDEs with Nonlinear Superposition Principles

Li Peng,[1,2] Liu Keying,[1] Pan Zuliang,[3] and Zhong Weizhou[2]

[1] *Department of Mathematics, North China University of Water Conservancy and Electric Power, Zhengzhou 450011, China*
[2] *School of Economics and Finance, Xi'an Jiaotong University, Xi'an 710061, China*
[3] *Department of Mathematics, Zhejiang University, Hangzhou 310027, China*

Correspondence should be addressed to Zhong Weizhou, weizhou@mail.xjtu.edu.cn

Received 29 November 2011; Revised 31 January 2012; Accepted 31 January 2012

Academic Editor: B. V. Rathish Kumar

Through assuming that nonlinear superposition principles (NLSPs) are embedded in a Lie group, a class of 3rd-order PDEs is derived from a general determining equation that determine the invariant group. The corresponding NLSPs and transformation to linearize the nonlinear PDE are found, hence the governing PDE is proved C-integrable. In the end, some applications of the PDEs are explained, which shows that the result has very subtle relations with linearization of partial differential equation.

1. Introduction

Construction of new solutions by superposition of known ones is a familiar tool in nonlinear partial differential equations. The idea of superpositions for nonlinear differential operators originated in 1893 by Vessiot [1]. It was immediately generalized by Guldberg [2]. Marius Sophus Lee pointed out that these are special cases of his own theory of "Fundamental Solutions of Differential Equations" [3, 4]. The problem was reconsidered in 1960 by Temple [5] who showed the existence of nonlinear d.e. and the general solution cannot be obtained as a finite number of particular solutions. In 1965, Inselberg showed that there exist classes of equations, involving nonlinear operators, where it is possible to "compose" two solutions to obtain a different solution [6]. This is different than the above works where the general solution is sought. In 2000, Ibragimov gives specific examples of the Vessiot-Guldberg-Lie algebra applied to some partial differential equations [7]. Jones and Ames introduce the definition "connecting function" to describe nonlinear superposition in [8].

Generally, these methods for finding NLSPs can be classified into two categories: one is based on ad hoc methods [9–12]; the other is Lie's classical symmetry algorithm [7, 13–15].

In contrast with the ad hoc methods, Lie's classical symmetry algorithm is systematic and automatic.

Following the definition of nonlinear superposition introduced by in [8, 15], formally, the simplest form of a superposition principle is an operation

$$F(u,v) = u * v, \quad F : V * V \longrightarrow W, \tag{1.1}$$

(where V and W are function spaces) which preserves some governing equations, so that if

$$\begin{aligned} f\left(x, u, u_{x_i}, u_{x_i x_j}, \ldots\right) &= 0, \\ f\left(x, v, v_{x_i}, v_{x_i x_j}, \ldots\right) &= 0 \end{aligned} \tag{1.2}$$

then

$$f\left(x, w, w_{x_i}, w_{x_i x_j}, \ldots\right) = 0, \tag{1.3}$$

where $w = u * v$. Of course, it is a simple matter to extend this concept to any n-ary operation that constructs new solutions from n old solutions.

Goard and Broadbridge have investigated how to use Lie symmetry algebras to find general superposition principles for nonlinear PDEs [15] and they have given the NLSPs of some first-order and second-order PDEs in two independent variables and linearized the PDEs. However, for higher-order nonlinear PDEs, its linearization and NLSPs have not been fully investigated. It is the purpose of the paper to demonstrate how to find the superposition principles for a class higher-order nonlinear PDEs.

We assume that for any pair of solutions $v(x)$ and $z(x)$ to the governing PDE, there exists a parameter ϵ so that we have a one-parameter solution

$$\overline{u} = F(v, z, \epsilon). \tag{1.4}$$

In further, suppose that the NLSP is surjective in the sense that, for any two solutions $u(x)$ and $z(x)$, there exists a solution $v(x)$ such that

$$u = F(v, z, 0). \tag{1.5}$$

From (1.4) then

$$\begin{aligned} \overline{u} &= u + \epsilon \frac{\partial F}{\partial \epsilon}(v, z, \epsilon)\bigg|_{\epsilon=0} + O\left(\epsilon^2\right) \\ &= u + \epsilon \overline{U}(v, z) + O\left(\epsilon^2\right). \end{aligned} \tag{1.6}$$

From (1.5), if $F_v \neq 0$, the implicit function theorem allows us to regard v as a function of u and z. Hence we write

$$\overline{U} = h(z, u) \equiv h(z(x, y), u), \tag{1.7}$$

which we find by solving the classical linear determining equations of the PDE. Then solving

$$\frac{d\overline{u}}{d\epsilon} = \overline{U}\left(z(x,y),\overline{u}\right)\big|_{\overline{u}=u,\,\epsilon=0}, \tag{1.8}$$

we find the NLSP for the PDE.

Goard and others have shown in their work that for first- and second-order PDEs of two independent variables, this will actually imply that the PDE will have a generator of the form

$$\overline{A}(x,y)B(u)\frac{\partial}{\partial u}, \quad \text{or} \quad \overline{A}(z)B(u)\frac{\partial}{\partial u}, \tag{1.9}$$

where $\overline{A}(x,y)$ is the general solution of a linear homogeneous PDE. Hence the PDE satisfies the necessary condition for the existence of a transformation to a linear PDE.

