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Research Article

Hyers-Ulam Stability of a System of First Order Linear Recurrences with Constant Coefficients

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We study the Hyers-Ulam stability in a Banach space X of the system of first order linear difference equations of the form $\mathbf{x}_{n+1} = A\mathbf{x}_n + \mathbf{d}_n$ for $n \in \mathbb{N}_0$ (nonnegative integers), where A is a given $r \times r$ matrix with real or complex coefficients, respectively, and $(\mathbf{d}_n)_{n\in\mathbb{N}_0}$ is a fixed sequence in X^r. That is, we investigate the sequences $(\mathbf{y}_n)_{n\in\mathbb{N}_0}$ in X^r such that $\delta := \sup_{n\in\mathbb{N}_0}\|\mathbf{y}_{n+1} - A\mathbf{y}_n - \mathbf{d}_n\| < \infty$ (with the maximum norm in X^r) and show that, in the case where all the eigenvalues of A are not of modulus 1, there is a positive real constant c (dependent only on A) such that, for each such a sequence $(\mathbf{y}_n)_{n\in\mathbb{N}_0}$, there is a solution $(\mathbf{x}_n)_{n\in\mathbb{N}_0}$ of the system with $\sup_{n\in\mathbb{N}_0}\|\mathbf{y}_n - \mathbf{x}_n\| \le c\delta$.

1. Introduction

The issue of stability of a functional equation can be expressed in the following way. *When must a function satisfying an equation approximately (in some sense) be near an exact solution to the equation?* It has been motivated by a question raised in 1940 by Ulam, concerning approximate homomorphisms of groups (see [1, 2]). The first partial answer to Ulam's question (in the case of Cauchy's functional equation in Banach spaces) was given by Hyers in [1]. After that result, a great number of papers on the subject have been published (see, e.g., monographs [3–5], survey articles [6–11], and the references given there), generalizing Ulam's problem and Hyers's theorem in various directions and to other equations (not necessarily functional) (see [12]). In particular, some results have been proved in [13], which concern the stability of linear difference equations of higher order of form (1). We describe them as follows.

Let T be either $\mathbb{N}_0$ (the set of nonnegative integers) or $\mathbb{Z}$ (the set of integers), let $\mathbb{K}$ be either the field of reals $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$, let $p \in \mathbb{N}$ (the set of positive integers), let $a_1, \ldots, a_p \in \mathbb{K}$ be fixed, and let $(b_n)_{n\in T}$ be a given sequence in a Banach space X over $\mathbb{K}$. The investigation of the Hyers-Ulam stability in X of the difference equation

$$x_{n+p} = a_1 x_{n+p-1} + \cdots + a_p x_n + b_n, \quad n \in T, \tag{1}$$

actually means a study of the sequences $(y_n)_{n\in T}$ in X, satisfying the condition

$$\delta := \sup_{n\in T} \left\| y_{n+p} - a_1 y_{n+p-1} - \cdots - a_p y_n - b_n \right\| < \infty. \tag{2}$$

Let $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ and $t_1, \ldots, t_p \in \mathbb{C}$ denote the roots of the characteristic equation of (1), which has the following form: $z^p = a_1 z^{p-1} + \cdots + a_p$. The following two theorems have been proved in [13] (see also [14, 15]).

Theorem 1. *Let $\delta > 0$ and $t_1, \ldots, t_p \in \mathbb{C} \setminus \mathbb{S}$. Suppose that $(y_n)_{n\in T}$ is a sequence in X such that (2) holds. Then, there exists a sequence $(x_n)_{n\in T}$ in X satisfying (1) such that*

$$\left\| y_n - x_n \right\| \le \frac{\delta}{\left|1 - \left|t_1\right|\right| \cdot \cdots \cdot \left|1 - \left|t_p\right|\right|}, \quad n \in T. \tag{3}$$

Moreover,

(a) $(x_n)_{n\in T}$ *is unique if and only if* $|t_i| > 1$ *for* $i = 1, \ldots, p$ *or* $T = \mathbb{Z}$;

(b) *if* $|t_i| > 1$ *for* $i \in \{1, \ldots, p\}$ *or* $T = \mathbb{Z}$, *then* $(x_n)_{n\in T}$ *is the unique sequence in* X *such that (1) holds and* $\sup_{n\in T}\|x_n - y_n\| < \infty$;

(c) *if* $T = \mathbb{N}_0$ *and* $|t_i| < 1$ *for some* $i \in \{1, \ldots, p\}$, *then the cardinality of the set of all sequences* $(x_n)_{n\in T}$ *in* X, *satisfying (1) and (3), equals the cardinality of* X.

