Research Article

Oscillations in Difference Equations with Deviating Arguments and Variable Coefficients

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New sufficient conditions for the oscillation of all solutions of difference equations with several deviating arguments and variable coefficients are presented. Examples illustrating the results are also given.

1. Introduction

In this paper we study the oscillation of all solutions of difference equation with several variable retarded arguments of the form

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) = 0, \quad n \in \mathbb{N}_0, \qquad (\mathrm{E_R})$$

and the (dual) difference equation with several variable advanced arguments of the form

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\sigma_i(n)) = 0, \quad n \in \mathbb{N}, \qquad (\mathrm{E_A})$$

where $m \in \mathbb{N}$, $(p_i(n))$, $1 \le i \le m$, are sequences of positive real numbers and $(\tau_i(n))$, $1 \le i \le m$, are sequences of integers such that

$$\tau_i(n) \le n-1, \quad \forall n \in \mathbb{N}_0,$$

$$\lim_{n \to \infty} \tau_i(n) = \infty, \quad 1 \le i \le m \qquad (1)$$

and $(\sigma_i(n))$, $1 \le i \le m$, are sequences of integers such that

$$\sigma_i(n) \ge n+1, \quad \forall n \in \mathbb{N}, \ 1 \le i \le m. \qquad (2)$$

Here, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ and ∇ denotes the backward difference operator $\nabla x(n) = x(n) - x(n-1)$.

Strong interest in $(\mathrm{E_R})$ is motivated by the fact that it represents a discrete analogue of the differential equation (see [1–3] and the references cited therein)

$$x'(t) + \sum_{i=1}^{m} p_i(t) x(\tau_i(t)) = 0, \quad t \ge 0, \qquad (3)$$

where, for every $i \in \{1, \dots, m\}$, p_i is a continuous real-valued function in the interval $[0, \infty)$ and τ_i is a continuous real-valued function on $[0, \infty)$ such that

$$\tau_i(t) \le t, \quad t \ge 0, \qquad \lim_{t \to \infty} \tau_i(t) = \infty, \qquad (4)$$

while $(\mathrm{E_A})$ represents a discrete analogue of the advanced differential equation (see [1, 2] and the references cited therein)

$$x'(t) - \sum_{i=1}^{m} p_i(t) x(\sigma_i(t)) = 0, \quad t \ge 1, \qquad (5)$$

where, for every $i \in \{1, \dots, m\}$, p_i is a continuous real-valued function in the interval $[1, \infty)$ and σ_i is a continuous real-valued function on $[1, \infty)$ such that

$$\sigma_i(t) \ge t, \quad t \ge 1. \qquad (6)$$

By a *solution* of (E_R), we mean a sequence of real numbers $(x(n))_{n \geq -w}$ which satisfies (E_R) for all $n \geq 0$. Here

$$w = -\min_{\substack{n \geq 0 \\ 1 \leq i \leq m}} \tau_i(n). \tag{7}$$

It is clear that, for each choice of real numbers c_{-w}, $c_{-w+1}, \ldots, c_{-1}, c_0$, there exists a unique solution $(x(n))_{n \geq -w}$ of (E_R) which satisfies the initial conditions $x(-w) = c_{-w}$, $x(-w + 1) = c_{-w+1}, \ldots, x(-1) = c_{-1}, x(0) = c_0$.

By a solution of (E_A), we mean a sequence of real numbers $(x(n))_{n \geq 0}$ which satisfies (E_A) for all $n \geq 1$.

A solution $(x(n))_{n \geq -w}$ (or $(x(n))_{n \geq 0}$) of (E_R) (or (E_A)) is called *oscillatory*, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

In the last few decades, the oscillatory behavior of the solutions of difference and differential equations with several deviating arguments and variable coefficients has been studied. See, for example, [1–14] and the references cited therein.

In 2006, Berezansky and Braverman [5] proved that if

$$\limsup_{n \to \infty} \sum_{i=1}^{m} p_i(n) > 0, \qquad \liminf_{n \to \infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n-1} p_i(j) > \frac{1}{e}, \tag{8}$$

where $\tau(n) = \max_{1 \leq i \leq m} \tau_i(n)$, for all $n \geq 0$, then all solutions of (E_R) oscillate.

Recently, Chatzarakis et al. [7–9] established the following theorems.

Theorem 1 (see [9]). *Assume that the sequences $(\tau_i(n))$ $[(\sigma_i(n))]$, $1 \leq i \leq m$, are increasing, (1) [(2)] holds, and*

$$\limsup_{n \to \infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) \left[\limsup_{n \to \infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} p_i(j) \right] > 1, \tag{9}$$

where $\tau(n) = \max_{1 \leq i \leq m} \tau_i(n)$, for all $n \geq 0$, $[\sigma(n) = \min_{1 \leq i \leq m} \sigma_i(n)$, for all $n \geq 1]$, or

$$\limsup_{n \to \infty} \sum_{i=1}^{m} p_i(n) > 0,$$

$$\liminf_{n \to \infty} \sum_{i=1}^{m} \sum_{j=\tau_i(n)}^{n-1} p_i(j) \left[\liminf_{n \to \infty} \sum_{i=1}^{m} \sum_{j=n+1}^{\sigma_i(n)} p_i(j) \right] > \frac{1}{e}, \tag{10}$$

then all solutions of (E_R) $[(E_A)]$ oscillate.

