Research Article

Completion of a Dislocated Metric Space

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We provide a construction for the completion of a dislocated metric space (abbreviated d-metric space); we also prove that the completion of the metric associated with a d-metric coincides with the metric associated with the completion of the d-metric.

1. Introduction

Completion of a metric space via Cauchy sequences can be achieved because of certain convergence properties enjoyed by the metric and the property that convergent sequences are Cauchy sequences. Lack of some of these properties in weaker forms of metric spaces comes in the way of completion process in the above lines. In semimetric spaces several new ways of completeness were invented, for example, Cauchy completeness, McAuley notions of strong and weak completeness [1], Moore completeness [2], and so on. Moshokoa [3] introduced the notion of convergence completeness for semimetric spaces and discussed completion on these lines.

For d-metric spaces adoption of Van der-Waerdens completion process through Cauchy sequences is possible but is not routine, the difficulty being the mischief created by the isolated points. Here we show how to overcome this problem.

In his study of programming languages, Hitzler [4] associated a metric d' with every d-metric by defining

$$d'(a,b) = \begin{cases} d(a,b), & \text{if } a \neq b \\ 0, & \text{if } a = b. \end{cases} \tag{1}$$

We establish that the metric associated with the completion of a d-metric is the completion of the metric associated with d.

We recall [4] where a distance function on a set X is said to be a d-metric on X if

(i) $d(x,y) = d(y,x)$;

(ii) $d(x,y) = 0 \Rightarrow x = y$;

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ for all x, y, z in X.

If d is a d-metric on X then (X,d) is called a d-metric space. Many authors (see, e.g. [5–9]) have studied fixed point theorems in d-metric spaces but topology and topological aspects on this space are discussed by Sarma and Kumari [10].

The class $\mathscr{B} = \{V_\epsilon(x)/x \in X \text{ and } \epsilon > 0\}$ is an open base for the topology $\mathfrak{I}_d$ induced by d, where $V_\epsilon(x) = \{y/d(x,y) < \epsilon\} \cup \{x\}$. In what follows whenever we talk about topological properties of a d-metric space, we refer to the topology $\mathfrak{I}_d$.

In [11], the authors highlighted some convergence properties and covers a huge range of implications and nonimplications among them. By using these convergence axioms many authors (see, e.g. [12–15]) have proved fixed point theorems in certain spaces.

The presence of the triangle inequality lends the Haussdorff property for d and some nice properties to (X,d). In particular (X,d) satisfies properties C_1 through C_5:

C_1: $\lim d(x_n, y_n) = 0 = \lim d(x_n, x) \Rightarrow \lim d(y_n, x) = 0$;

C_2: $\lim d(x_n, x) = 0 = \lim d(y_n, x) \Rightarrow \lim d(x_n, y_n) = 0$;

C_3: $\lim d(x_n, y_n) = 0 = \lim d(y_n, z_n) \Rightarrow \lim d(x_n, z_n) = 0$;

C_4: $\lim d(x_n, x) = 0 \Rightarrow \lim d(x_n, y) = d(x, y)$;

C_5: $\lim d(x_n, x) = \lim d(x_n, y) = 0 \Rightarrow x = y$; for all $x, y \in X$.

Above mentioned convergence axioms can be found in [11]. If the triangular inequality is deleted from the axioms on d then it is difficult to define the concept of completion of the resulting distance space. In such an amorphous space, even constant sequences may fail to converge. This and related difficulties compel us to retain the triangle inequality in the discussion of completeness.

Definition 1. Let (X, d) and (Y, d') be distance spaces. A map $f : X \rightarrow Y$ is called an *isodistance* if for any $x, y \in X$ one has $d\{f(x), f(y)\} = d'(x, y)$.

2. Completion

In what follows, d is a d-metric on a nonempty set X. A complete d-metric space is a d-metric space in which every Cauchy sequence converges. "Cauchy sequences" in d-metric spaces are defined exactly as in metric spaces.

