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Research Article

Global Exponential Stability of Impulsive Functional Differential Equations via Razumikhin Technique

Shiguo Peng[1] and Liping Yang[2]

[1] *College of Automation, Guangdong University of Technology, Guangzhou 510006, China*
[2] *College of Applied Mathematics, Guangdong University of Technology, Guangzhou 510006, China*

Correspondence should be addressed to Shiguo Peng, psg7202@126.com

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This paper develops some new Razumikhin-type theorems on global exponential stability of impulsive functional differential equations. Some applications are given to impulsive delay differential equations. Compared with some existing works, a distinctive feature of this paper is to address exponential stability problems for any finite delay. It is shown that the functional differential equations can be globally exponentially stabilized by impulses even if it may be unstable itself. Two examples verify the effectiveness of the proposed results.

1. Introduction

Functional differential equations (FDEs) which include delay differential equations (DDEs) play a very important role in formulation and analysis in mechanical, electrical, control engineering and physical sciences, economic, and social sciences [1, 2]. Therefore, the theory of FDEs has been developed very quickly. The investigation for FDEs has attracted the considerable attention of researchers and many qualitative theories of FDEs have been obtained. A large number of stability criteria of FDEs have been reported.

In addition to the delay effect, as is well known, impulsive effect is likely to exist in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time in the fields such as medicine and biology, economics, electronics, and telecommunications [3]. So far, a lot of interesting results on stability have been reported that have focused on the impulsive effect of FDEs (see, e.g., [4–16] and the references cited therein).

In particular, several papers devoted to the study of exponential stability of impulsive functional differential equations (IFDEs) have appeared during the past years. In [14, 15], the authors have investigated exponential stability of IFDEs by using the method of Lyapunov functions and Razumikhin techniques. In [16], the authors have also studied exponential stability by using the method of Lyapunov functional. However, some results in [15, 16] imposed a restrictive condition on time delays which were less than the length of all the impulsive intervals (see, e.g., [15, Theorems 3.1-3.2] and [16, Theorem 3.1]). The aim of this paper is to establish global exponential stability criteria for IFDEs by employing the Razumikhin technique which illustrate that impulses do contribute to the stability of some IFDEs and the restrictive condition that the time delays are less than the length of all the impulsive intervals can be removed in this paper.

2. Preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N} = \{1, 2, \dots\}$, I be the identity matrix, $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ be the maximal eigenvalue and the minimal eigenvalue of a matrix, respectively. If A is a vector or matrix, its transpose is denoted by A^T. For $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, let $|x| = \sqrt{x^T x}$ be Euclidean vector norm, and denote the induced matrix norm by

$$\|A\| = \sup_{x\neq 0} \frac{|Ax|}{|x|} = \sqrt{\lambda_{\max}(A^T A)}. \tag{2.1}$$

Let $\tau > 0$ and $C = C([-\tau, 0]; \mathbb{R}^n)$ denote the family of all bounded continuous $\mathbb{R}^n$-valued functions ϕ defined on $[-\tau, 0]$. $PC(\mathbb{I}; \mathbb{R}^n) = \{\psi : \mathbb{I} \to \mathbb{R}^n \mid \psi(s)$ is continuous for all but at most countable points $s \in \mathbb{I}$ and at these points $s \in \mathbb{I}$, $\psi(s^+)$ and $\psi(s^-)$ exist and $\psi(s^+) = \psi(s)\}$, where $\mathbb{I} \subset \mathbb{R}$ is an interval, $\psi(s^+)$ and $\psi(s^-)$ denote the right-hand and left-hand limits of the function $\psi(s)$ at time s, respectively. Especially, let $PC \triangleq PC([-\tau, 0]; \mathbb{R}^n)$ with norm $\|\psi\| = \sup_{-\tau\leqslant s\leqslant 0}|\psi(s)|$.