Theorem 2. *Let* $|t_j| = 1$ *for some* $j \in \{1, \ldots, p\}$. *Then, for any* $\delta > 0$, *there exists a sequence* $(y_n)_{n\in T}$ *in* X, *satisfying inequality (2), such that, for every sequence* $(x_n)_{n\in T}$ *in* X, *fulfilling recurrence (1),*

$$\sup_{n\in T}\|y_n - x_n\| = \infty. \quad (4)$$

Moreover, if $t_1, \ldots, t_p \in \mathbb{K}$ *or there is a bounded sequence* $(x_n)_{n\in T}$ *in* X *fulfilling (1), then* $(y_n)_{n\in T}$ *can be chosen unbounded.*

We somehow complement those results in this paper by the study of the Hyers-Ulam stability of the following system of first order linear difference equations in X with constant coefficients $a_{ij} \in \mathbb{K}$, $i, j = 1, \ldots, r$ ($r \in \mathbb{N}$ is fixed):

$$\begin{aligned} x^1_{n+1} &= a_{11}x^1_n + a_{12}x^2_n + \cdots + a_{1r}x^r_n + d^1_n, \\ x^2_{n+1} &= a_{21}x^1_n + a_{22}x^2_n + \cdots + a_{2r}x^r_n + d^2_n, \\ &\vdots \\ x^r_{n+1} &= a_{r1}x^1_n + a_{r2}x^2_n + \cdots + a_{rr}x^r_n + d^r_n, \end{aligned} \quad (5)$$

for all $n \in \mathbb{N}_0$, where $d^1_n, \ldots, d^r_n \in X$ for $n \in \mathbb{N}_0$ are given. If we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{bmatrix}, \qquad \mathbf{x}_n = \begin{bmatrix} x^1_n \\ x^2_n \\ \vdots \\ x^r_n \end{bmatrix}, \qquad \mathbf{d}_n = \begin{bmatrix} d^1_n \\ d^2_n \\ \vdots \\ d^r_n \end{bmatrix}, \quad (6)$$

then (5) can be expressed in the following simple form:

$$\mathbf{x}_{n+1} = A\mathbf{x}_n + \mathbf{d}_n, \quad n \in \mathbb{N}_0. \quad (7)$$

To simplify the notations, we consider $\mathbf{x}_n$ and $\mathbf{d}_n$ to be elements of X^r, when it is convenient (and when this makes no confusion); that is, we identify $\mathbf{x}_n$ with $(x^1_n, \ldots, x^r_n)$ and $\mathbf{d}_n$ with $(d^1_n, \ldots, d^r_n)$.

Our results correspond, in particular, not only to the outcomes in [13], but also to those in [14, 15], where similar problems have been studied for $r = 1$.

2. Some Auxiliary Results

By an elementary induction on n, we obtain the following simple observation.

Lemma 3. *If a sequence* $(\mathbf{x}_n)_{n\in\mathbb{N}_0}$ *in* X^r *satisfies (7), then*

$$\mathbf{x}_n = A^n\mathbf{x}_0 + \sum_{k=1}^{n} A^{n-k}\mathbf{d}_{k-1}, \quad n \in \mathbb{N}. \quad (8)$$