Lemma 2. x is an isolated point of X if and only if $X = x$ or $\inf_{y \neq x} d(x, y) > 0$.

Proof. Suppose x is an isolated point of X. Then there exists $r > 0$ such that $y \neq x \Rightarrow d(x, y) > r \Rightarrow X = \{x\}$ or $\inf_{y \neq x} d(x, y) \geq r > 0$. Conversely suppose $X = \{x\}$ or $\inf_{y \neq x} d(x, y) > 0$. If $X = \{x\}$, then clearly x is an isolated point of X. If $X \neq \{x\}$, then $\inf_{y \neq x} d(x, y) = r > 0$ which implies that $d(x, y) \geq r > r/2$ for all $y \neq x$. Hence x is an isolated point of X. $\square$

Corollary 3. If $d(x, x) > 0$, then x is an isolated point of X.

Proof. If $y \neq x$, then $d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$ and so $(1/2)d(x, x) \leq d(x, y)$ for all y in X. So $X = \{x\}$ or $\inf_{y \neq x} d(x, y) \geq (1/2)d(x, x) > 0$. $\square$

Theorem 4. Let (X, d) be a d-metric space. Then there exists a complete d-metric space (X^*, d^*) and an isodistance $T : (X, d) \rightarrow (X^*, d^*)$ such that $T(X)$ is dense in X^*.

Proof. Let I be the collection of isolated points of X and let $J = X - I$. Let $\bar{I}$ be the collection of sequences in X which are ultimately a constant element lying in I and $\bar{J}$ denote the class of Cauchy sequences in J. We define relations R_I and R_J, respectively, on $\bar{I}$ and $\bar{J}$ as follows.

If $(x_n)(y_n)$ are sequences in $\bar{I}$ then $(x_n)R_I(y_n)$ iff the ultimately constant value of (x_n) coincides with that of (y_n).

If $(x_n)(y_n)$ are sequences in $\bar{J}$ then $(x_n)R_J(y_n)$ iff $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$. Clearly R_I is an equivalence relation. We verify that R_J is an equivalence relation. Suppose $(x_n) \in \bar{J}$ and $\epsilon > 0$. Since (x_n) is a Cauchy sequence in J, $d(x_n, x_n) = 0$ and hence R_J is reflexive.

Suppose $(x_n)R_J(y_n)$ for $(x_n), (y_n) \in \bar{J}$. Then $\lim_{n \rightarrow \infty} d(x_n, y_n) = \lim_{n \rightarrow \infty} d(y_n, x_n) = 0$. Hence R_J is symmetric.

If $(x_n), (y_n), (z_n) \in \bar{J}, (x_n)R_J(y_n)$ and $(y_n)R_J(z_n)$. If $\epsilon > 0$, then there exists an integer n_1 such that $d(x_n, y_n) < \epsilon/2$ and $d(y_n, z_n) < \epsilon/2$, if $n > n_1$. Consider

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{if } n > n_1. \tag{2}$$

This proves that R_J is transitive and hence an equivalence relation. Let $\bar{X} = \bar{I} \cup \bar{J}$. Then $\sim = R_I \cup R_J$ is an equivalence relation on $\bar{X}$.

Let X^* denote $\bar{X}/\sim$. If $(x_n) \in \bar{X}, [(x_n)]$ denotes the equivalence class in X^* containing the sequence (x_n).

If $x \in X$ let (x) be the constant sequence (x_n) where $x_n = x$, $\forall n$ and $\hat{x} = [(x)]$, the equivalence class containing (x).

If $(x_n) \in \bar{J}, (y_n) \in \bar{J}$, it follows from the triangle inequality that $|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$. Since $(x_n), (y_n)$ are Cauchy sequences, given that $\epsilon > 0$, there exists a positive integer n_0 such that $d(x_n, x_m) < \epsilon/2$ and $d(y_n, y_m) < \epsilon/2$ for all $n, m \geq n_0$.