In the paper, we will start from a general third-order PDE and find the third-order PDE class with NLSPs. Furthermore, a discussion for higher-order PDEs is given.

2. A Class of Third-Order PDEs with NLSPs

Firstly, consider a general third-order determining equation:

$$\begin{aligned}
&\alpha_1(x,y,u)U_{uuu} + \alpha_2(x,y,u)U_{uux} + \alpha_3(x,y,u)U_{uxx} + \alpha_4(x,y,u)U_{xxx} + \alpha_5(x,y,u)U_{uu},\\
&\alpha_6(x,y,u)U_{ux} + \alpha_7(x,y,u)U_{xx} + \alpha_8(x,y,u)U_u + \alpha_9(x,y,u)U_x + \alpha_{10}(x,y,u)U_y\\
&\quad + \alpha_{11}(x,y,u)U_{uy} + \alpha_{12}(x,y,u)U_{yy} + \alpha_{13}(x,y,u)U_{xy} + \alpha_{14}(x,y,u)U_{uuy}\\
&\quad + \alpha_{15}(x,y,u)U_{uyy} + \alpha_{16}(x,y,u)U_{yyy} + \alpha_{17}(x,y,u)U_{uxy} + \alpha_{18}(x,y,u)U_{xyy}\\
&\quad + \alpha_{19}(x,y,u)U_{xxy} = \alpha_{20}(x,y,u)U.
\end{aligned} \tag{2.1}$$

Substitution of $U = h(z(x,y),u)$ into (2.1) gives

$$\begin{aligned}
&h_{uuu}\left[\alpha_1(x,y,u)\right] + h_{uuz}\left[\alpha_2(x,y,u)z_x + \alpha_{14}(x,y,u)z_y\right]\\
&\quad + h_{uzz}\left[\alpha_3(x,y,u)z_x^2 + \alpha_{15}(x,y,u)z_y^2 + \alpha_{17}(x,y,u)z_xz_y\right]\\
&\quad + h_{uu}\left[\alpha_{11}(x,y,u)\right] + h_{uz}\left[\alpha_6(x,y,u)z_x + \alpha_{11}(x,y,u)z_y\right.\\
&\qquad\qquad \left. +\alpha_3(x,y,u)z_{xx} + \alpha_{17}(x,y,u)z_{xy} + \alpha_{15}(x,y,u)z_{yy}\right]
\end{aligned}$$

$$
\begin{aligned}
&+ h_{zz}\Big[\alpha_7(x,y,u)z_x^2 + \alpha_{12}(x,y,u)z_y^2 + \alpha_{13}(x,y,u)z_x z_y \\
&\quad + 3\alpha_4(x,y,u)z_x z_{xx} + 3\alpha_{16}(x,y,u)z_y z_{yy} + \alpha_{18}(x,y,u)z_x z_{yy} \\
&\quad + 2\alpha_{18}(x,y,u)z_y z_{xy} + 2\alpha_{19}(x,y,u)z_x z_{xy} + \alpha_{19}(x,y,u)z_y z_{xx}\Big] \\
&+ h_z\big[\alpha_9(x,y,u)z_x + \alpha_{10}(x,y,u)z_y + \alpha_7(x,y,u)z_{xx}\alpha_{12}(x,y,u)z_{yy} \\
&\quad + \alpha_{13}(x,y,u)z_{xy} + \alpha_{16}(x,y,u)z_{yyy}\alpha_4(x,y,u)z_{xxx} \\
&\quad + \alpha_{18}(x,y,u)z_{xyy} + \alpha_{19}(x,y,u)z_{xxy}\big] = \alpha_{20}(x,y,u)h.
\end{aligned}
\tag{2.2}
$$

To obtain a solution of (2.2) where h depends on z, u alone, the coefficients of (2.2) need to be in z and u alone. Hence we consider the possible way in which (2.2) can be written as an equation in z and u. It is shown that the determining equation admits separable solutions

$$h(z,u) = A(z)B(u), \tag{2.3}$$

or, more generally,

$$\overline{U}(x,y,u) = \overline{A}(x,y)B(u). \tag{2.4}$$

In the case, at least one of $\alpha_4(x,y,u)$, $\alpha_{16}(x,y,u)$, $\alpha_{18}(x,y,u)$, $\alpha_{19}(x,y,u)$ is not zero. (We note that at least one of these coefficients must be nonzero for a third-order governing PDE.) In addition, we suppose

$$
\begin{gathered}
\alpha_1(x,y,u) = \overline{\alpha}(x,y)\gamma_1(u), \qquad \alpha_2(x,y,u) = 0, \qquad \alpha_3(x,y,u) = 0, \\
\alpha_4(x,y,u) = c_0(x,y)\alpha(x,y)\beta(u), \qquad \alpha_5(x,y,u) = \overline{\alpha}(x,y)\gamma_5(u), \qquad \alpha_6(x,y,u) = 0, \\
\alpha_7(x,y,u) = c_1(x,y)\alpha(x,y)\beta(u), \qquad \alpha_8(x,y,u) = \overline{\alpha}(x,y)\gamma_8(u), \\
\alpha_9(x,y,u) = c_2(x,y)\alpha(x,y)\beta(u), \qquad \alpha_{10}(x,y,u) = c_3(x,y)\alpha(x,y)\beta(u), \\
\alpha_{11}(x,y,u) = 0, \qquad \alpha_{12}(x,y,u) = c_4(x,y)\alpha(x,y)\beta(u), \\
\alpha_{13}(x,y,u) = c_5(x,y)\alpha(x,y)\beta(u), \qquad \alpha_{14}(x,y,u) = 0\ \alpha_{15}(x,y,u) = 0, \\
\alpha_{16}(x,y,u) = c_6(x,y)\alpha(x,y)\beta(u), \qquad \alpha_{17}(x,y,u) = 0, \\
\alpha_{18}(x,y,u) = c_7(x,y)\alpha(x,y)\beta(u), \qquad \alpha_{19}(x,y,u) = c_8(x,y)\alpha(x,y)\beta(u), \\
\alpha_{20}(x,y,u) = \overline{\alpha}(x,y)\gamma_{20}(u)
\end{gathered}
\tag{2.5}
$$

