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Research Article

Stochastic Dynamics of an SIRS Epidemic Model with Ratio-Dependent Incidence Rate

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We investigate the complex dynamics of an epidemic model with nonlinear incidence rate of saturated mass action which depends on the ratio of the number of infectious individuals to that of susceptible individuals. We first deal with the boundedness, dissipation, persistence, and the stability of the disease-free and endemic points of the deterministic model. And then we prove the existence and uniqueness of the global positive solutions, stochastic boundedness, and permanence for the stochastic epidemic model. Furthermore, we perform some numerical examples to validate the analytical findings. Needless to say, both deterministic and stochastic epidemic models have their important roles.

1. Introduction

Since the pioneer work of Kermack and McKendrick [1], mathematical models are used extensively in analyzing the spread, and control of infectious diseases qualitatively and quantitatively. The research results are helpful for predicting the developing tendencies of the infectious disease, for determining the key factors of the disease spreading, and for seeking the optimum strategies for preventing and controlling the spread of infectious diseases [2]. And in modeling communicable diseases, the incidence function has been considered to play a key role in ensuring that the models indeed give reasonable qualitative description of the transmission dynamics of the diseases [3–7].

Let $S(t)$ be the number of susceptible individuals, $I(t)$ the number of infective individuals, and $R(t)$ the number of removed individuals at time t, respectively. We consider the general SIRS epidemic model:

$$\begin{aligned}\frac{dS}{dt} &= b - dS - H(I,S) + \gamma R,\\ \frac{dI}{dt} &= H(I,S) - (d+\mu+\delta) I,\\ \frac{dR}{dt} &= \mu I - (d+\gamma) R,\end{aligned} \tag{1}$$

where b is the recruitment rate of the population, d is the natural death rate of the population, μ is the natural recovery rate of the infective individuals, γ is the rate at which recovered individuals lose immunity and return to the susceptible class, and δ is the disease-induced death rate. And the transmission of the infection is governed by an incidence rate $H(I,S)$.

In [8], Liu et al. proposed the general saturated nonlinear incidence rate:

$$H(I,S) = Sg(I), \qquad g(I) = \frac{kI^l}{1+\alpha I^h}, \tag{2}$$

where the parameters l and h are positive constants, k the proportionality constant, and α is a nonnegative constant, which measures the psychological or inhibitory effect. kI^l measures the infection force of the disease, and $1/(1+\alpha I^h)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. And the other nonlinear incidence rates are considered in [6, 9–19].

Note that the infectious force $g(I)$ of classical disease transmission models typically is only a function of infective individuals. But in the transmission of communicable diseases, it involves both infective individuals and susceptible individuals. Thus, Yuan et al. [18, 19] studied the infections force function with a ratio-dependent nonlinear incident rate which takes the following form:

$$g\left(\frac{I}{S}\right) = \frac{k(I/S)^l}{1+\alpha(I/S)^h}. \tag{3}$$

And in [19], Li et al. focus on an epidemic disease of SIRS type, in which they assume that the infectious force takes the form of (3) with $l = 1$ and $h = 1$, and the model is as follows:

$$\begin{aligned} \frac{dS}{dt} &= b - dS - \frac{kIS}{S+\alpha I} + \gamma R, \\ \frac{dI}{dt} &= \frac{kIS}{S+\alpha I} - (d+\mu) I, \\ \frac{dR}{dt} &= \mu I - (d+\gamma) R, \end{aligned} \tag{4}$$

where all the parameters are nonnegative and have the same definitions as in model (1).

From the standpoint of epidemiology, we are only interested in the dynamics of model (4) in the closed first quadrant $\mathbb{R}_+^3 = \{(S, I, R) : S \geq 0,\ I \geq 0,\ R \geq 0\}$. Thus, we consider only the epidemiological meaningful initial conditions $S(0) > 0$, $I(0) > 0$, $R(0) > 0$. Straightforward computation shows that model (4) is continuous and Lipschizian in $\mathbb{R}_+^3$ if we redefine that when $(S, I, R) = (0, 0, 0)$, $dS/dt = b$, $dI/dt = 0$, $dR/dt = 0$. Hence, the solution of model (4) with positive initial conditions exists and is unique.

It is clear that the limit set of model (4) is on the plane $S+I+R = b/d$, and the model can be reduced to the following:

$$\begin{aligned} \frac{dS}{dt} &= \left(b + \frac{\gamma b}{d}\right) - (d+\gamma) S - \gamma I - \frac{kIS}{S+\alpha I}, \\ \frac{dI}{dt} &= \frac{kIS}{S+\alpha I} - (d+\mu) I, \end{aligned} \tag{5}$$

when $(S, I) = (0, 0)$, $dS/dt = b + (\gamma b/d)$, $dI/dt = 0$.

For mathematical simplicity, let us nondimensionalize model (5) as in [19] with the following scaling:

$$x = \frac{d(d+\mu)}{b(d+\gamma)} S, \qquad y = \frac{d\gamma}{b(d+\gamma)} I, \qquad \tau = (d+\mu) t. \tag{6}$$

We still use variable t instead of τ, and model (5) takes the following form:

$$\begin{aligned} \frac{dx}{dt} &= 1 - qx - y - \frac{axy}{x+py}, \\ \frac{dy}{dt} &= \left(\frac{R_0 x}{x+py} - 1\right) y, \end{aligned} \tag{7}$$

where $q = (d+\gamma)/(d+\mu)$, $p = \alpha(d+\mu)/\gamma$, $a = k/\gamma$ are positive constants. $R_0 = k/(d+\mu)$ is the basic reproduction number. And when $(S, I) = (0, 0)$, $dx/dt = 1$, $dy/dt = 0$.

On the other hand, if the environment is randomly varying, the population is subject to a continuous spectrum of disturbances [20, 21]. That is to say, population systems are often subject to environmental noise; that is, due to environmental fluctuations, parameters involved in epidemic models are not absolute constants, and they may fluctuate around some average values. Based on these factors, more and more people began to be concerned about stochastic epidemic models describing the randomness and stochasticity [22–34], and the stochastic epidemic models can provide an additional degree of realism if compared to their deterministic counterparts [10, 35–47]. In Particular, Mao et al. [26] obtained the interesting and surprising conclusion: even a sufficiently small noise can suppress explosions in population dynamics. Beretta et al. [35] obtained the stability of epidemic model with stochastic time delays influenced by probability under certain conditions. Carletti [36] studied the stable properties of a stochastic model for phage-bacteria interaction in open marine environment analytically and numerically. In [37], establishing some stochastic models and studying of several endemic infections with demography, Nåsell found that some deterministic models are unacceptable approximations of the stochastic models for a large range of realistic parameter values. Dalal et al. [39, 40] showed that stochastic models had nonnegative solutions and carried out analysis on the asymptotic stability of models. In [41], Yu et al. presented stochastic asymptotic stability of the epidemic point of the two-group SIR model with random perturbation. It is shown in [45] that the SIR model has a unique global positive and asymptotic solution. But to our knowledge, the research on the stochastic dynamics of the epidemic model with ratio-dependent nonlinear incidence rate seems rare.