This implies that $|d(x_n, y_n) - d(x_m, y_m)| < \epsilon$ proving that $(d(x_n, y_n))$ is a Cauchy sequence of real numbers. By the completeness of R this sequence converges. The definition of R_J makes it obvious that $\lim_{n \rightarrow \infty} d(x_n, y_n)$ is independent of the choice of the representative sequences $(x_n), (y_n)$, respectively, from the classes $[(x_n)], [(y_n)]$.

We can prove similarly if $x \in X$ and $(y_n) \in \bar{J}, (z_n) \in \bar{J}, \lim d(x, y_n), \lim d(x, z_n)$ exists and or equal. Provided (y_n) and (z_n) belong to the same equivalence class.

We define $d^* : X^* \times X^* \rightarrow [0, \infty)$ as follows:

$d^*([(x_n)], [(y_n)]) = d(x, y)$ if $(x_n), (y_n) \in \bar{I}$ and x, y are, respectively, the ultimately constant terms of $(x_n), (y_n)$.

$d^*([(x_n)], [(y_n)]) = \lim_{n \rightarrow \infty} d(x, y_n)$ if $(x_n) \in \bar{I}$, $(y_n) \in \bar{J}$ and $x_n = x$ eventually.

If $(x_n) \in \bar{J}, (y_n) \in \bar{I}$, then define $d^*([(x_n)], [(y_n)]) = d^*([(Y_n)], [(X_n)])$.

If $(x_n) \in \bar{J}, (y_n) \in \bar{J}$, then define $d^*([(x_n)], [(y_n)]) = \lim_{n \rightarrow \infty} d(x_n, y_n)$.

Verification That d^* Is a d-Metric on X^*. Clearly $d^*(x^*, y^*) \geq 0$ and $d^*(x^*, y^*) = d^*(y^*, x^*)$ for $x^*, y^* \in X^*$.

Suppose $d^*(x^*, y^*) = 0$. Let $(x_n) \in x^*$ and $(y_n) \in y^*$. We first see that $(x_n), (y_n)$ either are both in $\bar{I}$ or are both in $\bar{J}$.

Suppose, on the contrary, $(x_n) \in \bar{I}$ and $(y_n) \in \bar{J}$. Let x be the ultimately constant value of (x_n). Consider

$$0 \leq d(x, x) \leq 2d(x, y_n) \quad \forall n,$$
$$\Longrightarrow 0 = d^*(x^*, y^*) = \lim_{n \rightarrow \infty} d(x, y_n). \tag{3}$$

Hence $0 \leq d(x, x) \leq \lim_{n \rightarrow \infty} 2d(x, y_n) = 0$, contrary to the fact that $x \in I$.

Suppose $x^*, y^* \in \bar{I}, (x_n) \in x^*$, and $(y_n) \in y^*$ with a, b the ultimately constant values of (x_n) and (y_n), respectively.

Then $d^*(x^*, y^*) = 0 \Rightarrow d(a, b) = 0 \Rightarrow a = b \Rightarrow (x_n) \sim (y_n) \Rightarrow x^* = y^*$.

Suppose $x^*, y^* \in \bar{J}$, $(x_n) \in x^*$ and $(y_n) \in y^*$. Consider

$$d^* (x^*, y^*) = 0 \implies \lim_{n \to \infty} d(x_n, y_n) = 0$$
$$\implies (x_n) \sim (y_n) \qquad (4)$$
$$\implies x^* = y^*.$$

Verification of the triangular inequality is routine.

Embedding of X in X^*. Define $T : X \to X^*$ by $T(x) = \hat{x}$. It is clear that T is an isodistance. We now verify that $T(x)$ is dense in X^*. Let $[(x_n)] \in X^*$ and $\epsilon > 0$.

Case (i) $((x_n) \in \bar{I})$. In this case let "a" be the ultimately constant value of (x_n).
Then by the definition of $T, \hat{a} = [(x_n)] \in T(X)$.
Then $\hat{a} = [(x_n)]$. Thus $[(x_n)] \in T(X)$ in this case.