with $\beta(u) \neq 0$, then from (2.2) we let

$$h_{zz} = k_1(z)h_z, \qquad h_{zzz} = k_2(z)h_z \tag{2.6}$$

for some two functions k_1, k_2 of z, and (2.2) becomes

$$\begin{aligned}
h_z\alpha(x,y)\beta(u)\Big[&c_0(x,y)k_2(z)z_x^3+3c_0(x,y)k_1(z)z_xz_{xx}+c_0(x,y)z_{xxx}\\
&+c_1(x,y)k_1(z)z_x^2+c_1(x,y)z_{xx}+c_2(x,y)x_x+c_3(x,y)z_y\\
&+c_4(x,y)k_1(z)z_x^2+c_4(x,y)z_{xx}+c_5(x,y)k_1(z)z_x^2+c_5(x,y)z_{xx}\\
&+c_6(x,y)k_2(z)z_y^3+3c_6(x,y)k_1(z)z_yz_y+c_6(x,y)z_{yyy}\\
&+c_7(x,y)k_2(z)z_xz_y^2+c_7(x,y)k_1(z)(z_xz_{yy}+2z_yz_{xy})\\
&+c_7(x,y)z_{xyy}c_8(x,y)k_2(z)z_x^2z_y+c_8(x,y)k_1(z)(z_yz_{xx}+2z_xz_{xy})+c_8z_{xxy}\Big]\\
+\overline{\alpha}(x,y)\gamma_8(u)h_u&+\overline{\alpha}(x,y)\gamma_5h_{uu}+\overline{\alpha}(x,y)\gamma_1(u)h_{uuu}=\overline{\alpha}(x,y)\gamma_{20}(u)h.
\end{aligned} \tag{2.7}$$

Then if we let

$$\begin{aligned}
&c_0(x,y)k_2(z)z_x^3+3c_0(x,y)k_1(z)z_xz_{xx}+c_0(x,y)z_{xxx}+c_1(x,y)k_1z_x^2\\
&\quad+c_1(x,y)z_{xx}+c_2(x,y)z_x+c_3(x,y)z_y+c_4(x,y)k_1(z)z_x^2+c_4(x,y)z_{xx}\\
&\quad+c_5(x,y)k_1(z)z_x^2+c_5(x,y)z_{xy}+c_6(x,y)k_2(z)z_y^3+3c_6(x,y)k_1(z)z_yz_{yy}\\
&\quad+c_6(x,y)z_{yyy}+c_7(x,y)k_2(z)z_xz_y^2+c_7(x,y)k_1(z)(z_xz_{yy}+2z_yz_{xy})\\
&\quad+c_7(x,y)z_{xyy}+c_8(x,y)k_2(z)z_x^2z_y+c_8(x,y)k_1(z)(z_yz_{xx}+2z_xz_{xy})\\
&\quad+c_8(x,y)z_{xxy}=g(z),
\end{aligned} \tag{2.8}$$

(2.2) becomes

$$\alpha(x,y)h_zg(z)+\overline{\alpha}(x,y)f_1(u)h_u+\overline{\alpha}(x,y)f_2(u)h_{uu}+\overline{\alpha}(x,y)f_3(u)h_{uuu}=\overline{\alpha}(x,y)f_4(u)h, \tag{2.9}$$

where

$$f_1(u)=\frac{\gamma_8(u)}{\beta(u)},\qquad f_2(u)=\frac{\gamma_5(u)}{\beta(u)},\qquad f_3(u)=\frac{\gamma_1(u)}{\beta(u)},\qquad f_4(u)=\frac{\gamma_{20}(u)}{\beta(u)}, \tag{2.10}$$

and notice that, from (2.6),

$$h(z,u)=A(z)B(u)+R(u), \tag{2.11}$$

where $A(z)=\int\exp(\int k(z)dz)dz$.

We consider the following possibilities for (2.7).