Theorem 2 (see [7, 8]). *Assume that the sequences $(\tau_i(n))$ $[(\sigma_i(n))]$, $1 \leq i \leq m$, are increasing and (1) [(2)] holds. Set*

$$\alpha = \min\{\alpha_i : 1 \leq i \leq m\},$$

$$\alpha_i = \liminf_{n \to \infty} \sum_{j=\tau_i(n)}^{n-1} p_i(j) \left[\liminf_{n \to \infty} \sum_{j=n+1}^{\sigma_i(n)} p_i(j) \right]. \tag{11}$$

If $0 < \alpha \leq 1/e$, and

$$\limsup_{n \to \infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) \left[\limsup_{n \to \infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} p_i(j) \right]$$

$$> 1 - \left(1 - \sqrt{1 - \alpha}\right)^2, \tag{12}$$

or

$$\limsup_{n \to \infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) \left[\limsup_{n \to \infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} p_i(j) \right]$$

$$> 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \tag{13}$$

then all solutions of (E_R) $[(E_A)]$ oscillate.

The authors study further (E_R) and (E_A) and derive new sufficient oscillation conditions. These conditions are the improved and generalized discrete analogues of the oscillation conditions for the corresponding differential equations, which were studied in 1982 by Ladas and Stavroulakis [2]. Examples illustrating the results are also given.

2. Oscillation Criteria

2.1. Retarded Difference Equations. We present new sufficient conditions for the oscillation of all solutions of (E_R).

Theorem 3. *Assume that $(\tau_i(n))$, $1 \leq i \leq m$, are increasing sequences of integers such that (1) holds and $(p_i(n))$, $1 \leq i \leq m$, are sequences of positive real numbers and define α_i, $1 \leq i \leq m$, by (11). If $\alpha_i > 0$, $1 \leq i \leq m$, and*

$$\left[\prod_{i=1}^{m} \left(\alpha_i + \sum_{j=1, j \neq i}^{m} \liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i(k) \right) \right]^{1/m} > \frac{1}{e}, \tag{14}$$

then all solutions of (E_R) oscillate.

Proof. Assume, for the sake of contradiction, that $(x(n))_{n \geq -w}$ is a nonoscillatory solution of (E_R). Then it is either eventually positive or eventually negative. As $(-x(n))_{n \geq -w}$ is also a solution of (E_R), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geq -w$ be an integer such that $x(n) > 0$ for all $n \geq n_1$. Then, there exists $n_2 \geq n_1$ such that

$$x(\tau_i(n)) > 0, \quad \forall n \geq n_2, \ 1 \leq i \leq m. \tag{15}$$

In view of this, (E_R) becomes

$$\Delta x(n) = -\sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \leq 0, \quad \forall n \geq n_2, \tag{16}$$

which means that the sequence $(x(n))$ is eventually decreasing.

Next choose a natural number $n_3 > n_2$ such that

$$x(n) < x(\tau_i(n)), \quad \forall n \geq n_3, \ 1 \leq i \leq m. \tag{17}$$

Set

$$z_i(n) = \frac{x(\tau_i(n))}{x(n)}, \quad \forall n \geq n_3, \ 1 \leq i \leq m,$$

$$\varphi_i = \liminf_{n \to \infty} z_i(n), \quad 1 \leq i \leq m. \tag{18}$$

It is obvious that

$$z_i(n) > 1, \qquad \varphi_i \geq 1 \quad \text{for } i = 1, 2, \ldots, m. \tag{19}$$

Now we will show that $\varphi_i < \infty$ for $i = 1, 2, \ldots, m$. Indeed, assume that $\varphi_i = \infty$ for some i, $i = 1, 2, \ldots, m$. For this i, by (E_R), we have

$$\Delta x(n) + p_i(n) x(\tau_i(n)) \leq 0, \quad \forall n \geq n_3. \tag{20}$$

At this point, we will establish the following claim.

Claim 1 (cf. [8]). For each $n \geq n_3$, there exists an integer $n_i^* \geq n$ for each $i = 1, 2, \ldots, m$ such that $\tau_i(n_i^*) \leq n - 1$, and

$$\sum_{j=n}^{n_i^*} p_i(j) \geq \frac{\alpha_i - \varepsilon}{2}, \tag{21}$$

$$\sum_{j=\tau_i(n_i^*)}^{n-1} p_i(j) > \frac{\alpha_i - \varepsilon}{2}, \tag{22}$$

where ε is an arbitrary real number with $0 < \varepsilon < \alpha_i$.