In this paper,

$$C^n_j := \binom{n}{j} = \frac{n!}{(n-j)!\,j!}, \quad n, j \in \mathbb{N}_0,\ n \ge j, \quad (9)$$

denote the binomial coefficients and $C^n_j := 0$ for $j > n$. The subsequent formula is well known:

$$\sum_{n=0}^{\infty} C^{n+j}_n w^n = \frac{1}{(1-w)^{j+1}}, \quad w \in [0,1),\ j \in \mathbb{N}_0. \quad (10)$$

Also, replacing w by $1/x$, we easily obtain that

$$\sum_{n=0}^{\infty} C^{n+j}_n x^{-(n+j+1)} = \frac{1}{(x-1)^{j+1}}, \quad x \in (1,\infty),\ j \in \mathbb{N}_0. \quad (11)$$

Further, write

$$\begin{aligned} \|A\| &:= \max_{1\le i\le r} \sum_{j=1}^{r} |a_{ij}|, \\ \|\mathbf{x}\| &:= \max_{1\le i\le r} \|x_i\|, \quad \mathbf{x} = (x_1, \ldots, x_r) \in X^r. \end{aligned} \quad (12)$$

Then, $(X^r, \|\cdot\|)$ is a Banach space and we have the following result, which will be useful in the proof of the main theorem.

Theorem 4. *Let* $J_{\lambda,r}$ *be a Jordan matrix of the form*

$$J_{\lambda,r} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}_{r\times r}, \quad (13)$$

with some $\lambda \in \mathbb{C} \setminus \mathbb{S}$. *If a sequence* $(\mathbf{y}_n)_{n\in\mathbb{N}_0}$ *in* X^r *satisfies*

$$\delta := \sup_{n\in\mathbb{N}_0} \|\mathbf{y}_{n+1} - J_{\lambda,r}\mathbf{y}_n - \mathbf{b}_n\| < \infty, \quad (14)$$

then there exists a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}_0}$ *in* X^r *such that*

$$\mathbf{x}_{n+1} = J_{\lambda,r}\mathbf{x}_n + \mathbf{b}_n, \quad n \in \mathbb{N}_0, \quad (15)$$

$$\sup_{n\in\mathbb{N}_0} \|\mathbf{y}_n - \mathbf{x}_n\| \le \delta \sum_{j=1}^{r} |1 - |\lambda||^{-j}. \quad (16)$$

Proof. Let $\mathbf{c}_n := \mathbf{y}_{n+1} - J_{\lambda,r}\mathbf{y}_n - \mathbf{b}_n$ for $n \in \mathbb{N}_0$. Then, by (14), $\|\mathbf{c}_n\| \le \delta$ for $n \in \mathbb{N}_0$ and (see Lemma 3)

$$\mathbf{y}_n = J_{\lambda,r}^n \mathbf{y}_0 + \sum_{k=1}^{n} J_{\lambda,r}^{n-k}\left(\mathbf{b}_{k-1} + \mathbf{c}_{k-1}\right), \quad n \in \mathbb{N}. \tag{17}$$

Case 1 ($|\lambda| < 1$). Define the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}_0}$ by (15) with $\mathbf{x}_0 = \mathbf{y}_0$. Then,

$$\|\mathbf{y}_n - \mathbf{x}_n\| = \left\|\sum_{k=1}^{n} J_{\lambda,r}^{n-k}\mathbf{c}_{k-1}\right\| \le \delta \sum_{k=1}^{n}\left\|J_{\lambda,r}^{n-k}\right\|, \quad n \in \mathbb{N}. \tag{18}$$

It is easy to show by induction on n that $J_{\lambda,r}^n$ is an upper (right) triangular matrix of the form

$$J_{\lambda,r}^n = \begin{bmatrix} \lambda^n & C_1^n\lambda^{n-1} & \cdots & C_{r-1}^n\lambda^{n-(r-1)} \\ 0 & \lambda^n & \cdots & C_{r-2}^n\lambda^{n-(r-2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_1^n\lambda^{n-1} \\ 0 & 0 & \cdots & \lambda^n \end{bmatrix}_{r\times r} \tag{19}$$

(it is enough to use the well-known formula $C_{k+1}^{n+1} = C_k^n + C_{k+1}^n$) whence we derive that $\|J_{\lambda,r}^n\| = \sum_{j=0}^{r-1} C_j^n |\lambda|^{n-j}$ for $n \in \mathbb{N}_0$.