There are different possible approaches to including random effects in the model, both from a biological and from a mathematical perspectives [48]. Our basic approach is analogous to that of Beddington and May [20], which is pursued in [48], and also, for example, in [45, 47] to epidemic models, in which they considered that the environmental noise was proportional to the variables. Following them, in this paper, we assume that stochastic perturbations are of a white noise type which is directly proportional to $x(t)$, $y(t)$, influenced on the $dx(t)/dt$ and $dy(t)/dt$ in model (4). In this way, we introduce stochastic perturbation terms into the growth equations of susceptible and infected individuals to incorporate the effect of randomly fluctuating environment, and the following stochastic differential equation is corresponding to model (7):

$$\begin{aligned} dx &= \left(1 - qx - y - \frac{axy}{x+py}\right) dt + \sigma_1 x dB_1(t), \\ dy &= \left(\frac{R_0 xy}{x+py} - y\right) dt + \sigma_2 y dB_2(t), \end{aligned} \tag{8}$$

where σ_1, σ_2 are real constants and known as the intensity of environmental fluctuations, and $B_1(t)$, $B_2(t)$ are independent standard Brownian motions.

The aim of this paper is to consider the dynamics of the epidemic models (7) and (8). The paper is organized as follows. In Section 2, we give some properties about deterministic model (7). In Section 3, we carry out the analysis of the dynamical properties of stochastic model (8). And in Section 4, we give some numerical examples and make a comparative analysis of the stability of the model with deterministic and stochastic environments and have some discussions.

2. Dynamics of the Deterministic Model

Let us begin to determine the location and number of the equilibria of model (7). It is easy to see that if $R_0 < 1$, the disease-free point $E_0 = (1/q, 0)$ is the unique equilibrium, corresponding to the extinction of the disease; if $R_0 > 1$, in addition to the disease-free point E_0, there is a unique endemic point $E^* = (x^*, y^*)$, corresponding to the survival of the disease, described by the following expressions:

$$x^* = \frac{pR_0}{pqR_0 + (R_0 + a)(R_0 - 1)}, \qquad y^* = \frac{R_0 - 1}{p}x^*. \tag{9}$$

The Jacobian matrix of model (7) at E_0 is as follows:

$$\begin{pmatrix} -q & -1-a \\ 0 & R_0 - 1 \end{pmatrix}. \tag{10}$$

It follows that E_0 is asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

The Jacobian matrix of model (7) at E^* is as follows:

$$J^* = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}, \tag{11}$$

where

$$\begin{gathered} J_{11} = -\frac{pqR_0^2 + a(R_0 - 1)^2}{pR_0^2}, \qquad J_{12} = -\frac{R_0^2 + a}{R_0^2}, \\ J_{21} = \frac{(R_0 - 1)^2}{pR_0}, \qquad J_{22} = -\frac{R_0 - 1}{R_0}. \end{gathered} \tag{12}$$

It is easy, by simple computations, to see that

$$\begin{gathered} \operatorname{tr}(J^*) = J_{11} + J_{22} < 0, \\ \det(J^*) = \frac{pqR_0 + (a + R_0)(R_0 - 1)}{pR_0^2} > 0. \end{gathered} \tag{13}$$

Summarizing the above, we have the following results on the dynamics of model (7).

Theorem 1. (i) *If $R_0 < 1$, then model* (7) *has a unique disease-free equilibrium E_0 which is asymptotically stable.*

(ii) *If $R_0 > 1$, then model* (7) *has two equilibria, a disease-free equilibrium E_0 which is an unstable saddle and an endemic equilibrium E^* which is asymptotically stable.*

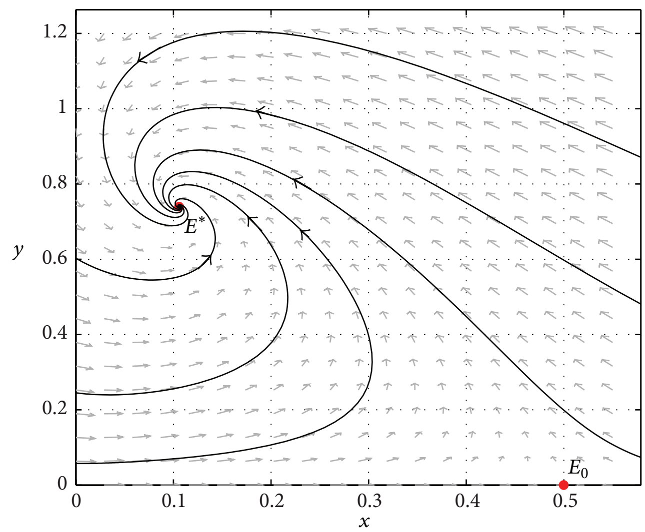

Figure 1: The dynamics of model (7). The parameters are taken as (14). $E_0 = (0.5, 0)$ is a saddle point. $E^* = (0.11, 0.74)$ is globally asymptotically stable.

As a matter of fact, we can prove that the endemic point $E^* = (x^*, y^*)$ is also global asymptotically stable. For more details, see [19].

In Figure 1, we show the dynamics of the deterministic model (7) with the following parameters:

$$a = 0.3, \qquad p = 0.5, \qquad q = 2, \qquad R_0 = 4.5. \tag{14}$$

In this case, $E_0 = (0.5, 0)$ is a saddle point. $E^* = (0.10563, 0.73944)$ is globally asymptotically stable.

In the following, we will focus on the boundedness, dissipation, and persistence of mode (7).