To prove this claim, let us consider an arbitrary real number ε with $0 < \varepsilon < \alpha_i$. Then by (11) we can choose an integer $n_3 \geq n_2$ such that

$$\sum_{j=\tau_i(n)}^{n-1} p_i(j) \geq \alpha_i - \varepsilon, \quad \forall n \geq n_3, \ 1 \leq i \leq m. \tag{23}$$

Assume, first, that $p_i(n) \geq (\alpha_i - \varepsilon)/2$ and choose $n_i^* = n$. Then $\tau_i(n_i^*) = \tau_i(n) \leq n - 1$. Moreover, we have

$$\sum_{j=n}^{n_i^*} p_i(j) = \sum_{j=n}^{n} p_i(j) = p_i(n) \geq \frac{\alpha_i - \varepsilon}{2} \tag{24}$$

and, by (23),

$$\sum_{j=\tau_i(n_i^*)}^{n-1} p_i(j) = \sum_{j=\tau_i(n)}^{n-1} p_i(j) > \frac{\alpha_i - \varepsilon}{2}. \tag{25}$$

So, (21) and (22) are fulfilled. Next, we suppose that $p_i(n) < (\alpha_i - \varepsilon)/2$. It is not difficult to see that (23) guarantees that $\sum_{j=0}^{\infty} p_i(j) = \infty$. In particular, it holds

$$\sum_{j=n}^{\infty} p_i(j) = \infty. \tag{26}$$

Thus, as $p_i(n) < (\alpha_i - \varepsilon)/2$, there always exists an integer $n_i^* > n$ so that

$$\sum_{j=n}^{n_i^* - 1} p_i(j) < \frac{\alpha_i - \varepsilon}{2} \tag{27}$$

and (21) holds. We assert that $\tau_i(n_i^*) \leq n - 1$. Otherwise, $\tau_i(n_i^*) \geq n$. We also have $\tau_i(n_i^*) \leq n_i^* - 1$. Hence, in view of (27), we get

$$\sum_{j=\tau_i(n_i^*)}^{n_i^*-1} p_i(j) \leq \sum_{j=n}^{n_i^*-1} p_i(j) < \frac{\alpha_i - \varepsilon}{2}. \tag{28}$$

On the other hand, (23) gives

$$\sum_{j=\tau_i(n_i^*)}^{n_i^*-1} p_i(j) \geq \alpha_i - \varepsilon > \frac{\alpha_i - \varepsilon}{2}. \tag{29}$$

We have arrived at a contradiction, which shows our assertion that $\tau_i(n_i^*) \leq n-1$. Furthermore, by using (23) (for the integer n_i^*) as well as (27), we obtain

$$\sum_{j=\tau_i(n_i^*)}^{n-1} p_i(j) = \sum_{j=\tau_i(n_i^*)}^{n_i^*-1} p_i(j)$$

$$- \sum_{j=n}^{n_i^*-1} p_i(j) > (\alpha_i - \varepsilon) - \frac{\alpha_i - \varepsilon}{2} \tag{30}$$

$$= \frac{\alpha_i - \varepsilon}{2}$$

and consequently (22) holds true. Our claim has been proved.

Now, summing up (20) from n to n_i^*, we find

$$x(n_i^* + 1) - x(n) + \sum_{j=n}^{n_i^*} p_i(j) x(\tau_i(j)) \leq 0, \quad \forall n \geq n_3, \tag{31}$$

or

$$x(n) \geq \left(\sum_{j=n}^{n_i^*} p_i(j)\right) x(\tau_i(n_i^*)) \geq \frac{\alpha_i - \varepsilon}{2} x(\tau_i(n_i^*)),$$

$$\forall n \geq n_3. \tag{32}$$

Summing up (20) from $\tau_i(n_i^*)$ to $n - 1$, we find

$$x(n) - x(\tau_i(n_i^*)) + \sum_{j=\tau_i(n_i^*)}^{n-1} p_i(j) x(\tau_i(j)) \leq 0,$$

$$\forall n \geq n_3, \tag{33}$$

or

$$x(\tau_i(n_i^*)) \geq \left(\sum_{j=\tau_i(n_i^*)}^{n-1} p_i(j)\right) x(\tau_i(n-1))$$

$$\geq \frac{\alpha_i - \varepsilon}{2} x(\tau_i(n)), \quad \forall n \geq n_3. \tag{34}$$

Combining (32) and (34), we obtain

$$x(n) \geq \left(\frac{\alpha_i - \varepsilon}{2}\right)^2 x(\tau_i(n)), \quad \forall n \geq n_3, \tag{35}$$

or

$$\frac{x\left(\tau_i\left(n\right)\right)}{x\left(n\right)} \leq \left(\frac{2}{\alpha_i - \varepsilon}\right)^2, \quad \forall n \geq n_3, \tag{36}$$

which means that $(z_i(n))$ is bounded. This contradicts our assumption that $\varphi_i = \infty$. Therefore $\varphi_i < \infty$ for every $i = 1, 2, \ldots, m$.