Case (ii) $((x_n) \in \bar{J})$. There exists a positive integer n_0 such that $d(x_n, x_m) < \epsilon$ if $n, m \geq n_0$. Let $x_{n_0} = a$. Then since $a \in J$, $d(a, a) = 0$,

$$d^* ([(x)], \hat{a}) = \lim_{n \to \infty} d(x_n, a) \leq \epsilon. \qquad (5)$$

Hence $T(X)$ is dense in X^*.

(X^*, d^*) Is Complete. Let (x_n^*) be a Cauchy sequence in X^*, and $\epsilon > 0$. There exists n_0 such that $n \geq m \geq n_0$ implies $d^*(x_n^*, x_m^*) < \epsilon/3$.
There is no harm in assuming that $n_0 > \epsilon/3$. Since $T(X)$ is dense in X^*, for each positive integer n, there exists z_n in X such that $d(x_n^*, \hat{z}_n) < 1/n$.
Hence

$$d^* (\hat{z}_n, \hat{z}_m) \leq d^* (\hat{z}_n, x_n^*) + d^* (x_n^*, x_m^*) + d^* (x_m^*, \hat{z}_m)$$
$$< \frac{1}{n} + \frac{1}{m} + \frac{\epsilon}{3} \qquad (6)$$
$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \text{if } n, m \geq n_0.$$

Hence $(\hat{z}_n)$ is a Cauchy sequence in $T(X)$. Since T is an isodistance, (z_n) is a Cauchy sequence in X.
Moreover, $d(z_n, z_m) = d^*(\hat{z}_n, \hat{z}_m) < \epsilon$, if $n \geq m \geq n_0$.
Let z^* denote $[(z_n)]$, by the triangle inequality:

$$< (1/n) + \lim_m d(z_n, z_m) < (2\epsilon/3) < \epsilon \text{ for } n \geq n_0;$$
$$d^*(x_n^*, z^*) \leq d^*(x_n^*, \hat{z}_n) + d^*(\hat{z}_n, z^*);$$
$$\implies \lim_m d^*(x_n^*, z^*) = 0 \text{ proving that } (X^*, d^*) \text{ is complete.} \qquad \square$$

Definition 5. Let (X, d) and (X_1, d_1) be d-metric spaces. (X_1, d_1) is said to be a completion of (X, d) if

(i) (X_1, d_1) is complete;

(ii) there is an isodistance $T : (X, d) \to (X_1, d_1)$ such that $T(X)$ is dense in X_1.

Note. If (X, d) is a complete metric space then its completion is (X, d) itself.

Lemma 6. *Let (X, d) be a d-metric space and let (X_1, d_1) be a completion of (X, d). Let $T : X \to X_1$ be isodistance embedding X in X_1 with $T(X)$ dense in X_1. Then a point y of X_1 is an isolated point if and only if $y = T(X)$ for some isolated point x of X_1.*

Proof. Suppose y is an isolated point of X_1. If y is not in $T(X)$, then since $T(X)$ is dense in X_1, there exists a sequence $T(x_n)$ in $T(X)$ such that $\lim_{n \to \infty} d(T(x_n), y) = 0$.
By Lemma 2, it follows that y is not an isolated point of X_1, a contradiction so that $y = T(X)$ for some $x \in X$. Hence Tx is an isolated point of X_1 and hence that of $T(X)$. Since X and $T(X)$ are isometric, x is an isolated point of X.
Conversely, suppose x is an isolated point of X. If $T(X)$ is not an isolated point of X_1, then for each positive integer k, there exists x_k in X_1 such that $0 < d_1(x_k, T(x)) \leq 1/2k$. Since $x_k \in X_1$, either $x_k \in T(X)$ or there exists y_n in $T(X)$ such that $0 < d_1((y_k), x_k) < d_1(x_k, T(x))$.
Now

$$0 < d_1 (y_k, T(x)) \leq d_1 ((y_k), x_k) + d_1 (x_k, T(x))$$
$$\leq \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}. \qquad (7)$$