Theorem 2. *All the solutions of model* (7) *with the positive initial condition $(x(0), y(0))$ are uniformly bounded within a region Γ, where*

$$\Gamma = \left\{ (x, y) \in \mathbb{R}_+^2 : x + \frac{a}{R_0}y \le \min\left\{\frac{1}{q}, \frac{R_0}{R_0 + a}\right\} \right\}. \tag{15}$$

Proof. Define function

$$N(t) = x(t) + \frac{a}{R_0}y(t). \tag{16}$$

Differentiating $N(t)$ with respect to time t along the solutions of model (7), we can get the following:

$$\frac{dN(t)}{dt} = \frac{dx}{dt} + \frac{a}{R_0}\frac{dy}{dt} = 1 - qx - \left(1 + \frac{a}{R_0}\right)y. \tag{17}$$

Thus, we obtain the following:

$$\frac{dN(t)}{dt} + \eta N(t) = 1 - (q - \eta)x - \left(1 + \frac{a}{R_0} - \eta\right)y < 1, \tag{18}$$

where $\eta < \min\{q, 1 + (a/R_0)\}$. And we obtain the following:

$$0 < N(x, y) \le \frac{1}{\eta} + N(x(0), y(0)) e^{-\eta t}. \tag{19}$$

As $t \to \infty$, $0 < N \le 1/\eta$. Therefore, all solutions of model (7) enter into the region Γ. This completes the proof. □

Theorem 3. *If $R_0 > 1$, model* (7) *is dissipative.*

Proof. Since all solutions of model (7) are positive, by the first equation of (7), we have the following:

$$\frac{dx}{dt} \le 1 - qx. \tag{20}$$

A standard comparison theorem shows that

$$\limsup_{t\to\infty} x(t) \le \frac{1}{q}. \tag{21}$$

Hence, for any $0 < \varepsilon \ll 1$ and large t, $x \le (1/q) + \varepsilon$. It then follows that y satisfies the following:

$$\frac{dy}{dt} \le \frac{y(((R_0 - 1)/q) + \varepsilon(R_0 - 1) - py)}{(1/q) + \varepsilon + py}. \tag{22}$$

The arbitrariness of ε then implies that

$$\limsup_{t\to\infty} y(t) \le \frac{R_0 - 1}{pq}. \tag{23}$$

□

Theorem 4. *If $R_0 > 1$ and $pq < (1 + a)(R_0 - 1)$, then model* (7) *is permanent; that is, there exists $\varepsilon > 0$ (independent of initial conditions), such that $\liminf_{t\to\infty} x(t) > \varepsilon$, $\liminf_{t\to\infty} y(t) > \varepsilon$.*

Proof. By the first equation in (7), we have the following:

$$\begin{aligned}\frac{dx}{dt} &= 1 - qx - (1 + a) y + \frac{apy^2}{x + py} \\ &> 1 - qx - (1 + a) y.\end{aligned} \tag{24}$$

If $R_0 > 1$ and $pq < (1 + a)(R_0 - 1)$, from the proof of Theorem 3, we see that $\limsup_{t\to\infty} y(t) \le (R_0 - 1)/pq$. Thus, for any $0 < \varepsilon \le (R_0 - 1)/pq$ and large t, $y(t) > ((R_0 - 1)/pq) - \varepsilon$. As a result, we have the following:

$$\frac{dx}{dt} > 1 - (1 + a)\left(\frac{R_0 - 1}{pq} - \varepsilon\right) - qx. \tag{25}$$

With the comparison principle, the arbitrariness of ε implies that

$$\liminf_{t\to\infty} x(t) \ge \frac{pq - (1 + a)(R_0 - 1)}{pq^2} \triangleq \underline{x}. \tag{26}$$

Hence, for any $0 < \varepsilon < (pq - (1 + a)(R_0 - 1))/pq^2$ and large t, $x(t) > \underline{x} - \varepsilon$.

And for large t, we have the following:

$$\frac{dy}{dt} > \frac{y((\underline{x} - \varepsilon)(R_0 - 1) - py)}{\underline{x} - \varepsilon + py}. \tag{27}$$

Therefore,

$$\liminf_{t\to\infty} x(t) \ge \frac{(\underline{x} - \varepsilon)(R_0 - 1)}{p}. \tag{28}$$

The arbitrariness of ε then implies that

$$\liminf_{t\to\infty} x(t) \ge \frac{\underline{x}(R_0 - 1)}{p} \triangleq \underline{y}. \tag{29}$$

Choosing a positive number ϵ such that $\epsilon < \min\{\underline{x}/2, \underline{y}/2\}$, we see that

$$\liminf_{t\to\infty} x(t) > \epsilon, \qquad \liminf_{t\to\infty} y(t) > \varepsilon. \tag{30}$$

This ends the proof. □

Noting that if the parameters of model (7) are fixed as (14), we can obtain the following:

$$R_0 > 1, \qquad pq = 0.6 < (1 + a)(R_0 - 1) = 4.55, \tag{31}$$

and from Theorems 3 and 4, we can conclude that model (7) is dissipation and persistence.

3. Dynamics of the Stochastic Model

In this subsection, we investigate the dynamical behavior of the stochastic model (8). Throughout this paper, let $(\Omega, \mathscr{F}, \mathscr{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\in\mathbb{R}_+}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathscr{F}_0$ contains all $\mathscr{P}$-null sets). $B_1(t)$, $B_2(t)$ are the Brownian motions defined on this probability space. We denote by $X(t) = ((x(t), y(t))$ and $|X(t)| = (x^2(t) + y^2(t))^{1/2}$. Denote $\Lambda = \{(x, y) \in \mathbb{R}_+^2 : x \ge a/R_0, y > 0\}$.

Denote by $C^{2,1}(\mathbb{R}^d \times (0, \infty); \mathbb{R}_+)$ the family of all nonnegative functions $V(x, t)$ defined on $\mathbb{R}^d \times (0, \infty)$ such that they are continuously twice differentiable in x and once in t. Define the differential operator L associated with d-dimensional stochastic differential equation:

$$dx(t) = f(x(t), t) dt + h(x(t), t) dB(t) \tag{32}$$

by

$$\begin{aligned}L &= \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x, t) \frac{\partial}{\partial x_i} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{d} \left[h^T(x, t) h(x, t)\right]_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.\end{aligned} \tag{33}$$

If L acts in a function $V \in C^{2,1}(\mathbb{R}^d \times (0, \infty); \mathbb{R}_+)$, then

$$\begin{aligned}LV(x, t) &= V_t(x, t) + V_x(x, t) f(x, t) \\ &\quad + \frac{1}{2}\mathrm{trace}\left[h^T(x, t) V_{xx}(x, t) h(x, t)\right],\end{aligned} \tag{34}$$

where T means transposition.