Dividing both sides of (E_R) by $x(n)$, for $n \geq n_3$, we obtain

$$\frac{\Delta x\left(n\right)}{x\left(n\right)} + \sum_{i=1}^{m} p_i\left(n\right) \frac{x\left(\tau_i\left(n\right)\right)}{x\left(n\right)} = 0, \tag{37}$$

or

$$\frac{\Delta x\left(n\right)}{x\left(n\right)} + \sum_{i=1}^{m} p_i\left(n\right) z_i\left(n\right) = 0. \tag{38}$$

Summing up (38) from $\tau_\rho(n)$ to $n-1$ for $\rho = 1, 2, \ldots, m$, we find

$$\sum_{j=\tau_\rho(n)}^{n-1} \frac{\Delta x\left(j\right)}{x\left(j\right)} + \sum_{i=1}^{m} \sum_{j=\tau_\rho(n)}^{n-1} p_i\left(j\right) z_i\left(j\right) = 0. \tag{39}$$

But

$$\sum_{j=\tau_\rho(n)}^{n-1} \frac{\Delta x\left(j\right)}{x\left(j\right)} = \sum_{j=\tau_\rho(n)}^{n-1} \left(\frac{x\left(j+1\right)}{x\left(j\right)} - 1\right)$$

$$\geq \sum_{j=\tau_\rho(n)}^{n-1} \left(1 + \ln \frac{x\left(j+1\right)}{x\left(j\right)} - 1\right) \tag{40}$$

$$= \ln \frac{x\left(n\right)}{x\left(\tau_\rho\left(n\right)\right)},$$

or

$$\sum_{j=\tau_\rho(n)}^{n-1} \frac{\Delta x\left(j\right)}{x\left(j\right)} \geq -\ln z_\rho\left(n\right), \quad \rho = 1, 2, \ldots, m. \tag{41}$$

Combining (39) and (41), we obtain

$$-\ln z_\rho\left(n\right) + \sum_{i=1}^{m} \sum_{j=\tau_\rho(n)}^{n-1} p_i\left(j\right) z_i\left(j\right) \leq 0, \quad \rho = 1, 2, \ldots, m, \tag{42}$$

or

$$\ln z_\rho\left(n\right) \geq \sum_{i=1}^{m} \sum_{j=\tau_\rho(n)}^{n-1} p_i\left(j\right) z_i\left(j\right), \quad \rho = 1, 2, \ldots, m. \tag{43}$$

Taking limit inferiors on both sides of the above inequalities (43), we obtain

$$\ln \varphi_\rho \geq \sum_{i=1}^{m} \varphi_i \left(\liminf_{n \to \infty} \sum_{k=\tau_\rho(n)}^{n-1} p_i\left(k\right)\right), \quad \rho = 1, 2, \ldots, m \tag{44}$$

and by adding we find

$$\sum_{i=1}^{m} \ln \varphi_i \geq \sum_{i=1}^{m} \varphi_i \left(\sum_{j=1}^{m} \liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i\left(k\right)\right). \tag{45}$$

Set

$$f\left(\varphi_1, \varphi_2, \ldots, \varphi_m\right)$$

$$\equiv \sum_{i=1}^{m} \ln \varphi_i - \sum_{i=1}^{m} \varphi_i \left(\sum_{j=1}^{m} \liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i\left(k\right)\right). \tag{46}$$

Clearly

$$f\left(\varphi_1, \varphi_2, \ldots, \varphi_m\right) \geq 0, \quad \forall \varphi_1, \varphi_2, \ldots, \varphi_m \geq 1. \tag{47}$$

Since

$$\frac{\partial f}{\partial \varphi_i} = \frac{1}{\varphi_i} - \sum_{j=1}^{m} \liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i\left(k\right) = 0, \tag{48}$$

for

$$\varphi_i = \frac{1}{\sum_{j=1}^{m} \liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i\left(k\right)}, \quad i = 1, 2, \ldots, m, \tag{49}$$

the function f has a maximum at the critical point

$$\left(\frac{1}{\sum_{j=1}^{m} \liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_1\left(k\right)}, \right.$$
$$\left. \ldots, \frac{1}{\sum_{j=1}^{m} \liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_m\left(k\right)}\right) \tag{50}$$

since the quadratic form

$$\sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial \varphi_i \partial \varphi_j} b_i b_j = -\sum_{i=1}^{m} \frac{b_i^2}{\varphi_i^2} < 0. \tag{51}$$

Since $f(\varphi_1, \varphi_2, \ldots, \varphi_m) \geq 0$, the maximum of f at the critical point should be nonnegative. Thus,

$$\sum_{i=1}^{m} \left[-\ln\left(\sum_{j=1}^{m} \liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i\left(k\right)\right)\right] - m \geq 0, \tag{52}$$

that is,

$$\max_{\varphi_i \geq 1} f\left(\varphi_1, \varphi_2, \ldots, \varphi_m\right)$$

$$= -\ln \prod_{i=1}^{m} \left(\sum_{j=1}^{m} \liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i\left(k\right)\right) - m. \tag{53}$$

Hence

$$\prod_{i=1}^{m} \left(\sum_{j=1}^{m} \liminf_{n \to \infty} \sum_{k=\tau j(n)}^{n-1} p_i\left(k\right)\right) \leq \frac{1}{e^m}, \tag{54}$$

or

$$\left[\prod_{i=1}^{m} \left(\alpha_i + \sum_{j=1, j\neq i}^{m} \liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i(k) \right) \right]^{1/m} \le \frac{1}{e}, \quad (55)$$

which contradicts (14).