In this paper, we consider the following IFDEs:

$$\begin{gathered} x'(t) = f(t, x_t), \quad t \neq t_k,\ t \geqslant 0, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k,\ k \in \mathbb{N}, \\ x(s) = \phi(s), \quad -\tau \leqslant s \leqslant 0, \end{gathered} \tag{2.2}$$

where $f : \mathbb{R}^+ \times PC \to \mathbb{R}^n$, $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$. The initial function $\phi \in PC$. The impulsive function $I_k \in C(\mathbb{R}^n; \mathbb{R}^n)$ $(k \in \mathbb{N})$, and the impulsive moments t_k $(k = 1, 2, \dots)$ satisfy $0 = t_0 < t_1 < t_2 < \cdots$, and $\lim_{k\to\infty} t_k = \infty$.

In this paper, we assume that functions f and I_k, $k \in \mathbb{N}$, satisfy all necessary conditions for the global existence and uniqueness of solutions for all $t \geqslant t_0$. Denote by $x(t) = x(t, t_0, \phi)$ the solution of (2.2) such that $x_{t_0} = \phi$. For the purpose of stability in this paper, we also assume that $f(t, 0) = 0$ and $I_k(0) = 0$, $k \in \mathbb{N}$. So system (2.2) admits a zero solution or trivial solution $x(t, t_0, 0) = 0$. We further assume that all the solutions $x(t)$ of (2.2) are continuous except at $t_k, k \in \mathbb{N}$, at which $x(t)$ is right continuous, that is, $x(t_k^+) = x(t_k),\ k \in \mathbb{N}$.

Definition 2.1. The trivial solution of system (2.2) is said to be globally exponentially stable, if there exist numbers $\lambda > 0$ and $M \geqslant 1$ such that

$$|x(t, t_0, \phi)| \leqslant M\|\phi\|e^{-\lambda t}, \quad t \geqslant 0, \tag{2.3}$$

whenever $\phi \in PC$.

Definition 2.2. $V : [-\tau, \infty)\times\mathbb{R}^n \to \mathbb{R}^+$ is said to belong to the class ν_0, if V is continuous on each of the sets $[t_{k-1}, t_k) \times \mathbb{R}^n$, $\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$ exists, $V(t, x)$ is locally Lipschitzian in all $x \in \mathbb{R}^n$, and $V(t, 0) \equiv 0$ for all $t \geq -\tau$.

Definition 2.3. $V : [-\tau, \infty) \times \mathbb{R}^n \to \mathbb{R}^+$ is said to belong to the class ν_1, if V is continuous on each of the sets $[t_{k-1}, t_k) \times \mathbb{R}^n$, $V(t, 0) \equiv 0$ for all $t \geq -\tau$, $\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$ exists, $V_t(t, x), V_x(t, x)$ are continuous, where $(t, x) \in [t_{k-1}, t_k) \times \mathbb{R}^n$, $k \in \mathbb{N}$,

$$V_t(t, x) = \frac{\partial V(t, x)}{\partial t}, \qquad V_x(t, x) = \left(\frac{\partial V(t, x)}{\partial x_1}, \ldots, \frac{\partial V(t, x)}{\partial x_n}\right). \tag{2.4}$$

Definition 2.4. Given a function $V : [-\tau, +\infty) \times \mathbb{R}^n \to \mathbb{R}^+$, the upper right-hand derivative of V with respect to system (2.2) is defined by

$$D^+V(t, \varphi(0)) = \limsup_{h\to 0^+} \frac{1}{h}\left[V(t + h, \varphi(0) + hF(t, \varphi)) - V(t, \varphi(0))\right] \tag{2.5}$$

for $(t, \varphi) \in \mathbb{R}^+ \times PC$.

3. Razumikhin-Type Theorems

In this section, we will present some Razumikhin-type theorems on global exponential stability for system (2.2) based on the Lyapunov-Razumikhin method.

Theorem 3.1. *Let $\rho = \sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} < \infty$ and c_1, c_2, p, q, γ all be positive numbers, c a real number, $q > 1/\gamma > 1$, and $c < \ln(1/\gamma)/\rho$. Suppose that there exists a function $V \in \nu_0$ such that*

(i) *$c_1|x|^p \leqslant V(t, x) \leqslant c_2|x|^p$ for all $t \geqslant t_0 - \tau$, and $x \in \mathbb{R}^n$;*

(ii) *$D^+V(t, \varphi(0)) \leqslant cV(t, \varphi(0))$ for all $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$, whenever $qV(t, \varphi(0)) \geqslant V(t + \theta, \varphi(\theta))$ for all $\theta \in [-\tau, 0]$;*

(iii) *$V(t_k, x + I_k(x)) \leqslant \gamma V(t_k^-, x)$, $k \in \mathbb{N}$, $x \in \mathbb{R}^n$.*

Then the trivial solution of system (2.2) is globally exponentially stable.