Since, in view of (10),

$$\sum_{n=0}^{\infty} C_j^n |\lambda|^{n-j} = \frac{1}{(1-|\lambda|)^{j+1}}, \quad j = 0, \dots, r, \tag{20}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left\|J_{\lambda,r}^n\right\| &= \sum_{n=0}^{\infty}\sum_{j=0}^{r-1} C_j^n |\lambda|^{n-j} = \sum_{j=0}^{r-1}\sum_{n=0}^{\infty} C_j^n |\lambda|^{n-j} \\ &= \sum_{j=1}^{r}\frac{1}{(1-|\lambda|)^j}\end{aligned} \tag{21}$$

and, by (18),

$$\|\mathbf{y}_n - \mathbf{x}_n\| \le \sum_{k=0}^{\infty}\left\|J_{\lambda,r}^k\right\|\delta = \delta\sum_{j=1}^{r}\frac{1}{(1-|\lambda|)^j}, \quad n \in \mathbb{N}_0. \tag{22}$$

That is, inequality (16) holds.

Case 2 ($|\lambda| > 1$). Since $J_{\lambda,r}^{-1}$ is an upper triangular matrix of the form

$$J_{\lambda,r}^{-1} = \begin{bmatrix} \lambda^{-1} & \dfrac{-1}{\lambda^2} & \cdots & \dfrac{(-1)^{r-1}}{\lambda^r} \\ 0 & \lambda^{-1} & \cdots & \dfrac{(-1)^{r-2}}{\lambda^{r-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dfrac{-1}{\lambda^2} \\ 0 & 0 & \cdots & \lambda^{-1} \end{bmatrix}_{r\times r}, \tag{23}$$

it is easy to check that $J_{\lambda,r}^{-n}$ is also an upper triangular matrix for each $n \in \mathbb{N}$ and has the form

$$J_{\lambda,r}^{-n} = \begin{bmatrix} \lambda^{-n} & \dfrac{-C_{n-1}^n}{\lambda^{n+1}} & \cdots & \dfrac{(-1)^{r-1}C_{n-1}^{n+r-2}}{\lambda^{n+r-1}} \\ 0 & \lambda^{-n} & \cdots & \dfrac{(-1)^{r-2}C_{n-1}^{n+r-3}}{\lambda^{n+r-2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dfrac{-C_{n-1}^n}{\lambda^{n+1}} \\ 0 & 0 & \cdots & \lambda^{-n} \end{bmatrix}_{r\times r}. \tag{24}$$

Hence,

$$\left\|J_{\lambda,r}^{-n}\right\| = \sum_{j=0}^{r-1} C_{n-1}^{n-1+j}|\lambda|^{-(n+j)}, \quad n \in \mathbb{N}. \tag{25}$$

Consequently, in view of (11),

$$\begin{aligned}\sum_{n=1}^{\infty}\left\|J_{\lambda,r}^{-n}\right\| &= \sum_{n=0}^{\infty}\sum_{j=0}^{r-1} C_n^{n+j}|\lambda|^{-(n+j+1)} \\ &= \sum_{j=0}^{r-1}\sum_{n=0}^{\infty} C_n^{n+j}|\lambda|^{-(n+j+1)} \\ &= \sum_{j=1}^{r}\frac{1}{(|\lambda|-1)^j}.\end{aligned} \tag{26}$$