The proof of the theorem is complete. $\qquad \square$

Theorem 4. *Assume that $(\tau_i(n))$, $1 \le i \le m$, are increasing sequences of integers such that (1) holds and $(p_i(n))$, $1 \le i \le m$, are sequences of positive real numbers and define α_i, $1 \le i \le m$, by (11). If $\alpha_i > 0$, $1 \le i \le m$, and*

$$\frac{1}{m}\sum_{i=1}^{m}\alpha_i + \frac{2}{m} \sum_{\substack{i<j \\ i,j=1}}^{m} \left(\liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i(k) \right.$$
$$\left. \times \liminf_{n \to \infty} \sum_{k=\tau_i(n)}^{n-1} p_j(k) \right)^{1/2} > \frac{1}{e}, \quad (56)$$

then all solutions of (E_R) oscillate.

Proof. Assume, for the sake of contradiction, that $(x(n))_{n \ge -w}$ is a nonoscillatory solution of (E_R). Then it is either eventually positive or eventually negative. As $(-x(n))_{n \ge -w}$ is also a solution of (E_R), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \ge -w$ be an integer such that $x(n) > 0$ for all $n \ge n_1$. Then, there exists $n_2 \ge n_1$ such that

$$x(\tau_i(n)) > 0, \quad \forall n \ge n_2, \ 1 \le i \le m. \quad (57)$$

In view of this, (E_R) becomes

$$\Delta x(n) = -\sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \le 0, \quad \forall n \ge n_2, \quad (58)$$

which means that the sequence $(x(n))$ is eventually decreasing.

Taking into account the fact that $\varphi_i < \infty$ for $i = 1, 2, \dots, m$ (see proof of Theorem 3), by using (44) and the fact that

$$\frac{1}{e} > \frac{\ln \varphi_\rho}{\varphi_\rho}, \quad \rho = 1, 2, \dots, m, \quad (59)$$

we obtain

$$\frac{1}{e} > \sum_{i=1}^{m} \left(\liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i(k) \right) \frac{\varphi_i}{\varphi_j}, \quad j = 1, 2, \dots, m. \quad (60)$$

Adding these inequalities we have

$$\frac{m}{e} \ge \sum_{i=1}^{m} \left(\liminf_{n \to \infty} \sum_{k=\tau_i(n)}^{n-1} p_i(k) \right)$$
$$+ \sum_{\substack{i<j \\ i,j=1}}^{m} \left[\left(\liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i(k) \right) \frac{\varphi_i}{\varphi_j} \right.$$
$$\left. + \left(\liminf_{n \to \infty} \sum_{k=\tau_i(n)}^{n-1} p_j(k) \right) \frac{\varphi_j}{\varphi_i} \right]$$

$$\ge \sum_{i=1}^{m} \left(\liminf_{n \to \infty} \sum_{k=\tau_i(n)}^{n-1} p_i(k) \right)$$
$$+ 2 \sum_{\substack{i<j \\ i,j=1}}^{m} \left[\left(\liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i(k) \right) \right.$$
$$\left. \times \left(\liminf_{n \to \infty} \sum_{k=\tau_i(n)}^{n-1} p_j(k) \right) \right]^{1/2}, \quad (61)$$

or

$$\frac{1}{m}\sum_{i=1}^{m}\alpha_i + \frac{2}{m} \sum_{\substack{i<j \\ i,j=1}}^{m} \left(\liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i(k) \right.$$
$$\left. \times \liminf_{n \to \infty} \sum_{k=\tau_i(n)}^{n-1} p_j(k) \right)^{1/2} \le \frac{1}{e}, \quad (62)$$

which contradicts (56).

The proof of the theorem is complete. $\qquad \square$

2.2. Advanced Difference Equations. Similar oscillation theorems for the (dual) advanced difference equation (E_A) can be derived easily. The proofs of these theorems are omitted, since they follow a similar procedure as in Section 2.1.

Theorem 5. *Assume that $(\sigma_i(n))$, $1 \le i \le m$, are increasing sequences of integers such that (2) holds and $(p_i(n))$, $1 \le i \le m$, are sequences of positive real numbers and define α_i, $1 \le i \le m$, by (11). If $\alpha_i > 0$, $1 \le i \le m$, and*

$$\left[\prod_{i=1}^{m} \left(\alpha_i + \sum_{j=1, j\neq i}^{m} \liminf_{n \to \infty} \sum_{k=n+1}^{\sigma_j(n)} p_i(k) \right) \right]^{1/m} > \frac{1}{e}, \quad (63)$$

then all solutions of (E_A) oscillate.

Theorem 6. *Assume that $(\sigma_i(n))$, $1 \le i \le m$, are increasing sequences of integers such that (2) holds and $(p_i(n))$, $1 \le i \le m$, are sequences of positive real numbers and define α_i, $1 \le i \le m$, by (11). If $\alpha_i > 0$, $1 \le i \le m$, and*

$$\frac{1}{m}\sum_{i=1}^{m}\alpha_i + \frac{2}{m} \sum_{\substack{i<j \\ i,j=1}}^{m} \left(\liminf_{n \to \infty} \sum_{k=n+1}^{\sigma_j(n)} p_i(k) \right.$$
$$\left. \times \liminf_{n \to \infty} \sum_{k=n+1}^{\sigma_i(n)} p_j(k) \right)^{1/2} > \frac{1}{e}, \quad (64)$$

then all solutions of (E_A) oscillate.