Also $y_k \neq x$ since $d_1((y_k), x_k) < d_1(x_k, T(x))$.
Hence $0 < d_1(y_k, T(x)) < 1/k$ which, by Lemma 2, contradicts the fact that $T(x)$ is an isolated point of $T(X)$. $\square$

Theorem 7. *Let (X, d) be a d-metric space, (X_1, d_1) and (X_2, d_2) completion of (X, d), and $T_i : (X, d) \to (X_i, d_i)$ $(i = 1, 2)$ isometrics such that $T_i(x)$ is dense in X_i. Then there exists an isodistance $T : (X_1, d_1) \xrightarrow{\text{on to}} (X_2, d_2)$ such that following diagram is commutative.*

Proof. Consider the following:

$$(X, d) \xrightarrow{T_1} (X_1, d_1) \qquad T_2 \searrow \qquad \downarrow T \qquad (X_2, d_2) \qquad (8)$$
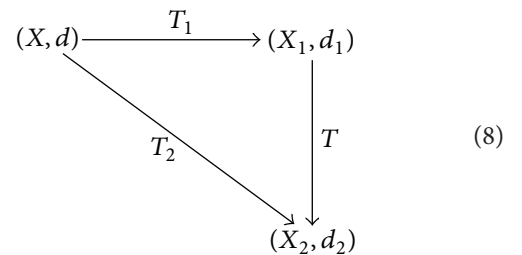
Definition of T. If $x \in X_1$ and x is an isolated point of X_1, then $T_1^{-1}(x)$ is an isolated point of X; hence $T_2(T_1^{-1}(x))$ is an isolated point of X_2.
Define $T(x) = T_2(T_1^{-1}(x))$. If $x \in X_1$ and is not an isolated point, there exists a sequence (z_n) in X such that $\{T_1 z_n\}$ converges to x in (X_1, d_1).
Since T_1 is an isodistance and $\{T_1 z_n\}$ is convergent and hence a Cauchy sequence, it follows that $\{z_n\}$ is a Cauchy sequence in X. Since T_2 is an isodistance and $\{z_n\}$ is a Cauchy

sequence, it follows that $\{T_2 z_n\}$ is a Cauchy sequence in (X_2, d_2). Since (X_2, d_2) is complete, there exists $z \in X_2$ such that $\lim d_2(T_2 z_n, z) = 0$. Clearly this z is independent of the choice of the sequence $\{z_n\}$ in X.

Define $T(x) = z$. Clearly $TT_1 = T_2$ and bijection.

T Is an Isodistance. If $x, y \in X$, $T(T_1(x)) = T_2(x)$ and $T(T_1(y)) = T_2(y)$.

So $d_2(T(T_1(x)), T(T_1(y))) = d_2((T_2(x)), T_2(y)) = d_2(x, y) = d_1((T_1(x)), T_1(y))$.

If $x, y \in X_1 - X$ and $x = \lim T_1 x_n$, $y = \lim T_1 y_n$ where $x_n, y_n \in X$, then

$$\begin{aligned} d_2(Tx, Ty) &= d_2\left(\lim T_2 x_n, \lim T_2 y_n\right) \\ &= \lim d_2\left(\lim T_2 x_n, \lim T_2 y_n\right) \\ &= \lim d\left(x_n, y_n\right) \\ &= d_1\left(\lim T_1 x_n, \lim T_1 y_n\right) \\ &= d_1\left(x, y\right). \end{aligned} \tag{9}$$

The arguments for the cases when $x \in X_1 - X$ and $y \in X$ or $x \in X$ and $y \in X_1 - X$ are similar. Hence T is an isodistance. Interchanging the places of X_1 and X_2, we get in a similar way an isodistance $S : X_2 \to X_1$ such that $ST_2 = T_1$.

Since $ST_2 = T_1$ and $TT_1 = T_2$, we have $TST_2 = TT_1$ and $STT_1 = ST_2 = T_1$.

Since $T(X)$ is dense in X_1 and $T_2(x)$ in X_2, we get $TS =$ identity on X_1 and ST is identity on X_2.