Taking into account that $\|J_{\lambda,r}^{-k}\mathbf{c}_{k-1}\| \le \|J_{\lambda,r}^{-k}\|\|\mathbf{c}_{k-1}\| \le \delta\|J_{\lambda,r}^{-k}\|$ for $k \in \mathbb{N}$, we deduce that the series $\sum_{k=1}^{\infty} J_{\lambda,r}^{-k}\mathbf{c}_{k-1}$ is convergent. Take

$$\mathbf{s} := \sum_{k=1}^{\infty} J_{\lambda,r}^{-k}\mathbf{c}_{k-1} \tag{27}$$

and define $(\mathbf{x}_n)_{n\in\mathbb{N}_0}$ by (15) with $\mathbf{x}_0 = \mathbf{y}_0 + \mathbf{s}$. Then, (see Lemma 3)

$$\mathbf{x}_n = J_{\lambda,r}^n\left(\mathbf{y}_0 + \mathbf{s}\right) + \sum_{k=1}^{n} J_{\lambda,r}^{n-k}\mathbf{b}_{k-1}, \quad n \in \mathbb{N}. \tag{28}$$

Next, by (26), for every $n \in \mathbb{N}$, we have

$$\begin{aligned}\|\mathbf{y}_n - \mathbf{x}_n\| &= \left\|\sum_{k=1}^{n} J_{\lambda,r}^{n-k}\mathbf{c}_{k-1} - J_{\lambda,r}^n\mathbf{s}\right\| \\ &= \left\|\sum_{k=1}^{n} J_{\lambda,r}^{n-k}\mathbf{c}_{k-1} - J_{\lambda,r}^n\sum_{k=1}^{\infty} J_{\lambda,r}^{-k}\mathbf{c}_{k-1}\right\| \\ &= \left\|\sum_{k=1}^{n} J_{\lambda,r}^{n-k}\mathbf{c}_{k-1} - \sum_{k=1}^{\infty} J_{\lambda,r}^{n-k}\mathbf{c}_{k-1}\right\| \\ &= \left\|\sum_{k=n+1}^{\infty} J_{\lambda,r}^{n-k}\mathbf{c}_{k-1}\right\|,\end{aligned} \tag{29}$$

whence

$$\|\mathbf{y}_n - \mathbf{x}_n\| \le \sum_{k=1}^{\infty}\left\|J_{\lambda,r}^{-k}\right\|\delta = \delta\sum_{j=1}^{r}\frac{1}{(|\lambda|-1)^j}, \quad n \in \mathbb{N}_0. \tag{30}$$

□

3. The Main Result

Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of A with multiplicities $r_1, \dots, r_m$, respectively. There exists a nonsingular matrix Q in $\mathbb{C}^{r\times r}$ with $A = QJQ^{-1}$, where

$$J = J_{\lambda_1,r_1} \oplus \cdots \oplus J_{\lambda_m,r_m},$$
$$J_{\lambda_j,r_j} = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix}_{r_j\times r_j}, \quad j = 1, \dots, m. \tag{31}$$

The next theorem is the main result of this paper.

Theorem 5. *Assume that $\lambda_j \in \mathbb{C} \setminus \mathbb{S}$ for $j = 1, \dots, m$. For any sequence $(\mathbf{z}_n)_{n\in\mathbb{N}_0}$ in X^r, satisfying*

$$\delta := \sup_{n\in\mathbb{N}_0} \|\mathbf{z}_{n+1} - A\mathbf{z}_n - \mathbf{d}_n\| < \infty, \tag{32}$$

there exists a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}_0}$ in X^r such that

$$\mathbf{x}_{n+1} = A\mathbf{x}_n + \mathbf{d}_n, \quad n \in \mathbb{N}_0, \tag{33}$$

$$\sup_{n\in\mathbb{N}_0} \|\mathbf{z}_n - \mathbf{x}_n\| \le \delta \|Q\| \left\|Q^{-1}\right\| \max_{j=1,\dots,m} \sum_{k=1}^{r_j} \frac{1}{\left|1 - \left|\lambda_j\right|\right|^k}. \tag{34}$$

Proof. Let $(\mathbf{z}_n)_{n\in\mathbb{N}_0}$ be a sequence in X^r such that (32) holds.