2.3. Special Cases. In the case where p_i, $i = 1, 2, \dots, m$, are positive real constants and τ_i are constant retarded arguments of the form $\tau_i(n) = n - k_i$, [σ_i are constant advanced arguments

of the form $\sigma_i(n) = n + k_i]$, $k_i \in \mathbb{N}$, $i = 1, 2, \ldots, m$, equation (E_R) $[(E_A)]$ takes the form

$$\Delta x(n) + \sum_{i=1}^{m} p_i x(n - k_i) = 0, \quad n \in \mathbb{N}_0,$$

$$\left[\nabla x(n) - \sum_{i=1}^{m} p_i x(n + k_i) = 0, n \in \mathbb{N} \right].$$

(E)

For this equation, as a consequence of Theorems 3 [5] and 4 [6], we have the following corollary.

Corollary 7. *Assume that*

$$\left[\prod_{i=1}^{m} p_i \right]^{1/m} \left(\sum_{i=1}^{m} k_i \right) > \frac{1}{e},$$

(65)

or

$$\frac{1}{m} \left(\sum_{i=1}^{m} \sqrt{p_i k_i} \right)^2 > \frac{1}{e}.$$

(66)

Then all solutions of (E) oscillate.

Remark 8. A research question that arises is whether Theorems 3–6 are valid, even in the case where the coefficients $p(n)$ oscillate (see [15, 16]). Then our results would be comparable to those in [15, 16]. This is a question that we currently study and expect to have some results soon.

3. Examples

The following two examples illustrate that the conditions for oscillations (65) and (66) are independent. They are chosen in such a way that only one of them is satisfied.

Example 1. Consider the retarded difference equation

$$\Delta x(n) + \frac{1}{5e} x(n-1) + \frac{1}{6e} x(n-2)$$

$$+ \frac{1000}{7198e} x(n-3) = 0, \quad n \in \mathbb{N}_0.$$

(67)

Here $m = 3$, $\tau_1(n) = n - 1$, $\tau_2(n) = n - 2$, $\tau_3(n) = n - 3$, and

$$\tau(n) = \max_{1 \leq i \leq 3} \tau_i(n) = n - 1.$$

(68)

It is easy to see that

$$\left[\prod_{i=1}^{3} p_i \right]^{1/3} \left(\sum_{i=1}^{3} k_i \right)$$

$$= \sqrt[3]{\frac{1}{5e} \cdot \frac{1}{6e} \cdot \frac{1000}{7198e}} \cdot (1 + 2 + 3) \simeq 0.3679135$$

$$> \frac{1}{e}.$$

(69)

That is, condition (65) of Corollary 7 is satisfied and therefore all solutions of equation (67) oscillate.

However,

$$\frac{1}{3} \left(\sum_{i=1}^{3} \sqrt{p_i k_i} \right)^2$$

$$= \frac{1}{3} \left(\sqrt{\frac{1}{5e} \cdot 1} + \sqrt{\frac{1}{6e} \cdot 2} + \sqrt{\frac{1000}{7198e} \cdot 3} \right)^2 \simeq 0.3420547$$

$$< \frac{1}{e}.$$

(70)

That is, condition (66) of Corollary 7 is not satisfied.

Observe that

$$\alpha_1 = \liminf_{n \to \infty} \sum_{j=n-1}^{n-1} p_1 = \frac{1}{5e} \simeq 0.0735758,$$

$$\alpha_2 = \liminf_{n \to \infty} \sum_{j=n-2}^{n-1} p_2 = \frac{2}{6e} \simeq 0.1226264,$$

(71)

$$\alpha_3 = \liminf_{n \to \infty} \sum_{j=n-3}^{n-1} p_3 = \frac{3000}{7198e} \simeq 0.1533256.$$

Thus

$$\alpha = \min \{ \alpha_i : 1 \leq i \leq 3 \} \simeq 0.0735758 < \frac{1}{e}.$$

(72)

Also,

$$\limsup_{n \to \infty} \sum_{i=1}^{3} \sum_{j=n-1}^{n} p_i$$

$$= \limsup_{n \to \infty} \left[2 \cdot \frac{1}{5e} + 2 \cdot \frac{1}{6e} + 2 \cdot \frac{1000}{7198e} \right] \simeq 0.3719953$$

$$< 1,$$

$$0.3719953 < 1 - \left(1 - \sqrt{1 - \alpha} \right)^2 \simeq 0.9985944,$$

$$0.3719953 < 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2} \simeq 0.997069,$$

$$\liminf_{n \to \infty} \sum_{i=1}^{3} \sum_{j=n-1}^{n-1} p_i = \liminf_{n \to \infty} \left[\frac{1}{5e} + \frac{1}{6e} + \frac{1000}{7198e} \right]$$

$$\simeq 0.1859976 < \frac{1}{e},$$

$$\liminf_{n \to \infty} \sum_{i=1}^{3} \sum_{j=\tau_i(n)}^{n-1} p_i(j) = \liminf_{n \to \infty} \left[\frac{1}{5e} + \frac{2}{6e} + \frac{3000}{7198e} \right]$$

$$\simeq 0.349528 < \frac{1}{e}$$

(73)

and therefore none of the conditions (9), (12), (13), (8), and (10) are satisfied.