(i) $g(z) = 0$, $\overline{\alpha}(x, y) \neq 0$, not all $f_i(u) = 0$, $i = 1, 2, 3, 4$. From (2.9) we have

$$f_1(u)h_u + f_2(u)h_{uu} + f_3(u)h_{uuu} = f_4(u)h. \tag{2.12}$$

Equation (2.12) admits the separable solution (2.3), where $B(u)$ satisfies

$$f_1(u)B'(u) + f_2(u)B''(u) + f_3(u)B'''(u) = f_4(u)B(u). \tag{2.13}$$

The linear PDE which $\overline{A}(x, y)$ satisfies is

$$\begin{aligned} &c_0(x, y)\overline{A}_{xxx} + c_1(x, y)\overline{A}_x + c_2(x, y)\overline{A}_y + c_4(x, y)\overline{A}_{xx} + c_5(x, y)\overline{A}_{xx} \\ &\quad + c_6(x, y)\overline{A}_{yyy} + c_7(x, y)\overline{A}_{xyy} + c_8\overline{A}_{xxy} = 0. \end{aligned} \tag{2.14}$$

(ii) $g(z) \neq 0$, $\alpha(x, y) \neq 0$, $f_i(u)$, $i = 1, 2, 3, 4$. We then require $\alpha(x, y) = \overline{\alpha}(x, y)$ in (2.9).

Equation (2.9) admits separable solutions where

$$A'(z)g(z) = \lambda A(z) \tag{2.15}$$

and $B(u)$ satisfies

$$f_4(u)B(u) - f_1(u)B'(u) - f_2(u)B''(u) - f_3(u)B'''(u) = \lambda B(u), \tag{2.16}$$

for some nonzero constant λ.

From (2.15), we have

$$k(z) = \frac{\lambda - g'(z)}{g(z)} \tag{2.17}$$

$\overline{A}(x, y)$ satisfies

$$\begin{aligned} &c_0(x, y)\overline{A}_{xxx} + c_1(x, y)\overline{A}_x c_2(x, y)\overline{A}_y + c_4(x, y)\overline{A}_{xx} + c_5(x, y)\overline{A}_{xx} \\ &\quad + c_6(x, y)\overline{A}_{yyy} + c_7(x, y)\overline{A}_{xyy} + c_8\overline{A}_{xxy} = \lambda\overline{A}. \end{aligned} \tag{2.18}$$

So, we obtain (2.8) which has symmetries with generator

$$\overline{A}(x, y)B(u)\frac{\partial}{\partial u}. \tag{2.19}$$

The PDEs are linearizable with

$$\varphi = \int \frac{1}{B(u)} du \tag{2.20}$$

(by [14, Theorem 6.4, 2-2]). Equation (2.8) represents a class of third-order PDEs with various choices of $c_i(x,y)$ $(i = 0,1,\ldots,8)$, $k(u)$ and $F(u)$. A class of general third-order PDEs in two independent variables with NLSPs of (1.1) embedded in a Lie group are identified in (2.8).

3. Symmetry Analysis of (2.8)

We rewrite (2.8) in the following formulation:

$$\begin{aligned}
&c_0(x,y)k_2(u)u_x^3 + 3c_0(x,y)k_1(u)u_xu_{xx} + c_0(x,y)u_{xxx} + c_1(x,y)k_1(u)u_x^2 \\
&\quad + c_1(x,y)u_{xx} + c_2(x,y)u_x + c_3(x,y)u_y + c_4(x,y)k_1(u)u_y^2 + c_4(x,y)u_{yy} \\
&\quad + c_5(x,y)k_1(u)u_xu_y + c_5(x,y)u_{xy} + c_6(x,y)k_2(u)u_y^3 + 3c_6(x,y)k_1(u)u_yu_{yy} \\
&\quad + c_6(x,y)u_{yyy} + c_7(x,y)k_2(u)u_xu_y^2 + c_7(x,y)k_1(u)(u_xu_{yy} + 2u_yu_{xy}) \\
&\quad + c_7(x,y)u_{xyy} + c_8(x,y)k_2(u)u_x^2u_y + c_8(x,y)k_1(u)(u_yu_{xx} + 2u_xu_{xy}) + c_8u_{xxy} = F(u),
\end{aligned} \tag{3.1}$$

where for $F \neq 0$, $k(u) = (\lambda - F'(u))/F(u)$, λ is constant, and for $F = 0$, $k(u)$ is arbitrary.

Equation (3.1) has classical symmetry with generator

$$\Gamma = U(x,y,u)\frac{\partial}{\partial u} \tag{3.2}$$

with its third-order prolongation

$$\begin{aligned}
\Gamma^* = {}& U\frac{\partial}{\partial u} + D_xU\frac{\partial}{\partial u_x} + D_yU\frac{\partial}{\partial u_y} + D_{xx}U\frac{\partial}{\partial u_{xx}} + D_{xy}U\frac{\partial}{\partial u_{xy}} + D_{yy}U\frac{\partial}{\partial u_{yy}} \\
& + D_{xxx}U\frac{\partial}{\partial u_{xxx}} + D_{xxy}U\frac{\partial}{\partial u_{xxy}} + D_{xyy}U\frac{\partial}{\partial u_{xyy}} + D_{yyy}U\frac{\partial}{\partial u_{yyy}},
\end{aligned} \tag{3.3}$$