3.1. Existence and Uniqueness of Global Positive Solutions. To investigate the dynamical behavior of model (8), the first thing concerned is whether the solution is global existent. In this section, using the Lyapunov analysis method (mentioned in [24]), we will show the solution of model (8) is global and nonnegative.

Lemma 5. *There is a unique local positive solution $(x(t), y(t))$ for $t \in [0, \tau_e)$ to model (8) almost surely (a.s.) for the initial value $(x(0), y(0)) \in \Lambda$, where τ_e is the explosion time.*

Proof. Set

$$u(t) = \ln x(t), \qquad v(t) = \ln y(t), \tag{35}$$

by Itô formula, we have the following:

$$\begin{aligned} du &= \left(\frac{1}{e^u} - q - \frac{e^v}{e^u} - \frac{ae^v}{e^u + pe^v} - \frac{\sigma_1^2}{2}\right) dt + \sigma_1 dB_1(t), \\ dv &= \left(\frac{R_0 e^u}{e^u + pe^v} - 1 - \frac{\sigma_2^2}{2}\right) dt + \sigma_2 dB_2(t), \end{aligned} \tag{36}$$

at $t \geq 0$ with initial value $u(0) = \ln x(0)$, $v(0) = \ln y(0)$.

It is easy to see that the coefficients of model (36) satisfy the local Lipschitz condition, and there is a unique local solution $u(t)$, $v(t)$ on $[0, \tau_e)$ [24]. Therefore, $x(t) = e^{u(t)}$, $y(t) = e^{v(t)}$ are the unique positive local solutions to model (36) with the initial value $(x(0), y(0)) \in \Lambda$. □

Lemma 5 only tells us that there exists a unique local positive solution to model (8). In the following, we show this solution is global; that is, $\tau_e = \infty$, which is motived by the work of Luo and Mao [29].

Theorem 6. *Consider model (8), for any given initial value $(x(0), y(0)) \in \Lambda$, there is a unique solution $(x(t), y(t))$ on $t \geq 0$ and the solution will remain in Λ with probability 1.*

Proof. Let $n_0 > 0$ be sufficiently large for $x(0)$ and $y(0)$ lying within the interval $[1/n_0, n_0]$. For each integer $n > n_0$, define the stopping times:

$$\tau_n = \inf\left\{t \in [0, \tau_e] : x(t) \notin \left(\frac{1}{n}, n\right) \text{ or } y(t) \notin \left(\frac{1}{n}, n\right)\right\}. \tag{37}$$

We set $\inf \emptyset = \infty$ ($\emptyset$ represents the empty set) in this paper. τ_n is increasing as $n \to \infty$. Let $\tau_\infty = \lim_{n\to\infty} \tau_n$; then $\tau_\infty \leq \tau_e$ a.s..

In the following, we need to show $\tau_\infty = \infty$ a.s. If this statement is violated, there exist constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathcal{P}\{\tau_\infty \leq T\} > \varepsilon$. As a consequence, there exists an integer $n_1 \geq n_0$ such that

$$\mathcal{P}\{\tau_n \leq T\} \geq \varepsilon, \quad n \geq n_1. \tag{38}$$

Define a function $V_1 : \Lambda \to \mathbb{R}_+$ by the following:

$$V_1(x, y) = \left(\frac{R_0}{a}x - 1 - \ln \frac{R_0}{a}x\right) + (y - 1 - \ln y), \tag{39}$$

which is a non-negativity function.

If $(x(t), y(t)) \in \Lambda$, by the Itô formula, we compute the following:

$$\begin{aligned} dV_1 &= \left[\left(\frac{R_0}{a} - \frac{1}{x}\right)\left(1 - qx - y - \frac{axy}{x + py}\right) + \frac{\sigma_1^2}{2}\right] dt \\ &\quad + \sigma_1 \left(\frac{R_0}{a}x - 1\right) dB_1(t) \\ &\quad + \left[\left(1 - \frac{1}{y}\right)\left(\frac{R_0 x}{x + py} - 1\right) y + \frac{\sigma_2^2}{2}\right] dt \\ &\quad + \sigma_2 (y - 1) dB_2(t) \\ &= LV_1 dt + \sigma_1 \left(\frac{R_0}{a}x - 1\right) dB_1(t) \\ &\quad + \sigma_2 (y - 1) dB_2(t), \end{aligned} \tag{40}$$

where

$$\begin{aligned} LV_1 &= q + \frac{ay - R_0 x}{x + py} - \frac{R_0 q}{a} x \\ &\quad + \left(\frac{R_0}{a} - \frac{1}{x}\right)(1 - y) + 1 - y + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} \\ &\leq q + \frac{a}{p} + \left(\frac{R_0}{a} - \frac{1}{x}\right)(1 - y) \\ &\quad + 1 - y + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}. \end{aligned} \tag{41}$$

Case 1 (assume $a \geq R_0$). In this case, we have $x \geq 1$. It follows that

$$\begin{aligned} &\left(\frac{R_0}{a} - \frac{1}{x}\right)(1 - y) + 1 - y \\ &\quad \leq \frac{R_0}{a} + y\left(\frac{1}{x} - 1\right) - \frac{1}{x} + 1 \leq 1 + \frac{R_0}{a}. \end{aligned} \tag{42}$$

Case 2 (assume $a < R_0$). If $a/R_0 \leq x < 1$, one has the following:

(a) $((R_0/a) - (1/x))(1 - y) + 1 - y \leq (R_0/a) + ((y - 1)/x) + 1 - y \leq 1 + (R_0/a)$ provided that $0 < y \leq 1$;

(b) $((R_0/a) - (1/x))(1 - y) + 1 - y \leq 1$ provided that $y > 1$.