Example 2. Consider the advanced difference equation

$$\nabla x(n) - \frac{5}{100} x(n+1) - \frac{22}{100} x(n+2) = 0, \quad n \in \mathbb{N}. \quad (74)$$

Here $m = 2$, $\sigma_1(n) = n + 1$, $\sigma_2(n) = n + 2$, and

$$\sigma(n) = \min_{1 \le i \le 2} \sigma_i(n) = n + 1. \quad (75)$$

It is easy to see that

$$\frac{1}{2} \left(\sum_{i=1}^{2} \sqrt{p_i k_i} \right)^2 = \frac{1}{2} \left(\sqrt{\frac{5}{100} \cdot 1} + \sqrt{\frac{22}{100} \cdot 2} \right)^2 \\ \simeq 0.393323969 > \frac{1}{e}. \quad (76)$$

That is, condition (66) of Corollary 7 is satisfied and therefore all solutions of (74) oscillate.

However,

$$\left[\prod_{i=1}^{2} p_i \right]^{1/2} \left(\sum_{i=1}^{2} k_i \right) = \sqrt{\frac{5}{100} \cdot \frac{22}{100}} \cdot (1 + 2) \\ \simeq 0.314642654 < \frac{1}{e}. \quad (77)$$

That is, condition (65) of Corollary 7 is not satisfied.

Observe that

$$\alpha_1 = \liminf_{n \to \infty} \sum_{j=n+1}^{n+1} p_1 = \frac{5}{100} = 0.05,$$

$$\alpha_2 = \liminf_{n \to \infty} \sum_{j=n+1}^{n+2} p_2 = 2 \cdot \frac{22}{100} = 0.44. \quad (78)$$

Thus

$$\alpha = \min \{ \alpha_i : 1 \le i \le 2 \} = 0.05 < \frac{1}{e}. \quad (79)$$

Also,

$$\limsup_{n \to \infty} \sum_{i=1}^{2} \sum_{j=n}^{n+1} p_i = \limsup_{n \to \infty} \left[2 \cdot \frac{5}{100} + 2 \cdot \frac{22}{100} \right] = 0.54 < 1,$$

$$0.54 < 1 - \left(1 - \sqrt{1 - \alpha} \right)^2 \simeq 0.999358869,$$

$$0.54 < 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2} \simeq 0.998682383 \quad (80)$$

and therefore none of the conditions (9), (12), and (13) are satisfied.

At this point, we give an example with general retarded arguments illustrating the main result of Theorem 3. Similarly, one can construct examples to illustrate Theorems 4–6.

Example 3. Consider the delay difference equation

$$\Delta x(n) + \frac{c}{3n} x([0.5n]) + \frac{2c}{3n} x([0.2n]) = 0, \quad n \ge 10, \quad (81)$$

with $c = 17/50$.

Here $\tau_1(n) = [0.5n]$ and $\tau_2(n) = [0.2n]$ denote the integer parts of $0.5n$ and $0.2n$. Observe that the sequences $\tau_1(n)$ and $\tau_2(n)$ are increasing, $\lim_{n \to \infty} \tau_1(n) = +\infty$, $\lim_{n \to \infty} \tau_2(n) = +\infty$, and

$$\tau_1(n) = [0.5n] \le 0.5n < n \quad \text{or} \quad \tau_1(n) \le n - 1,$$

$$\tau_2(n) = [0.2n] \le 0.2n < n \quad \text{or} \quad \tau_2(n) \le n - 1, \quad (82)$$

$$\forall n \ge 10.$$

Observe that, for a positive decreasing function $f(x)$, the following inequality holds:

$$\int_{b-1}^{b} f(x)\, dx \ge f(b) \ge \int_{b}^{b+1} f(x)\, dx. \quad (83)$$

Based on the above inequality, we will show that

$$\liminf_{n \to \infty} \sum_{k=[(a/b)n]}^{n-1} \frac{Q}{k} = Q \cdot \ln \frac{b}{a} \quad (84)$$

for any $a, b \in \mathbb{N}$, $a < b$, and any real number Q. Indeed,

$$\sum_{k=[(a/b)n]}^{n-1} \frac{Q}{k} \ge Q \sum_{k=[(a/b)n]}^{n-1} \int_{k}^{k+1} \frac{ds}{s}$$

$$= Q \int_{[(a/b)n]}^{n} \frac{ds}{s} = Q \cdot \ln \frac{n}{[(a/b)\, n]},$$

$$\sum_{k=[(a/b)n]}^{n-1} \frac{Q}{k} \le Q \sum_{k=[(a/b)n]}^{n-1} \int_{k-1}^{k} \frac{ds}{s}$$

$$= Q \int_{[(a/b)n]-1}^{n-1} \frac{ds}{s} = Q \cdot \ln \frac{n-1}{[(a/b)\, n] - 1}. \quad (85)$$