Proof. For any $\phi \in PC$, we denote the solution $x(t, t_0, \phi)$ of (2.2) by $x(t)$. Without loss of generality, we assume that $\|\phi\| \neq 0$.

Since $q > 1/\gamma$ and $c < -(\ln\gamma/\rho)$, there exist positive numbers μ and h such that

$$q > \frac{e^{\mu\tau}}{\gamma} > \frac{1}{\gamma}, \qquad c < c + \mu < -\frac{\ln(\gamma + h)}{\rho}. \tag{3.1}$$

Set $\gamma_1 = -\ln(\gamma + h/\rho)$, then $\gamma < e^{-\gamma_1\rho} < 1$.

Set $W(t) = e^{\mu t}V(t, x(t))$, we have

$$D^{+}W(t) = \mu W(t) + e^{\mu t}D^{+}V(t, x(t)), \quad t \in [t_{k-1}, t_k), \ k \in \mathbb{N}. \tag{3.2}$$

Let $\overline{M} > c_2/c_1\gamma$ be a fixed number. In the following, we will prove that

$$W(t) < c_1\overline{M}\|\phi\|^{p}, \quad t \geqslant t_0 - \tau. \tag{3.3}$$

We first prove that

$$W(t) < c_1\overline{M}\|\phi\|^{p}, \quad t \in [t_0 - \tau, t_1). \tag{3.4}$$

It is noted that $W(t_0 + \theta) \leqslant c_2\|\phi\|^p < \gamma c_1\overline{M}\|\phi\|^p < c_1\overline{M}\|\phi\|^p$, $\theta \in [-\tau, 0]$. So, it only needs to prove that $W(t) < c_1\overline{M}\|\phi\|^p$ for $t \in (t_0, t_1)$. On the contrary, there exist some $t \in (t_0, t_1)$ such that $W(t) \geqslant c_1\overline{M}\|\phi\|^p$. Set $t^* = \inf\{t \in (t_0, t_1) : W(t) \geqslant c_1\overline{M}\|\phi\|^p\}$, then we have $t^* \in (t_0, t_1)$ and $W(t^*) \geqslant c_1\overline{M}\|\phi\|^p$. Set $\bar{t} = \sup\{t \in [t_0, t^*) : W(t) \leqslant \gamma c_1\overline{M}\|\phi\|^p\}$. Then $\bar{t} \in (t_0, t^*)$ and $W(\bar{t}) = \gamma c_1\overline{M}\|\phi\|^p$. For $t \in [\bar{t}, t^*]$, we have

$$W(t) \geqslant \gamma c_1\overline{M}\|\phi\|^{p} \geqslant \gamma W(t + \theta), \quad \forall\theta \in [-\tau, 0]. \tag{3.5}$$

Hence

$$V(t, x(t)) \geqslant \gamma e^{-\mu\tau}V(t + \theta, x(t + \theta)) \geqslant \frac{1}{q}V(t + \theta, x(t + \theta)), \quad \forall\theta \in [-\tau, 0]. \tag{3.6}$$

By condition (ii) and (3.2), it follows that for $t \in [\bar{t}, t^*]$

$$D^{+}W(t\ ) = \mu W(t) + e^{\mu t}D^{+}V(t, x(t)) \leqslant (\mu + c)W(t) \leqslant \gamma_1 W(t). \tag{3.7}$$

So, we obtain $W(t^*) \leqslant W(\bar{t})e^{\gamma_1(t^* - \bar{t})} \leqslant \gamma c_1\overline{M}\|\phi\|^p e^{\gamma_1\rho} < c_1\overline{M}\|\phi\|^p$. This is a contradiction, so (3.4) holds.