where U satisfies

$$\begin{aligned}
&c_0(x,y)U_{xxx} + c_1(x,y)U_{xx} + c_2(x,y)U_x + c_3(x,y)U_y + c_4(x,y)U_{yy} + c_5(x,y)U_{xy} \\
&\quad + c_6(x,y)U_{yyy} + c_7(x,y)U_{xyy} + c_8(x,y)U_{xxy} + U_u g(u) \\
&\quad + \left[U_{uuu} + 3k_1(u)U_{uu} + k'_2(u)U + 2k_2(u)U_u\right] \\
&\quad * \left[c_0(x,y)u_x^3 + c_8(x,y)u_x^2u_y + c_7(x,y)u_xu_y^2 + c_6(x,y)u_y^3\right] + \left[U_{uu} + k'_1(u)U + k_1(u)U_u\right] \\
&\quad * \left[3c_0(x,y)u_xu_{xx} + c_8(x,y)(2u_xu_{xy} + u_xu_{xx}) + c_7(x,y)(2u_yu_{xy} + u_xu_{yy}) \right. \\
&\qquad \left. +3c_6(x,y)u_yu_{yy}\right] \\
&\quad + \left[U_{uux} + 2k_1(u)U_{ux} + k_2(u)U_x\right] * \left[3c_0(x,y)u_x^2 + 2c_8(x,y)u_xu_y + c_7(x,y)u_y^2\right] \\
&\quad + \left[U_{uuy} + 2k_1(u)U_{uy} + k_2(u)U_y\right] * \left[c_8(x,y)u_x^2 + 2c_7(x,y)u_xu_y + 3c_6(x,y)u_y^2\right] \\
&\quad + \left[U_{uu} + k'_1(u)U + k_1(u)U_u\right] * \left[c_1(x,y)u_x^2 + c_4(x,y)u_xu_y + c_5(x,y)u_y^2\right] \\
&\quad + \left[U_{uxx} + k_1(u)U_{xx}\right] * \left[3c_0(x,y)u_x + c_8(x,y)u_y\right] + \left[U_{uxy} + k_1(u)U_{xy}\right] \\
&\quad * \left[2c_8(x,y)u_x + 2c_7(x,y)u_y\right] + \left[U_{uyy} + k_1(u)U_{yy}\right] * \left[c_7(x,y)u_x + 3c_6(x,y)u_y\right] \\
&\quad + \left[U_{ux} + k_1(u)U_x\right] * \left[3c_0(x,y)u_{xx} + 2c_8(x,y)u_{xy} + c_7(x,y)u_{yy} + 2c_1(x,y)u_x + c_5(x,y)u_y\right] \\
&\quad + \left[U_{uy} + k_1(u)U_y\right] * \left[c_8(x,y)u_{xx} + 2c_7(x,y)u_{xy} + 3c_6(x,y)u_{yy} \right. \\
&\qquad\qquad \left. + c_5(x,y)u_x + 2c_4(x,y)u_y\right] = F'(u)U.
\end{aligned} \tag{3.4}$$

We require

$$\begin{gathered}
U_{uuu} + 3k_1(u)U_{uu} + k'_2(u)U + 2k_2(u)U_u = 0, \\
U_{uu} + k'_1(u)U + k_1(u)U_u = 0, \\
U_{uux} + 2k_1(u)U_{ux} + k_2(u)U_x = 0, \\
U_{uuy} + 2k_1(u)U_{uy} + k_2(u)U_y = 0, \\
U_{uxx} + k_1(u)U_{xx} = 0, \\
U_{uxy} + k_1(u)U_{xy} = 0, \\
U_{uyy} + k_1(u)U_{yy} = 0, \\
U_{ux} + k_1(u)U_x = 0, \\
U_{uy} + k_1(u)U_y = 0,
\end{gathered} \tag{3.5}$$

where $k_1(u) = k(u),\ k_2(u) = k^2(u) + k'(u)$. Let

$$U_u + k(u)U = 0, \tag{3.6}$$

then,

$$\begin{aligned}
&U_{uuu} + 3k_1(u)U_{uu} + k'_2(u)U + 2k_2(u)U_u \\
&\quad = U_{uuu} + 3k(u)U_{uu} + \left(2k(u)k'(u) + k''(u)\right)U + 2\left(k^2(u) + k'(u)\right)U_u \\
&\quad = U_{uuu} + k''(u) + 2k'(u)U_u + k(u)U_{uu} + 2k(u)\left(U_{uu} + k'(u)U + k(u)U_u\right) \\
&\quad = U_{uu}(U_u + k(u)U) + 2k(u)D_u(U_u + k(u)U) = 0.
\end{aligned} \tag{3.7}$$

It is easy to verify other equations if (3.6) is satisfied. Then, (3.4) degenerates into

$$\begin{aligned}
&U_u + k(u)U = 0, \\
&c_0(x,y)U_{xxx} + c_1(x,y)U_{xx} + c_2(x,y)U_x + c_3(x,y)U_y + c_4(x,y)U_{yy} + c_5(x,y)U_{xyy} \\
&\quad + c_6(x,y)U_{yyy} + c_7(x,y)U_{xyy} + c_8(x,y)U_{xxy} = UF'(u) - U_uF(u).
\end{aligned} \tag{3.8}$$