Hence S and T are bijections. $\qquad\square$

3. Completion of the Metric Associated with a d-Metric

If d is a d-metric on X then d' is a metric on X if d' is defined by $d'(x, y) = d(x, y)$ when $x \neq y$ and $d'(x, y) = 0$ for all x, y in X.

Suppose $(\overline{X}, \overline{d})$ is the completion of (X, d); then $\overline{d}$ gives rise to a metric $\overline{d}'$ defined by $\overline{d}'(x, y) = \overline{d}(x, y)$ for all $x, y \in \overline{X}$ and $\overline{d}(x, x) = 0$ for all $x, y \in \overline{X}$.

Also, the metric space (X, d') has a metric space (X_0, d_0) as its completion. In this section, we prove that the metric spaces $(\overline{X}, \overline{d}')$ and (X_0, d_0) are isometric.

Definition 8. Let X, d be a d-metric space. Define ρ on $X \times X$ by

$$\rho(x, y) = \begin{cases} d(x, y), & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases} \tag{10}$$

ρ is a metric on X and is called the metric associated with d.

Clearly $0 \leq \rho(x, y) \leq d(x, y) \ \forall x, y$ and $d(x, x) \leq 2\rho(x, y)$ whenever $x \neq y$. If $s \in \{\rho, d\}$, $r > 0$ and $x \in X$.

Write $\mathscr{B}_r^s(x) = \{y / s(x, y) < r\}$. Then $\mathscr{B}_r^\rho(x) = \mathscr{B}_r^d(x) \cup \{x\}$ and $\mathscr{V}_r^s(x) = \mathscr{B}_r^s(x) \cup \{x\}$.

The collection $\{\mathscr{V}_r^d(x) / x \in X, r > 0\}$ and $\mathscr{V}_r^\rho(x) = \{\mathscr{V}_r^\rho(x) / x \in X, r > 0\}$ generate the same topology on X.

However, convergent sequences in X are not necessarily the same since constant sequences are convergent sequences with respect to ρ, while this holds with respect to d for x with $d(x, x) = 0$ only.

Existence of points with positive self-distance leads to unpleasantness in the extension of the concept of continuity in metric spaces as well. This is evident from the following.

Example 9. Let d be a d-metric on a set X which is not a metric. So that the set $A = \{x / d(x, x) \neq 0\}$ is nonempty. If ρ is a metric associated with d then the identity map $i : (X, \rho) \to (X, d)$ is continuous in the usual sense. But if $x \in A$, the constant sequence (x) converges in (X, ρ) while it does not converge in (X, d).

If $(X, d), (Y, \rho)$ we call $f : X \to Y$ sequentially d-continuous if $\lim d(x_n, x) = 0 \Rightarrow \lim \rho(f(x_n), f(x)) = 0$.

If $s \in \{\rho, d\}$ and $\{x_n\}$ is a sequence in X, we say that X is s-Cauchy sequence or simply s-Cauchy if $\{x_n\}$ is a Cauchy sequence in (X, s).

Proposition 10. $\lim \rho(x_n, x) = 0 \Leftrightarrow$ *either*

(i) $x_n = x$ *eventually or*

(ii) (x_n) *can be split into subsequences* (y_n) *and* (z_n) *where* $y_n = x$ *for every n, $z_n \neq x$ for any n and $\lim d(z_n, x) = 0$.*

Proof. Routine. $\qquad\square$

Proposition 11. *If a sequence* (x_n) *in X is d-Cauchy then* (x_n) *is ρ-Cauchy. Conversely if* (x_n) *is ρ-Cauchy and is not eventually constant, then* (x_n) *is d-Cauchy.*

Proof. Since $0 \leq \rho(x_n, x_m) \leq d(x_n, x_m)$, d-Cauchy $\Rightarrow \rho$-Cauchy.