Now, we assume that for some $m \in \mathbb{N}$,

$$W(t) < c_1\overline{M}\|\phi\|^{p}, \quad t \in [t_0 - \tau, t_m). \tag{3.8}$$

We will prove that

$$W(t) < c_1\overline{M}\|\phi\|^{p}, \quad t \in [t_m, t_{m+1}). \tag{3.9}$$

Suppose not, there exist some $t \in [t_m, t_{m+1})$ such that $W(t) \geqslant c_1\overline{M}\|\phi\|^p$. Set $t^* = \inf\{t \in [t_m, t_{m+1}) : W(t) \geqslant c_1\overline{M}\|\phi\|^p\}$. From condition (iii) and (3.8), we have $W(t_m) \leqslant \gamma W(t_m^-) \leqslant \gamma c_1\overline{M}\|\phi\|^p < c_1\overline{M}\|\phi\|^p$. Hence $t^* \in (t_m, t_{m+1})$ and $W(t^*) = c_1\overline{M}\|\phi\|^p$. Set $\bar{t} = \sup\{t \in [t_m, t^*) : W(t) \leqslant \gamma c_1\overline{M}\|\phi\|^p\}$. Then we have $W(\bar{t}) = \gamma c_1\overline{M}\|\phi\|^p$. Furthermore, we have $W(t) \geqslant \gamma c_1\overline{M}\|\phi\|^p \geqslant \gamma W(t+\theta)$ for $t \in [\bar{t}, t^*]$. Thus, (3.7) holds. Then $W(t^*) \leqslant \gamma c_1\overline{M}\|\phi\|^p e^{\gamma_1\rho} < c_1\overline{M}\|\phi\|^p$, which yields a contradiction. Therefore, (3.9) holds.

By mathematical induction, we have

$$W(t) < c_1\overline{M}\|\phi\|^p, \quad t \geqslant 0. \tag{3.10}$$

Hence

$$|x(t)| < M\|\phi\|e^{-\lambda t}, \quad t \geqslant 0, \tag{3.11}$$

where $M = \overline{M}^{1/p}$, $\lambda = \mu/p$. The proof is therefore complete. □

Theorem 3.2. *Let $\varrho = \inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\} > 0$ and $c_1, c_2, p, q, \gamma, c$ all be positive numbers, $q > \gamma \geqslant 1$, and $c > \ln\gamma/\varrho$. Suppose that there exists a function $V \in \nu_0$ such that*

(i) *$c_1|x|^p \leqslant V(t,x) \leqslant c_2|x|^p$ for all $t \geqslant t_0 - \tau$, and $x \in \mathbb{R}^n$,*

(ii) *$D^+V(t,\varphi(0)) \leqslant -cV(t,\varphi(0))$ for all $t \in [t_{k-1}, t_k)$ $(k \in \mathbb{N})$, whenever $qV(t,\varphi(0)) \geqslant V(t+\theta,\varphi(\theta))$ for all $\theta \in [-\tau, 0]$;*

(iii) *$V(t_k, x + I_k(x)) \leqslant \gamma V(t_k^-, x)$, $k \in \mathbb{N}$, $x \in \mathbb{R}^n$.*

Then the trivial solution of system (2.2) is globally exponentially stable.

Proof. Since $q > \gamma$ and $c > \ln\gamma/\varrho$, there exist positive numbers μ and h such that

$$q > (\gamma + h)e^{\mu\tau} > \gamma, \qquad c > c - \mu > \frac{\ln(\gamma + 2h)}{\varrho}. \tag{3.12}$$

Set $\overline{q} = \gamma + h$ and $\gamma_2 = \ln(\gamma + 2h)/\varrho$, then $1 \leqslant \gamma < \overline{q} < e^{\gamma_2\varrho}$. Set $W(t) = e^{\mu t}V(t, x(t))$, where $x(t)$ is defined as in the proof of Theorem 3.1.

Now, let $\overline{M} > \overline{q}c_2/c_1$, we will prove that (3.3) holds.