Hence, there exists a positive number M independent on x, y and t such that $LV_1 \leq M$. Substituting this inequality into (40), we can get the following:

$$\begin{aligned} dV_1 &\leq M dt + \sigma_1 \left(\frac{R_0}{a}x - 1\right) dB_1(t) \\ &\quad + \sigma_2 (y - 1) dB_2(t). \end{aligned} \tag{43}$$

Integrating both sides of the above inequality from 0 to $\tau_n \wedge T$ and taking expectations leads to the following:

$$EV_1(x(\tau_n \wedge T), y(\tau_n \wedge T)) \leq V_1(x(0), y(0)) + MT. \tag{44}$$

Set $\Omega_n = \{\tau_n \leq T\}$, for $n \geq n_1$ and consider inequality (38), we can get $\mathscr{P}(\Omega_n) \geq \varepsilon$. Note that for every $\omega \in \Omega_n$, there exists some i such that $x_i(\tau_n, \omega)$ equals either n or $1/n$ for $i = 1, 2$; hence,

$$\begin{aligned} & V_1\left(x\left(\tau_n, \omega\right), y\left(\tau_n, \omega\right)\right) \\ & \quad \geq \min \left\{(n-1-\ln n),\left(\frac{1}{n}-1-\ln \frac{1}{n}\right)\right\}. \end{aligned} \tag{45}$$

It then follows from (44) that

$$\begin{aligned} & V_1(x(0), y(0)) + MT \\ & \quad \geq E\left[I_{\Omega_n(\omega)} V_1\left(x\left(\tau_n\right), y\left(\tau_n\right)\right)\right] \\ & \quad \geq \epsilon \min \left\{(n-1-\ln n),\left(\frac{1}{n}-1-\ln \frac{1}{n}\right)\right\}, \end{aligned} \tag{46}$$

where I_{Ω_n} is the indicator function of Ω_n.

As $n \to \infty$ we have the following:

$$\infty > V_1(x(0), y(0)) + MT = \infty \quad \text{a.s.}, \tag{47}$$

which leads to the contradiction. This completes the proof. □

3.2. Stochastic Boundedness and Permanence. Theorem 6 shows that the solutions to model (8) will remain in Λ. Generally speaking, the nonexplosion property, the existence, and the uniqueness of the solution are not enough but the property of boundedness and permanence are more desirable since they mean the long-time survival in the population dynamics. Now, we present the definition of stochastic ultimate boundedness and stochastic permanence [31].

Definition 7. The solutions $X(t) = (x(t), y(t))$ of model (8) are said to be stochastically ultimately bounded, if for any $\varepsilon \in (0, 1)$, there is a positive constant $\delta = \delta(\varepsilon)$, such that for any initial value $(x(0), y(0)) \in \Lambda$, the solution $X(t)$ of model (8) has the property that

$$\limsup_{t \to \infty} P\{|X(t)| > \delta\} < \varepsilon. \tag{48}$$

Definition 8. The solutions $X(t) = (x(t), y(t))$ of model (8) are said to be stochastically permanent if for any $\varepsilon \in (0, 1)$, there exists a pair of positive constants $\delta = \delta(\varepsilon)$ and $\chi = \chi(\varepsilon)$, such that for any initial value $(x(0), y(0)) \in \Lambda$, the solution $X(t)$ of model (8) has the property that

$$\begin{aligned} & \liminf_{t \to \infty} P\{|X(t)| \geq \delta\} \geq 1-\varepsilon, \\ & \liminf_{t \to \infty} P\{|X(t)| \leq \chi\} \geq 1-\varepsilon. \end{aligned} \tag{49}$$

Theorem 9. *The solutions of model* (8) *are stochastically ultimately bounded for any initial value* $(x(0), y(0)) \in \Lambda$.

Proof. Denote functions

$$V_2 = e^t x^\theta, \qquad V_3 = e^t y^\theta \tag{50}$$

for $(x, y) \in \Lambda$ and $0 < \theta < 1$.

Applying the Itô formula leads to the following:

$$\begin{aligned} dV_2 &= LV_2 dt + \sigma_1 \theta e^t x^\theta dB_1(t), \\ dV_3 &= LV_3 dt + \sigma_2 \theta e^t y^\theta dB_2(t), \end{aligned} \tag{51}$$

where

$$\begin{aligned} LV_2 &= e^t x^\theta\left(1+\theta\left(\frac{1}{x}-q-\frac{y}{x}-\frac{ay}{x+py}\right)+\frac{\sigma_1^2 \theta(\theta-1)}{2}\right), \\ LV_3 &= e^t y^\theta\left(1+\theta\left(\frac{R_0 x}{x+py}-1\right)+\frac{\sigma_2^2 \theta(\theta-1)}{2}\right). \end{aligned} \tag{52}$$

Thus, there exists the positive constants M_1 and M_2 such that we have $LV_2 < M_1 e^t$ and $LV_3 < M_2 e^t$. It follows that $e^t Ex^\theta - Ex(0)^\theta \leq M_1 e^t$ and $e^t Ey^\theta - Ey(0)^\theta \leq M_2 e^t$. Then we get the following:

$$\begin{aligned} & \limsup_{t \to \infty} Ex^\theta \leq M_1 < +\infty, \\ & \limsup_{t \to \infty} Ey^\theta \leq M_2 < +\infty. \end{aligned} \tag{53}$$

Note that

$$\begin{aligned} |X(t)|^\theta &= \left(x^2(t)+y^2(t)\right)^{\theta/2} \\ & \leq 2^{\theta/2} \max\left\{x^\theta(t), y^\theta(t)\right\} \\ & \leq 2^{\theta/2}\left(x^\theta+y^\theta\right). \end{aligned} \tag{54}$$

Therefore, we obtain the following:

$$\limsup_{t \to \infty} E|X(t)|^\theta \leq 2^{\theta/2}\left(M_1+M_2\right) < +\infty. \tag{55}$$

As a result, there exists a positive constant δ_1 such that

$$\limsup_{t \to \infty} E\left(\sqrt{|X(t)|}\right) < \delta_1. \tag{56}$$

Now, for any $\varepsilon > 0$, let $\delta = \delta_1^2/\varepsilon^2$; then by Chebyshev's inequality,

$$\mathscr{P}\{|X(t)| > \delta\} \leq \frac{E\left(\sqrt{|X(t)|}\right)}{\sqrt{\delta}}. \tag{57}$$

Hence,

$$\limsup_{t \to \infty} \mathscr{P}\{|X(t)| > \delta\} \leq \frac{\delta_1}{\sqrt{\delta}} = \varepsilon, \tag{58}$$

which yields the required assertion. □

We are now in the position to show the stochastic permanence. Let us present some hypothesis and a useful lemma.