Letting $\overline{U} = h(z(x,y),u)$ and substituting $\overline{U}$ into (3.8) give

$$h = A(z)\exp\left(-\int k(u)du\right), \tag{3.9}$$

where $A(z)$ needs to satisfy

$$\begin{aligned}
&c_0(x,y)\left[A_{zzz}z_x^3 + 3A_{zz}z_xz_{xx} + A_zz_xxx\right] + c_1(x,y)\left[A_{zz}z_x^2 + A_zz_{xx}\right] + c_2(x,y)[A_zz_x] \\
&\quad + c_3(x,y)[A_zz_x] + c_4(x,y)\left[A_{zz}z_y^2 + A_zz_{yy}\right] + c_5(x,y)[A_{zz}z_xz_y + A_zz_{xy}] \\
&\quad + c_6(x,y)\left[A_{zzz}z_y^3 + 3A_{zz}z_yz_{yy} + A_zz_{yyy}\right] \\
&\quad + c_7(x,y)\left[A_{zzz}z_xz_y^2 + A_{zz}(z_xz_{yy} + 2z_yz_{xy}) + A_zz_{xyy}\right] \\
&\quad + c_8(x,y)\left[A_{zzz}z_{xx}z_y + A_{zz}(z_yz_{xx} + 2z_xz_{xy}) + A_zz_{xxy}\right] = \lambda A,
\end{aligned} \tag{3.10}$$

where $\lambda = 0$ if $F = 0$. Hence we choose $A(z) = \int \exp(\int k(z)dz)dz$ as in (2.11), and we can find that $A(z)$ make (3.10) satisfied by substituting it into (3.10).

Then solving (1.8), that is,

$$\frac{d\overline{u}}{d\epsilon} = \int \exp\left(\int k(z)dz\right)dz\exp\left(-\int k(\overline{u})d\overline{u}\right)\Big|_{\overline{u}=u,\epsilon=0}, \tag{3.11}$$

gives the NLSP for (3.4) as

$$\overline{u} = P^{-1}[\epsilon P(z) + P(u)], \tag{3.12}$$

where

$$P(u) = \int \exp\left(\int k(u)du\right)du. \tag{3.13}$$

If we let

$$\overline{U} = h(x, y, u) = \overline{A}(x, y)B(u) \tag{3.14}$$

and substitute it into (3.8), then

$$B(u) = \exp\left(-\int k(u)du\right), \tag{3.15}$$

and $\overline{A}$ satifies

$$\begin{aligned} &c_0(x,y)\overline{A}_{xxx} + c_1(x,y)\overline{A}_x c_2(x,y)\overline{A}_y + c_4(x,y)\overline{A}_{xx} + c_5(x,y)\overline{A}_{xx} \\ &\quad + c_6(x,y)\overline{A}_{yyy} + c_7(x,y)\overline{A}_{xyy} + c_8\overline{A}_{xxy} = \lambda\overline{A}, \end{aligned} \tag{3.16}$$

where again $\lambda = 0$ when $F = 0$. By solving

$$\frac{d\overline{u}}{d\epsilon} = \overline{A}(x,y)\, B(\overline{u})|_{\overline{u}=u,\, \epsilon=0}, \tag{3.17}$$

we obtain

$$\overline{u} = P^{-1}\left[\epsilon\overline{A}(x,y) + P(u)\right], \tag{3.18}$$

where P is given in (3.13). By using (2.20) and (3.15), the transformation to linearize (3.1) is obtained as

$$v = P(u) = \int \exp\left(\int k(u)du\right)du, \tag{3.19}$$

which transforms (3.1) into the linear equation

$$\begin{aligned} &c_0(x,y)v_{xxx} + c_1(x,y)v_{xx} + c_2(x,y)v_x + c_3(x,y)v_y + c_4(x,y)v_{yy} \\ &\quad + c_5(x,y)u_{xy} + c_6(x,y)v_{yyy} + c_7(x,y)v_{xyy} + c_8 v_{xxy} = \lambda v + \alpha \end{aligned} \tag{3.20}$$

for some constant α and where $\lambda = 0$ if $F = 0$. It is indicated that the governing PDE (3.1) is C-integrable [16].

It is noticed that there is no explicit general solution for the linearized equation except few special cases. Thus, it is generally hard to obtain explicit general solution of the original nonlinear PDE, although the NLSP and the transformation to linearise the system are explicit. Equation (3.1) can also be used to determine whether some equations are C-integrable. For example, the kdv equation is only S-integrable [16], but not C-integrable, since it is not included in the class. Many numerical methods (such as Finite Difference Method, Finite Element Method) can be used to solve the linearized equation, and the solution of (3.1) can be obtained through the corresponding variable transformation.

4. Some Explanations of the Result

By explicitly constructing the full class of PDEs with a Lie group of NLSPs, we arrive at a single expression (3.1) for a class of third-order variable-coefficient equations.

4.1. Relations with ODE

First of all, we should point out that when letting $c_0(x,y) = c_6(x,y) = c_7(x,y) = c_8(x,y) = 0$, (3.1) is second-order PDE, which is (2.44) in [15]

$$\begin{aligned} &c_1(x,y)k_1(u)u_x^2 + c_1(x,y)u_{xx} + c_2(x,y)u_x + c_3(x,y)u_y \\ &\quad + c_4(x,y)k_1(u)u_y^2 + c_4(x,y)u_{yy} + c_5(x,y)k_1(u)u_xu_y + c_5(x,y)u_{xy} = F(u). \end{aligned} \tag{4.1}$$

Second, letting $c_0(x,y) = c_3(x,y) = c_4(x,y) = c_5(x,y) = c_6(x,y) = c_7(x,y) = c_9(x,y) = 0$, $c_1(x,y)$, $c_2(x,y)$ be only related to variable x, an ODE is obtained

$$c_1(x)k_1(u)u_x^2 + c_1(x)u_{xx} + c_2(x)u_x = F(u) \tag{4.2}$$

which is a special case of the *Painlevé* equation [17]

$$y'' + P(x,y)y'^2 + Q(x,y)y' + R(x,y) = 0. \tag{4.3}$$

The special situation of *Painlevé* equation has the NLSPs and can be transformed into linear equation.