We first prove that (3.4) holds. In fact, it is noticed that $W(t_0 + \theta) \leqslant c_2\|\phi\|^p < (1/\overline{q})c_1\overline{M}\|\phi\|^p < c_1\overline{M}\|\phi\|^p$, for $\theta \in [-\tau, 0]$. So it only needs to prove $W(t) < c_1\overline{M}\|\phi\|^p$ for $t \in (t_0, t_1)$. On the contrary, there exists $t \in (t_0, t_1)$ such that $W(t) \geqslant c_1\overline{M}\|\phi\|^p$. Set $t^* = \inf\{t \in (t_0, t_1) : W(t) \geqslant c_1\overline{M}\|\phi\|^p\}$, then we have $t^* \in (t_0, t_1)$. Set $\bar{t} = \sup\{t \in [t_0, t^*) : W(t) \leqslant (1/\overline{q})c_1\overline{M}\|\phi\|^p\}$. Then $\bar{t} \in (t_0, t^*)$ and $W(\bar{t}) = (1/\overline{q})c_1\overline{M}\|\phi\|^p$. For $t \in [\bar{t}, t^*]$, we have

$$\overline{q}W(t) \geqslant c_1\overline{M}\|\phi\|^p \geqslant W(t+\theta), \quad \forall\theta \in [-\tau, 0]. \tag{3.13}$$

Hence

$$V(t, x(t)) \geqslant \frac{1}{\overline{q}}e^{-\mu\tau}V(t+\theta, x(t+\theta)) \geqslant \frac{1}{q}V(t+\theta, x(t+\theta)), \quad \forall\theta \in [-\tau, 0]. \tag{3.14}$$

It follows that for $t \in [\bar{t}, t^*]$

$$D^+W(t) \leqslant (\mu - c)W(t) \leqslant -\gamma_2 W(t), \tag{3.15}$$

which leads to $W(t^*) \leqslant W(\bar{t})$. This is a contradiction to the fact $W(t^*) = c_1\overline{M}\|\phi\|^p > (1/\overline{q})c_1\overline{M}\|\phi\|^p = W(\bar{t})$, so (3.4) holds.

Now, we assume that for some $m \in \mathbb{N}$, (3.8) holds. We will prove that (3.9) holds. In order to do this, we first claim that

$$W(t_m^-) \leqslant \frac{1}{\overline{q}} c_1\overline{M}\|\phi\|^p. \tag{3.16}$$

Suppose not, then we have $W(t_m^-) > (1/\overline{q})c_1\overline{M}\|\phi\|^p$. There are two cases to be considered.

Case 1. $W(t) > (1/\overline{q})c_1\overline{M}\|\phi\|^p$ for all $t \in [t_{m-1}, t_m)$.

By (3.8), we have $\overline{q}W(t) > c_1\overline{M}\|\phi\|^p > W(t+\theta)$ for $\theta \in [-\tau, 0]$ and $t \in [t_{m-1}, t_m)$. Thus, we get $D^+W(t) \leqslant -\gamma_2 W(t)$ for $t \in [t_{m-1}, t_m)$ which leads to $W(t_m^-) \leqslant W(t_{m-1})e^{-\gamma_2(t_m - t_{m-1})} < c_1\overline{M}\|\phi\|^p e^{-\gamma_2 \varrho} < (1/\overline{q})c_1\overline{M}\|\phi\|^p$. This is a contradiction.

Case 2. There is some $t \in [t_{m-1}, t_m)$ such that $W(t) \leqslant (1/\overline{q})c_1\overline{M}\|\phi\|^p$.

Set $\bar{t} = \sup\{t \in [t_{m-1}, t_m) : W(t) \leqslant (1/\overline{q})c_1\overline{M}\|\phi\|^p\}$. Then $\bar{t} \in [t_{m-1}, t_m)$ and $W(\bar{t}) = (1/\overline{q})c_1\overline{M}\|\phi\|^p$. Since for $t \in [\bar{t}, t_m)$, $\overline{q}W(t) \geqslant c_1\overline{M}\|\phi\|^p \geqslant W(t+\theta)$, $\theta \in [-\tau, 0]$. By (3.15), we have $D^+W(t) \leqslant 0$ for $t \in [\bar{t}, t_m)$, which gives $W(t_m^-) \leqslant W(\bar{t}) = (1/\overline{q})c_1\overline{M}\|\phi\|^p$. This is also a contradiction.