Lemma 10. *Assume $R_0 > a + \max\{4, 2pq\}$. For any initial value $(x(0), y(0)) \in \Lambda$, the solution $(x(t), y(t))$ satisfies that*

$$\limsup_{t\to\infty} E\left(\frac{1}{|X(t)|^{\rho}}\right) \le H, \tag{59}$$

where ρ is an arbitrary positive constant satisfying

$$\frac{\rho+1}{2}(\max\{\sigma_1,\sigma_2\})^2 < 1 + \min\left\{\frac{R_0-a}{2p} - q, \frac{R_0-a}{2} - 2\right\}, \tag{60}$$

$$H = \frac{2^{\rho}(C_2 + 4kC_1)}{4kC_1} \times \max\left\{1, \left(\frac{2C_1 + C_2 + \sqrt{C_2^2 + 4C_1C_2}}{2C_1}\right)^{\rho-2}\right\} \tag{61}$$

in which k is an arbitrary positive constant satisfying

$$\frac{\rho(\rho+1)}{2}(\max\{\sigma_1,\sigma_2\})^2 + k < \rho + \rho\min\left\{\frac{R_0-a}{2p} - q, \frac{R_0-a}{2} - 2\right\} \tag{62}$$

with

$$\begin{aligned} C_1 &= \rho + \rho\min\left\{\frac{R_0-a}{2p} - q, \frac{R_0-a}{2} - 2\right\} - \frac{\rho(\rho+1)}{2}(\max\{\sigma_1,\sigma_2\})^2 - k > 0, \\ C_2 &= \rho\max\{q, 2+a\} + \frac{\rho R_0(R_0-1)\max\{1,p^2\}}{2ap} + \rho(\max\{\sigma_1,\sigma_2\})^2 + 2k > 0. \end{aligned} \tag{63}$$

Proof. Set $U(x,y) = 1/(x+y)$ for $(x(t), y(y)) \in \Lambda$, by the Itô formula, we have the following:

$$\begin{aligned} dU &= -U^2\left[1 - qx - y - \frac{axy}{x+py} + \frac{R_0xy}{x+py} - y\right]dt \\ &\quad + U^3\left(\sigma_1^2x^2 + \sigma_2^2y^2\right)dt \\ &\quad - U^2\left(\sigma_1 x dB_1(t) + \sigma_2 y dB_2(t)\right) \\ &= LU\,dt - U^2\left(\sigma_1 x dB_1(t) + \sigma_2 y dB_2(t)\right), \end{aligned} \tag{64}$$

where

$$LU = -U^2\left(1 - qx - 2y + \frac{(R_0-a)xy}{x+py}\right) + U^3\left(\sigma_1^2x^2 + \sigma_2^2y^2\right). \tag{65}$$

Choose a positive constant ρ such that it satisfies (60). Applying the Itô formula again, we can get the following:

$$\begin{aligned} L\left[(1+U)^{\rho}\right] &= \rho(1+U)^{\rho-1}LU + \frac{\rho(\rho-1)}{2}U^4(1+U)^{\rho-2}\left(\sigma_1^2x^2 + \sigma_2^2y^2\right) \\ &= (1+U)^{\rho-2}\Phi, \end{aligned} \tag{66}$$

where

$$\begin{aligned} \Phi &= -\rho U^2\left(1 - qx - 2y + \frac{(R_0-a)xy}{x+py}\right) \\ &\quad - \rho U^3\left(1 - qx - 2y + \frac{(R_0-a)xy}{x+py}\right) \\ &\quad + \rho U^3\left(\sigma_1^2x^2 + \sigma_2^2y^2\right) \\ &\quad + \frac{\rho(1+\rho)U^4}{2}\left(\sigma_1^2x^2 + \sigma_2^2y^2\right) \\ &\le -\rho U^2 + \rho U^2(qx + (2+a)y) \\ &\quad - \rho U^3\left(\left(\frac{R_0-a}{2p} - q\right)x + \left(\frac{R_0-a}{2} - 2\right)y\right) \\ &\quad + \rho U^3\left(\frac{(R_0-a)(x^2+p^2y^2)}{2p(x+py)}\right) \\ &\quad + \rho U^3\left(\sigma_1^2x^2 + \sigma_2^2y^2\right) \\ &\quad + \frac{\rho(1+\rho)U^4}{2}\left(\sigma_1^2x^2 + \sigma_2^2y^2\right). \end{aligned} \tag{67}$$

Using the facts that

$$\begin{aligned} U^3\left(\sigma_1^2x^2 + \sigma_2^2y^2\right) &< (\max\{\sigma_1,\sigma_2\})^2U, \\ U^4\left(\sigma_1^2x^2 + \sigma_2^2y^2\right) &< (\max\{\sigma_1,\sigma_2\})^2U^2, \end{aligned} \tag{68}$$

so,

$$\begin{aligned} \Phi \le {} & -U^2\left(\rho + \rho\min\left\{\frac{R_0-a}{2p} - q, \frac{R_0-a}{2} - 2\right\} - \frac{\rho(\rho+1)}{2}(\max\{\sigma_1,\sigma_2\})^2\right) \\ & + U\left(\rho\max\{q, 2+a\} + \frac{\rho R_0(R_0-1)\max\{1,p^2\}}{2ap} + \rho(\max\{\sigma_1,\sigma_2\})^2\right). \end{aligned} \tag{69}$$

Now, let $k > 0$ sufficiently small such that it satisfies (62), by the Itô formula; then

$$\begin{aligned} L\left[e^{kt}(1+U)^{\rho}\right] &= ke^{kt}(1+U)^{\rho} + e^{kt}L(1+U)^{\rho} \\ &= e^{kt}(1+U)^{\rho-2}\left(k(1+U)^2+\Phi\right) \\ &\leq e^{kt}(1+U)^{\rho-2}\left(-C_1U^2+C_2u+k\right) \\ &\leq H_1e^{kt}, \end{aligned} \tag{70}$$

where $H_1 = ((C_2 + 4kC_1)/4C_1)\max\{1,((2C_1+C_2+\sqrt{C_2^2+4C_1C_2})/2C_1)^{\rho-2}\}$ and C_1, C_2 have been defined in the statement of the theorem. Thus,

$$E\left[e^{kt}(1+U)^{\rho}\right] \leq (1+U(0))^{\rho} + \frac{H_1}{k}e^{kt}. \tag{71}$$

So we can have the following:

$$\limsup_{t\to\infty} E\left[U(t)^{\rho}\right] \leq \limsup_{t\to\infty} E(1+U)^{\rho} \leq \frac{H_1}{k}. \tag{72}$$

In addition, we know that $(x+y)^{\rho} \leq 2^{\rho}(x^2+y^2)^{\rho/2} = 2^{\rho}|X(t)|^{\rho}$; consequently,

$$\begin{aligned} \limsup_{t\to\infty} E\left[\frac{1}{|X(t)|^{\rho}}\right] &\leq 2^{\rho}\limsup_{t\to\infty} E\left[U(t)^{\rho}\right] \\ &\leq \frac{2^{\rho}H_1}{k} = H, \end{aligned} \tag{73}$$

which complets the proof. □

Consider Chebyshev inequality, Theorem 9, and Lemma 10 together, we immediately obtain the following result.