Conversely suppose that (x_n) is ρ-Cauchy, given $\epsilon > 0 \ \exists \ \mathscr{N}(\epsilon)$ such that $\rho(x_n, x_m) < \epsilon$ if $n \geq \mathscr{N}(\epsilon)$ and $m \geq \mathscr{N}(\epsilon)$. So if $m \geq \mathscr{N}(\epsilon)$, $n \geq \mathscr{N}(\epsilon)$, and $x_n \neq x_m$, then $d(x_m, x_n) < \epsilon$.

Since (x_n) is not eventually constant and $n \geq \mathscr{N}(\epsilon)$, there exists $m \geq \mathscr{N}(\epsilon)$ such that $x_m \neq x_n$. Then

$$\begin{aligned} d(x_n, x_n) &\leq d(x_n, x_m) \, d(x_m, x_n) \\ &= 2d(x_m, x_n) \\ &= 2d(x_n, x_n) \\ &< 2\epsilon. \end{aligned} \tag{11}$$

Thus if (x_n) is not eventually constant then for all $n \geq \mathscr{N}(\epsilon)$ and $m \geq \mathscr{N}(\epsilon)$, $d(x_n, x_m) < 2\epsilon$. Hence (x_n) is d-Cauchy. $\qquad\square$

Example 12. Let $X = (0, \infty)$ and $d(x, y) = x + y$; then

$$\rho(x, y) = \begin{cases} x + y, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases} \tag{12}$$

If (x_n) is any eventually nonconstant sequence in $(0, \infty)$, then (x_n) is d-Cauchy if and only if $\forall \epsilon > o$ there exists $\mathscr{N}(\epsilon)$ such that $x_n + x_m < \epsilon$ for $n \geq m \geq \mathscr{N}(\epsilon)$. This implies that $\lim x_n = 0$.

However, if $\lim x_n = 0$, then $\forall \epsilon > 0 \; \exists \; \mathcal{N}(\epsilon)$, such that $x_n < \epsilon/2$ for $m \geq \mathcal{N}(\epsilon)$, $n \geq \mathcal{N}(\epsilon)$.

Hence $x_n + x_m < \epsilon$ for $m \geq \mathcal{N}(\epsilon)$, $n \geq \mathcal{N}(\epsilon)$. However, constant sequences are not d-Cauchy but ρ-Cauchy.

Theorem 13. *Let (X, d) be a metric space, ρ the metric associated with d on X, (X^*, d^*) the completion of (X, d), and ρ^* the metric associated with d^* on X^*. Then (X^*, ρ^*) is the completion of (X, ρ). In particular if (X, d) is a complete metric space then (X, ρ) is a complete metric space. We prove that*

(i) *X is dense in (X^*, ρ^*);*

(ii) *every ρ^*-Cauchy sequence in X^* is ρ^*-convergent.*

Proof of (i). Let $x^* \in X^* - X$. Then there exists a sequence (x_n) in X such that $\lim d^*(x_n^*, x^*) = 0$ since $x_n \in X$, $x_n \neq x^*$ $\forall n$. So that $\lim \rho^*(x_n^*, x^*) = 0$.

This implies that X is dense in (X^*, ρ^*). □

Proof of (ii). Let (x_n^*) be ρ^*-Cauchy in X^*. If (x_n^*) is eventually constant, then there exist N and $x^* \in X^*$ such that $x_n^* = x^*$ for $n \geq N$.

In this case $\lim \rho^*(x_n^*, x^*) = 0$ for $n \geq N$; hence (x_n^*) is ρ^*-convergent.

Suppose (x_n^*) is not eventually constant. Then (x_n^*) is a d^*-Cauchy sequence. Since (X^*, d^*) is complete, there exists $x^* \in X^*$ such that $\lim d^*(x_n^*, x^*) = 0$. Since $0 \leq \rho^*(x_n^*, x^*) \leq d^*(x_n^*, x^*) = 0$, $\lim \rho^*(x_n^*, x^*) = 0$.

Hence (x_n^*) is ρ^*-convergent to x^*. This completes the proof of (ii). □

Disclosure

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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