Furthermore, we can get a class of third-order ODEs that have NLSPs

$$\begin{aligned} &c_0(x)k_2(u)u_x^3 + 3c_0(x)k_1(u)u_xu_{xx} + c_0(x)u_{xxx} \\ &\quad + c_1(x)k_1(u)u_x^2 + c_1(x)u_{xx} + c_2(x)u_x = F(u). \end{aligned} \tag{4.4}$$

4.2. Relations with PDEs Whose Potential Equations Are in the Form (3.1)

Letting $F(u) = 0$, $k_1(u) = k_2(u) = 1$, $c_i(x,y) = 0$, $i = 4,5,6,7,8$, and differentiate it with variable x, we get

$$\frac{\partial}{\partial x}\Big\{c_0(x,y)u_x^3 + 3c_0(x,y)u_x u_{xx} + c_0(x,y)u_{xxx} + c_1(x,y)u_x^2 + c_1(x,y)u_{xx} + c_2(x,y)u_x + c_3(x,y)u_y\Big\} = 0. \tag{4.5}$$

Making the replacement $u_x \to v$, we get a class nonlinear PDEs whose potential equations can be linearized. Here we only give out the form with that coefficients are constants for simplicity

$$c_0\left(3v^2 v_2 + 3v_x^2 + 3vv_{xx}\right) + c_1(2vv_x + v_{xx}) + c_2 v_x + c_3 v_y = 0. \tag{4.6}$$

The transformation to linearize the PDE is

$$w = \exp(u) = \exp\left(\int v\,dx\right). \tag{4.7}$$

In fact, it is the Cole-Hopf transform [18, 19]

$$v = (\ln w)_x = \frac{w_x}{w}. \tag{4.8}$$

It is easy to calculate that the NLSP is

$$\frac{\partial}{\partial x}\left(\ln\left(\epsilon\exp\left(\int z\,dx\right) + \exp\left(\int v\,dx\right)\right)\right). \tag{4.9}$$

When $c_0 = c_2 = 0$ the potential Burgers equation is included in (3.1) [15]. And at the same time, we can also get the Burgers equation from(4.6)

$$c_1(2vv_x + v_{xx}) + c_3 v_y = 0. \tag{4.10}$$

4.3. Some Examples of NLSPs and Transformations to Linearize Nonlinear PDEs

Some more concrete examples are given below. Letting $c_0(x,y) = c_6(x,y) = 1$, $c_i(x,y) = 0$, $i \neq 0,6$ in (3.1), then

$$k_2(u)u_x^3 + 3k_1(u)u_x u_{xx} + u_{xxx} + k_2(u)u_y^3 + 3k_1(u)u_y u_{yy} + u_{yyy} = F(u). \tag{4.11}$$

We discuss the two following cases.

Case 1. In the case of $F(u) = u^2$, let $\lambda = 0$, then $k_1(u) = -2/u$, $k_2(u) = 6/u^2$ and

$$6u_x^3 - 6uu_x u_{xx} + u^2 u_{xxx} + 6u_y^3 - 6uu_y u_{yy} + u^2 u_{yyy} = u^4. \tag{4.12}$$

In this case, $P(u) = -1/u$, hence the NLSP is

$$\overline{u} = -\frac{1}{-\epsilon(1/z) - 1/u}. \tag{4.13}$$

The transformation to linearize the PDE is

$$v = -\frac{1}{u}. \tag{4.14}$$

The corresponding linearized PDE is

$$v_{xxx} + v_{yyy} = 1. \tag{4.15}$$

Case 2. In the case of $F(u) = 0$, let $k_1(u) = 1$, $k_2(u) = 1$, then

$$u_x^3 + 3u_x u_{xx} + u_{xxx} + u_y^3 + 3u_y u_{yy} + u_{yyy} = 0. \tag{4.16}$$

In the case $P(u) = \exp(u)$, the NLSP is

$$\overline{u} = \ln\left[\epsilon \exp(u) + \exp(z)\right]. \tag{4.17}$$

The transformation to linearize the PDE is

$$v = \exp(u). \tag{4.18}$$

The corresponding linearized PDE is

$$v_{xxx} + v_{yyy} = 0. \tag{4.19}$$

Let $k_1(u) = u$, $k_2(u) = u^2 + 1$

$$\left(u^2 + 1\right)u_x^3 + 3uu_x u_{xx} + u_{xxx} + \left(u^2 + 1\right)u_y^3 + 3uu_y u_{yy} + u_{yyy} = 0. \tag{4.20}$$

In the case $P(u) = \int \exp(u^2/2)du$, the NLSP is

$$\overline{u} = \left\{2\ln\left[\frac{d}{du}\left[\epsilon\int \exp\left(\frac{u^2}{2}\right)du + \int \exp\left(\frac{z^2}{2}\right)dz\right]\right]\right\}^{1/2}. \tag{4.21}$$

The transformation to linearize the PDE is

$$v = \int \exp\left(\frac{u^2}{2}\right) du. \tag{4.22}$$

The corresponding linearized PDE is

$$v_{xxx} + v_{yyy} = 0. \tag{4.23}$$

5. Conclusion

We have given out a class of nonlinear PDEs with NLSPs by using the idea suggested by Goard and Broadbridge in [15]. By explicitly constructing the full class of PDEs with a Lie group of NLSPs, we arrived at a single expression (3.1) for a class of third-order variable-coefficient equations that includes previously second-order PDEs obtained in [15]. The class of PDEs includes the *Painlevé* type equations that can be linearized. We derive Cole-Hopf transformation of (4.3) whose potential equations are included in (3.1). PDE (3.1) is C-integrable since it can be linearized.