It is easy to see that

$$\lim_{n \to \infty} Q \cdot \ln \frac{n}{[(a/b)\, n]} = \lim_{n \to \infty} Q \cdot \ln \frac{n-1}{[(a/b)\, n] - 1} = Q \cdot \ln \frac{b}{a}. \quad (86)$$

From the above, it follows that

$$\liminf_{n \to \infty} \sum_{k=[(a/b)n]}^{n-1} \frac{Q}{k} = Q \cdot \ln \frac{b}{a}. \quad (87)$$

Therefore

$$\alpha_1 = \liminf_{n \to \infty} \sum_{k=[0.5n]}^{n-1} \frac{c}{3k} = \frac{c}{3} \ln 2 > 0,$$

$$\alpha_2 = \liminf_{n \to \infty} \sum_{k=[0.2n]}^{n-1} \frac{2c}{3k} = \frac{2c}{3} \ln 5 > 0,$$

$$\liminf_{n \to \infty} \sum_{k=[0.2n]}^{n-1} \frac{c}{3k} = \frac{c}{3} \ln 5, \tag{88}$$

$$\liminf_{n \to \infty} \sum_{k=[0.5n]}^{n-1} \frac{2c}{3k} = \frac{2c}{3} \ln 2.$$

Hence

$$\left[\prod_{i=1}^{2} \left(\alpha_i + \sum_{j=1,\, j \neq i}^{2} \liminf_{n \to \infty} \sum_{k=\tau_j(n)}^{n-1} p_i(k) \right) \right]^{1/2}$$

$$= \sqrt{\left[\frac{c}{3} \ln 2 + \frac{c}{3} \ln 5 \right] \cdot \left[\frac{2c}{3} \ln 2 + \frac{2c}{3} \ln 5 \right]} \tag{89}$$

$$= \frac{c\sqrt{2}\ln 10}{3} \simeq 0.369052667 > \frac{1}{e}.$$

That is, condition (14) of Theorem 3 is satisfied and therefore all solutions of (81) oscillate.

Observe, however, that

$$\sum_{i=1}^{2} \sum_{j=\tau(n)}^{n-1} p_i(j) = \sum_{j=[0.5n]}^{n-1} \frac{c}{3j} + \sum_{j=[0.5n]}^{n-1} \frac{2c}{3j}$$

$$= \sum_{j=[0.5n]}^{n-1} \frac{c}{j} \longrightarrow c \ln 2 \simeq 0.235670041$$

$$\text{as } n \longrightarrow \infty, \tag{90}$$

$$\sum_{i=1}^{2} \sum_{j=\tau(n)}^{n} p_i(j) = \sum_{j=[0.5n]}^{n} \frac{c}{3j} + \sum_{j=[0.5n]}^{n} \frac{2c}{3j}$$

$$= \sum_{j=[0.5n]}^{n} \frac{c}{j} \longrightarrow c \ln 2 \simeq 0.235670041$$

$$\text{as } n \longrightarrow \infty.$$

Thus

$$\liminf_{n \to \infty} \sum_{i=1}^{2} \sum_{j=\tau(n)}^{n-1} p_i(j)$$

$$= \limsup_{n \to \infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) \simeq 0.235670041 < \frac{1}{e}. \tag{91}$$

Also,

$$\alpha = \min\{\alpha_i : 1 \leq i \leq 2\} = \alpha_1 = \frac{c}{3} \ln 2 \simeq 0.07855668,$$

$$0.235670041 < 1 - \left(1 - \sqrt{1-\alpha}\right)^2 \simeq 0.998393464,$$

$$0.235670041 < 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2} \simeq 0.996639107 \tag{92}$$

and therefore none of the conditions (8), (9), (12), and (13) are satisfied.

Conflict of Interests

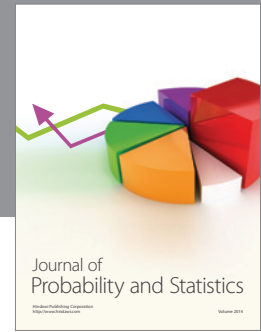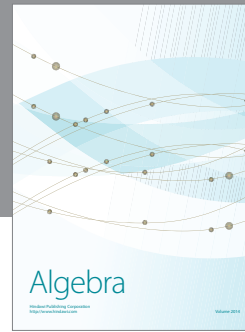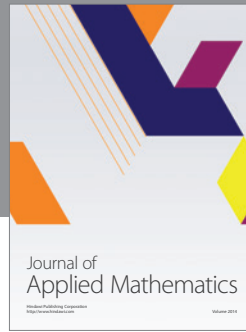The authors declare that there is no conflict of interests regarding the publication of this paper.
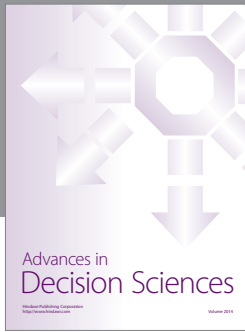
Acknowledgment

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