Theorem 11. *If the following conditions are satisfied*

(i) $a+\max\{2pq,4\} < R_0$;

(ii) $(1/2)(\max\{\sigma_1,\sigma_2\})^2 < 1+\min\{((R_0-a)/2p)-q,((R_0-a)/2)-2\}$,

then the solutions of model (8) *is stochastically permanent.*

4. Conclusions and Discussions

In this paper, by using the theory of stochastic differential equation, we investigate the dynamics of an SIRS epidemic model with a ratio-dependent incidence rate. The value of this study lies in two aspects. First, it presents some relevant properties of the deterministic model (7), including boundedness, dissipation, persistence, and the stability of the disease-free and endemic points. Second, it verifies the existence of global positive solutions, stochastic boundedness, and permanence for the stochastic model (8).

As an example, we give some numerical examples to illustrate the dynamical behavior of stochastic model (8) by using the Milstein method mentioned in Higham [49]. In this way, model (8) can be rewritten as the following discretization equations:

$$\begin{aligned} x_{k+1} &= x_k + \left(1-qx_k-y_k-\frac{ax_ky_k}{x_k+py_k}\right)\Delta t \\ &\quad + \sigma_1x_k\sqrt{\Delta t}\xi_k + \frac{\sigma_1^2}{2}x_k^2\left(\xi_k^2-1\right)\Delta t, \\ y_{k+1} &= y_k + \left(\frac{R_0x_ky_k}{x_k+py_k}-y_k\right)\Delta t \\ &\quad + \sigma_2y_k\sqrt{\Delta t}\eta_k + \frac{\sigma_2^2}{2}y_k^2\left(\eta_k^2-1\right)\Delta t, \end{aligned} \tag{74}$$

where ξ_k and η_k, $k = 1,2,\ldots,n$, are the Gaussian random variables $N(0,1)$.

The parameters of model (8) are fixed as (14). In this case, model (7) has the endemic point $E^* = (0.11, 0.74)$. And model (8) becomes as follows:

$$\begin{aligned} dx &= \left(1-2x-y-\frac{0.3xy}{x+0.5y}\right)dt + \sigma_1xdB_1(t), \\ dy &= \left(\frac{4.5xy}{x+0.5y}-y\right)dt + \sigma_2ydB_2(t). \end{aligned} \tag{75}$$

Simple computations show that

$$\begin{aligned} & a+\max\{2pq,4\} = 4.3 < 4.5 = R_0, \\ & \frac{0.03^2}{2} = \frac{1}{2}(\max\{\sigma_1,\sigma_2\})^2 \\ &\qquad < 1+\min\left\{\frac{R_0-a}{2p}-q,\frac{R_0-a}{2}-2\right\} = 1.1, \\ &\qquad\qquad \text{if } (\sigma_1,\sigma_2) = (0.03,0.01), \\ & \frac{0.5^2}{2} = \frac{1}{2}(\max\{\sigma_1,\sigma_2\})^2 \\ &\qquad < 1+\min\left\{\frac{R_0-a}{2p}-q,\frac{R_0-a}{2}-2\right\} = 1.1, \\ &\qquad\qquad \text{if } (\sigma_1,\sigma_2) = (0.5,0.3). \end{aligned} \tag{76}$$

It is easy to see that, all the conditions of Theorem 11 are satisfied, and we can therefore conclude that, with $(\sigma_1,\sigma_2) = (0.03,0.01)$ and $(\sigma_1,\sigma_2) = (0.5,0.3)$, the solutions of model (8) is stochastically permanent. The numerical examples shown in Figures 2 and 3 clearly support these results. In Figure 2, with $(\sigma_1,\sigma_2) = (0.03,0.01)$, the solutions of model (8) will be oscillating slightly around the endemic point $E^* = (0.11,0.74)$ of model (7). And in Figure 3, with $(\sigma_1,\sigma_2) = (0.5,0.3)$, the solutions of model (8) will be oscillating strongly around the endemic point $E^* = (0.11,0.74)$ of model (7).

It is worthy to note that, throughout this paper, the parameters for model (7), also for model (8), are fixed as the set

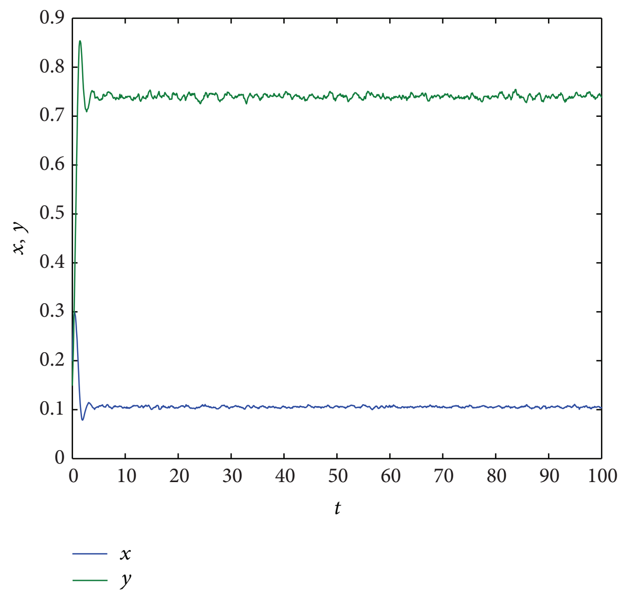

FIGURE 2: The solution of the stochastic model (8) with initial values $x(0) = 0.2$, $y(0) = 0.15$. The parameters are taken as (14), $\sigma_1 = 0.03$, $\sigma_2 = 0.01$.

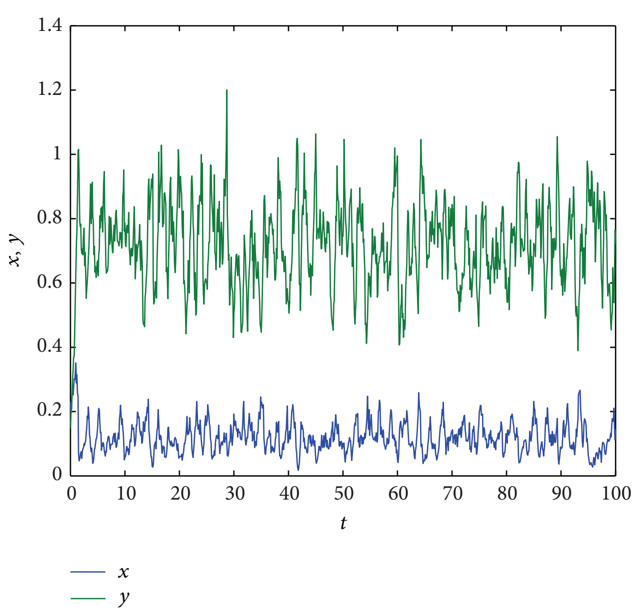

FIGURE 3: The solution of the stochastic model (8) with initial values $x(0) = 0.2$, $y(0) = 0.15$. The parameters are taken as (14), $\sigma_1 = 0.5$, $\sigma_2 = 0.3$.