It is should be pointed out that higher-order nonlinear PDEs with NLSPs can be obtained in the same way, although much effort for symbolic computation is needed. For example, a fourth-order PDE with NLSP is in the following:

$$k_3(u)u_x^4 + 6k_2(u)u_x^2 u_{xx} + 3k_1(u)u_{xx}^2 + 6k_1(u)u_x u_{xxx} + u_{xxxx} = F(u). \tag{5.1}$$

Here $k_1(u) = k(u)$, $k_2(u) = k^2(u) + k'(u)$, $k_3(u) = k'''(u) + 3k(u)k'(u) + k^3(u)$, $k(u) = \lambda - F'(u)/F(u)$ $(F(u) \neq 0)$, λ is constant. When $F(u) = 0$, $k(u)$ is arbitrary. Here the C-integrable fourth-order ODE (5.1) can be obtained by the proposed Lie algorithm method relative easily on the computer.

Acknowledgment

The authors gratefully acknowledge fruitful discussions with Dr. Shen Shoufeng.

References

[1] E. Vessiot, “On a class of differential equations,” *Annales Scientifiques de l’École Normale Supérieure*, vol. 10, p. 53, 1893 (French).

[2] A. Guldberg, “On differential equations possessing a fundamental system of integrals,” *Comptes Rendus de l’Académie des Sciences*, vol. 116, p. 964, 1893 (French).

[3] S. Lie, “General studies on differential equations admitting finite continuous groups,” *Mathematische Annalen*, vol. 25, no. 1, pp. 71–151, 1885 (German), reprinted in Lie’s Collected Works, vol. 6, paper III, pp. 139–223.

[4] S. Lie, “On ordinary differential equations possessing fundamental systems of integrals,” *Comptes Rendus de l’Académie des Sciences*, vol. 116, 1893 (French), reprinted in Lie’s Collected Works, vol. 4, paper VII, pp. 314–316.

[5] G. Temple, “A superposition for ordinary differential equationns,” Lect. Topics in Nonlinear D.E. Report 1415, David Taylor Model Basin, Carderock, Md, USA, 1960.

[6] A. Inselberg, *On classification and superposition principles for nonlinear operators*, Ph.D. thesis, University of Illinois at Urbana-Champaign, Champaign, Ill, USA, 1965.
[7] N. H. Ibragimov, "Discussion of Lie's nonlinear superposition theory," in *Proceedings of the International Conference of Modern Group Analysis for the New Millennium (MOGRAN '00)*, Ufa, Russia, October 2000.
[8] S. E. Jones and W. F. Ames, "Nonlinear superposition," *Journal of Mathematical Analysis and Applications*, vol. 17, pp. 484–487, 1967.
[9] A. Inselberg, "Noncommutative superpositions for nonlinear operators," *Journal of Mathematical Analysis and Applications*, vol. 29, pp. 294–298, 1970.
[10] S. A. Levin, "Principles of nonlinear superposition," *Journal of Mathematical Analysis and Applications*, vol. 30, pp. 197–205, 1970.
[11] M. N. Spijker, "Superposition in linear and nonlinear ordinary differential equations. ," *Journal of Mathematical Analysis and Applications*, vol. 30, pp. 206–222, 1970.
[12] M. Ito, "An extension of nonlinear evolution equations of the K-dV (mK-dV) type to higher orders," *Journal of the Physical Society of Japan*, vol. 49, no. 2, pp. 771–778, 1980.
[13] P. J. Olver, *Applications of Lie Groups to Differential Equations*, vol. 107, Springer, New York, NY, USA, 1986.
[14] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, vol. 81 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1989.
[15] J. M. Goard and P. Broadbridge, "Nonlinear superposition principles obtained by Lie symmetry methods," *Journal of Mathematical Analysis and Applications*, vol. 214, no. 2, pp. 633–657, 1997.
[16] F. Calogero, "Universal C-integrable nonlinear partial differential equation in $N+1$ dimensions," *Journal of Mathematical Physics*, vol. 34, no. 7, pp. 3197–3209, 1993.
[17] J. B. McLeod and P. J. Olver, "The connection between partial differential equations soluble by inverse scattering and ordinary differential equations of *Painlevé* type," *SIAM Journal on Mathematical Analysis*, vol. 14, no. 3, pp. 488–506, 1983.
[18] E. Hopf, "The partial differential equation $u_t+uu_x = \mathrm{I}u_{xx}$," *Communications on Pure and Applied Mathematics*, vol. 3, pp. 201–230, 1950.
[19] J. D. Cole, "On a quasi-linear parabolic equation occurring in aerodynamics," *Quarterly of Applied Mathematics*, vol. 9, pp. 225–236, 1951.