First, consider the case where $\mathbb{K} = \mathbb{C}$. Write $\mathbf{y}_n := Q^{-1}\mathbf{z}_n$ and $\mathbf{b}_n := Q^{-1}\mathbf{d}_n$ for $n \in \mathbb{N}_0$. Then,

$$\begin{aligned} \|\mathbf{y}_{n+1} - J\mathbf{y}_n - \mathbf{b}_n\| &\le \left\|Q^{-1}\right\| \|\mathbf{z}_{n+1} - A\mathbf{z}_n - \mathbf{d}_n\| \\ &\le \left\|Q^{-1}\right\| \delta =: \delta_0, \quad n \in \mathbb{N}_0. \end{aligned} \tag{35}$$

Define projections $p_j : X^r \to X^{r_j}$ (for $j = 1, \dots, m$) by

$$\begin{aligned} p_1(w_1, \dots, w_r) &:= \left(w_1, \dots, w_{r_1}\right), \\ p_2(w_1, \dots, w_r) &:= \left(w_{r_1+1}, \dots, w_{r_1+r_2}\right), \\ &\vdots \\ p_{m-1}(w_1, \dots, w_r) &:= \left(w_{r_1+\cdots+r_{m-2}+1}, \dots, w_{r_1+\cdots+r_{m-1}}\right), \\ p_m(w_1, \dots, w_r) &:= \left(w_{r_1+\cdots+r_{m-1}+1}, \dots, w_{r_1+\cdots+r_m}\right) \end{aligned} \tag{36}$$

for $(w_1, \dots, w_r) \in X^r$. For simplicity, we write $\mathbf{y}_n = (p_1(\mathbf{y}_n), \dots, p_m(\mathbf{y}_n))$ and $\mathbf{b}_n = (p_1(\mathbf{b}_n), \dots, p_m(\mathbf{b}_n))$ for $n \in \mathbb{N}_0$. It is easily seen that (in analogous notation)

$$\begin{aligned} J\mathbf{y}_n &= \left(J_{\lambda_1,r_1} p_1(\mathbf{y}_n), \dots, J_{\lambda_m,r_m} p_m(\mathbf{y}_n)\right), \quad n \in \mathbb{N}_0, \\ &\left\|p_j(\mathbf{y}_{n+1}) - J_{\lambda_j,r_j} p_j(\mathbf{y}_n) - p_j(\mathbf{b}_n)\right\| \\ &\le \|\mathbf{y}_{n+1} - J\mathbf{y}_n - \mathbf{b}_n\| \le \delta_0, \\ &n \in \mathbb{N}_0, \quad j = 1, \dots, m. \end{aligned} \tag{37}$$

According to Theorem 4 (applied for each $j \in \{1, \dots, m\}$, separately), there exists a sequence $(\mathbf{u}_n)_{n\in\mathbb{N}_0}$ in X^r such that

$$\begin{aligned} p_j(\mathbf{u}_{n+1}) &= J_{\lambda_j,r_j} p_j(\mathbf{u}_n) + p_j(\mathbf{b}_n), \quad n \in \mathbb{N}_0, \\ \left\|p_j(\mathbf{u}_n) - p_j(\mathbf{y}_n)\right\| &\le \delta_0 \sum_{k=1}^{r_j} \frac{1}{\left|1 - \left|\lambda_j\right|\right|^k}, \\ &n \in \mathbb{N}_0, \quad j = 1, \dots, m. \end{aligned} \tag{38}$$

Clearly,

$$\begin{aligned} \mathbf{u}_{n+1} &= J\mathbf{u}_n + \mathbf{b}_n, \quad n \in \mathbb{N}_0, \\ \|\mathbf{y}_n - \mathbf{u}_n\| &\le \delta_0 \max\left\{\sum_{k=1}^{r_j} \frac{1}{\left|1 - \left|\lambda_j\right|\right|^k} : j = 1, \dots, m\right\}, \\ &n \in \mathbb{N}_0. \end{aligned} \tag{39}$$