(14). The reason is that with this parameter set, the conditions of our theoretical results hold. Of course, one can adopt other parameters set to show the numerical results.

From the theoretical and numerical results, we can know that, when the noise density is not large, the stochastic model (8) preserves the property of the stability of the deterministic model (7). To a great extent, we can ignore the noise and use the deterministic model (7) to describe the population dynamics. However, when the noise is sufficiently large, it can force the population to become largely fluctuating. In this case, we cannot use deterministic model (7) but stochastic model (8) to describe the population dynamics. Needless to say, both deterministic and stochastic epidemic models have their important roles.

Furthermore, from the numerical results in Figure 2, one can see that model (8) is stochastically stable. But we cannot prove the stochastic stability because of the complexity of model (8). This can be further investigated.

On the other hand, we know that there are different possible approaches to including random effects in the epidemic models affected by environmental white noise, here we consider another method to introduce random effects in the epidemic model (7). The martingale approach was initiated by Beretta et al. [35] and applied in [27, 30, 45, 47]. They introduced stochastic perturbation terms into the growth equations to incorporate the effect of a randomly fluctuating environment. In detail, assume that the stochastic perturbations of the state variables around their steady-state E^* are of a white noise type which is proportional to the distances of x, y from their steady-state values x^* and y^*, respectively. In this way, model (7) will be reduced to the following form:

$$\begin{aligned} dx &= \left(1 - qx - y - \frac{axy}{x + py}\right) dt + \sigma_1 \left(x - x^*\right) dB_1(t), \\ dy &= \left(\frac{R_0 xy}{x + py} - y\right) dt + \sigma_2 \left(y - y^*\right) dB_2(t), \end{aligned} \tag{77}$$

where the definitions of σ_1, σ_2 and $B_1(t)$, $B_2(t)$ are the same as in (8).

If $R_0 > 1$, stochastic model (77) can center at its endemic point E^*, with the change of variables $u = x - x^*$, $v = y - y^*$. The linearized version of model (77) is as follows:

$$dz(t) = f_1(z(t))\, dt + f_2(z(t))\, dB(t), \tag{78}$$

where

$$\begin{gathered} z(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \qquad f_1 = \begin{pmatrix} J_{11}u(t) + J_{12}v(t) \\ J_{21}u(t) + J_{22}v(t) \end{pmatrix}, \\ f_2 = \begin{pmatrix} \sigma_1 u(t) & 0 \\ 0 & \sigma_2 v(t) \end{pmatrix}, \end{gathered} \tag{79}$$

where J_{11}, J_{12}, J_{21}, J_{22} are defined as (12).

It is easy to see that the stability of the endemic point E^* of model (77) is equivalent to the stability of zero solution of model (78).

Before proving the stochastic stability of the zero solution of model (78), we put forward a lemma in [50].

Lemma 12. *Suppose there exists a function $V(z,t) \in C^2(\Omega)$ satisfying the following inequalities:*

$$K_1|z|^{\omega} \le V(z,t) \le K_2|z|^{\omega}, \tag{80}$$

$$LV(z,t) \le -K_3|z|^{\omega}, \tag{81}$$

where $\omega > 0$ and K_i $(i = 1, 2, 3)$ is positive constant. Then the zero solution of mode (78) *is exponentially ω-stable for all time $t \ge 0$.*

From the lemma above, note that if $\omega = 2$ in (80) and (81), then the zero solution of model (78) is stochastically asymptotically stable in probability. Thus, we obtain the following theorem.

Theorem 13. *Assume that* $\sigma_1^2 < 2(pqR_0^2 + a(R_0 - 1)^2)/pR_0^2$, $\sigma_2^2 < 2(R_0 - 1)/R_0$ *hold; then the zero solution of model* (78) *is asymptotically mean square stable. And the endemic point* E^* *of model* (77) *is asymptotically mean square stable.*

The details of the proof are shown in the Appendix.

We should point out that the results obtained in this paper are only for the simple case when $l = h = 1$ of the incidence rate (3). The dynamical behaviors of the stochastic epidemic model with general ratio-dependent incidence rate (3) are desirable in future studies.

Appendix

The proof of Theorem 13

Proof. Let us consider the Lyapunov function

$$V_5(z(t)) = \frac{1}{2}\left(u^2 + \kappa v^2\right), \tag{A.1}$$

where $\kappa = (R_0^2 + a)/R_0(R_0 - 1)^2$.

It is easy to check that inequality (80) holds with $\omega = 2$. Moreover,

$$\begin{aligned} LV_5(z(t)) &= u\left(J_{11}u + J_{12}v\right) + \kappa v\left(J_{21}u + J_{22}v\right) \\ &\quad + \frac{1}{2}\left(\sigma_1^2 u^2 + \kappa\sigma_2^2 v^2\right) \\ &= \left(J_{11} + \frac{\sigma_1^2}{2}\right)u^2 + \kappa\left(J_{22} + \frac{\sigma_2^2}{2}\right)v^2 \\ &= -z^T Q z, \end{aligned} \tag{A.2}$$

where

$$Q = \begin{pmatrix} J_{11} + \dfrac{\sigma_1^2}{2} & 0 \\ 0 & \kappa\left(J_{22} + \dfrac{\sigma_2^2}{2}\right) \end{pmatrix}. \tag{A.3}$$

When $\sigma_1^2 < 2(pqR_0^2 + a(R_0 - 1)^2)/pR_0^2, \sigma_2^2 < 2(R_0 - 1)/R_0$, the two eigenvalues λ_1, λ_2 of the matrix Q will be positive. Set $\lambda_{\min} = \min\{\lambda_1, \lambda_2\}$, it follows from (A.2) immediately that

$$LV_5(z(t)) \le -\lambda_{\min}|z(t)|^2. \tag{A.4}$$

We therefore have the assertion. □

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