Let $\mathbf{x}_n := Q\mathbf{u}_n$ for $n \in \mathbb{N}_0$. Then,

$$\begin{aligned} \mathbf{x}_{n+1} &= Q\mathbf{u}_{n+1} = QJ\mathbf{u}_n + Q\mathbf{b}_n = QJQ^{-1}\mathbf{x}_n + \mathbf{d}_n, \quad n \in \mathbb{N}_0, \\ \|\mathbf{z}_n - \mathbf{x}_n\| &= \|Q\mathbf{y}_n - Q\mathbf{u}_n\| \le \|Q\| \|\mathbf{y}_n - \mathbf{u}_n\| \\ &\le \|Q\| \delta_0 \max\left\{\sum_{k=1}^{r_j} \frac{1}{\left|1 - \left|\lambda_j\right|\right|^k} : j = 1, \dots, m\right\}, \\ &n \in \mathbb{N}_0. \end{aligned} \tag{40}$$

Now, consider the case $\mathbb{K} = \mathbb{R}$. Define the linear structure in $\widehat{X} := X^2$ by $(x, y) + (z, w) := (x + z, y + w)$ and $(\alpha + i\beta)(x, y) := (\alpha x - \beta y, \beta x + \alpha y)$ for $x, y, z, w \in X, \alpha, \beta \in \mathbb{R}$. Then, $\widehat{X}$ is a complex Banach space (see, e.g., [16, page 39], [17], or [18, 1.9.6, page 66]), when endowed with the Taylor norm $\|\cdot\|_T$ given by

$$\|(x, y)\|_T := \sup_{0\le\theta\le 2\pi} \|(\cos\theta) x + (\sin\theta) y\|, \quad x, y \in X. \tag{41}$$

Note that

$$\max\{\|x\|, \|y\|\} \le \|(x, y)\|_T \le \|x\| + \|y\|, \quad x, y \in X. \tag{42}$$

Define $\widehat{p}_1 : X^2 \to X$ by $\widehat{p}_1(w_1, w_2) := w_1$ for $w_1, w_2 \in X$. Let $\widehat{\mathbf{z}}_n := (\mathbf{z}_n, 0)$ and $\widehat{\mathbf{d}}_n := (\mathbf{d}_n, 0)$ for $n \in \mathbb{N}_0$. Then, $(\widehat{\mathbf{z}}_n)_{n\in\mathbb{N}_0}$ is a sequence in $\widehat{X}$ and

$$\left\|\widehat{\mathbf{z}}_{n+1} - A\widehat{\mathbf{z}}_n - \widehat{\mathbf{d}}_n\right\|_T = \left\|\mathbf{z}_{n+1} - A\mathbf{z}_n - \mathbf{d}_n\right\| \le \delta, \quad n \in \mathbb{N}_0. \tag{43}$$

So, by the first part of the proof, there is a sequence $(\widehat{\mathbf{x}}_n)_{n\in\mathbb{N}_0}$ in $\widehat{X}$ such that

$$\widehat{\mathbf{x}}_{n+1} = A\widehat{\mathbf{x}}_n + \widehat{\mathbf{d}}_n, \quad n \in \mathbb{N}_0,$$
$$\sup_{n\in\mathbb{N}_0} \left\|\widehat{\mathbf{z}}_n - \widehat{\mathbf{x}}_n\right\|_T \le \delta \|Q\| \left\|Q^{-1}\right\| \max_{j=1,\ldots,m} \sum_{k=1}^{r_j} \frac{1}{\left|1 - \left|\lambda_j\right|\right|^k}. \tag{44}$$

Write $\mathbf{x}_n := \widehat{p}_1(\widehat{\mathbf{x}}_n)$ for $n \in \mathbb{N}_0$. Then, it is easily seen that (33) and (34) are valid (in view of (42)). □

Remark 6. The assumption that $|\lambda_j| \neq 1$ for $j = 1, \ldots, m$ cannot be omitted in the general case (at least when $r = 1$), in view of Theorem 2.

It seems that our method cannot be easily applied to the systems of linear difference equations of higher orders, because it is difficult in such cases to obtain a formula analogous as (8).

Open Problems. There arises a natural question if some results similar to Theorem 2 and statements (a)–(c) of Theorem 1 can be obtained for difference equation (33) with $r > 1$ (also with $\mathbb{N}_0$ replaced by $\mathbb{Z}$).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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