Hence, (3.16) holds. It follows from (iii) that $W(t_m) \leqslant \gamma W(t_m^-) \leqslant (\gamma/\overline{q})c_1\overline{M}\|\phi\|^p < c_1\overline{M}\|\phi\|^p$. Now, we assume that (3.9) is not true, set $t^* = \inf\{t \in [t_m, t_{m+1}) : W(t) \geqslant c_1\overline{M}\|\phi\|^p\}$. Hence $t^* \in (t_m, t_{m+1})$ and $W(t^*) = c_1\overline{M}\|\phi\|^p$. If $W(t) > (1/\overline{q})c_1\overline{M}\|\phi\|^p$ for $t \in [t_m, t^*]$, set $\bar{t} = t_m$, otherwise, set $\bar{t} = \sup\{t \in [t_m, t^*) : W(t) \leqslant (1/\overline{q})c_1\overline{M}\|\phi\|^p\}$. Thus, we have $\overline{q}W(t) \geqslant c_1\overline{M}\|\phi\|^p \geqslant W(t+\theta)$ for $t \in [\bar{t}, t^*]$. Hence, by (3.15), $D^+W(t) \leqslant 0$ for $t \in [\bar{t}, t^*]$. Then $W(t^*) \leqslant W(\bar{t}) < c_1\overline{M}\|\phi\|^p$, which yields a contradiction. Therefore, (3.9) holds. The rest is the same as in the proof of Theorem 3.1. □

Remark 3.3. By Theorems 3.1 and 3.2, we can design impulsive control $\{I_k(x(t_k^-)),\ k \in \mathbb{N}\}$ for the following FDEs

$$\begin{gathered} x'(t) = f(t, x_t), \quad t \geqslant t_0, \\ x_{t_0}(s) = \phi(s), \quad -\tau \leqslant s \leqslant 0,\ \phi \in C, \end{gathered} \tag{3.17}$$

such that the system can be impulsively stabilized to its trivial solution. In Theorem 3.1, the constant c may be chosen as a positive number. In the stability theory of FDEs, the condition $D^+V(t, \varphi(0)) \leqslant cV(t, \varphi(0))$ allows the derivative of the Lyapunov function to be positive which may not even guarantee the stability of functional differential system (see, e.g., [4, 15]). However, as we can see from Theorem 3.1, impulses play an important role in making a functional differential system globally exponentially stable even if it may be unstable itself.

Remark 3.4. It is important to emphasize that, in contrast with some existing exponential stability results for IFDEs in the literature [15, 16], Theorems 3.1 and 3.2 are also valid for any finite delay. Therefore, our new results are more practically applicable than those in the literature, since the restrictive condition that the supper bound of time delay is less than the length of all the impulsive intervals is actually removed here.

4. Applications and Examples

Now, we will apply the general Razumikhin-type theorems established in Section 3 to deal with the global exponential stability of impulsive delay differential equations (IDDEs).

Consider a delay system of the form

$$\begin{aligned} x'(t) &= F(t, x(t), x(t-\delta_1(t)), \dots, x(t-\delta_m(t))), \quad t \neq t_k,\ t \geqslant 0, \\ \Delta x(t_k) &= x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k,\ k \in \mathbb{N}, \\ x(s) &= \phi(s), \quad -\tau \leqslant s \leqslant 0, \end{aligned} \tag{4.1}$$

where $\phi \in PC$, $\delta_i : \mathbb{R}^+ \to [0, \tau],\ 1 \leqslant i \leqslant m$, are all continuous, and

$$F : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \longrightarrow \mathbb{R}^n \tag{4.2}$$

is continuous. We also assume that (4.1) has a global solution which is again denoted by $x(t) = x(t, t_0, \phi)$, $F(t, 0, \dots, 0) = 0$ and $I_k(0) = 0,\ k \in \mathbb{N}$.

Theorem 4.1. *Let $\rho = \sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} < \infty$, λ_0 be a real number, and $\lambda_1, \dots, \lambda_m, c_1, c_2, p, \gamma$ be all positive numbers, $0 < \gamma < 1$. Suppose that there exists a function $V \in \nu_1$ such that*

(i) *$c_1|x|^p \leqslant V(t, x) \leqslant c_2|x|^p$ for all $t \geqslant t_0 \geqslant -\tau$, and $x \in \mathbb{R}^n$;*

(ii)

$$V_t(t, x) + V_x(t, x)F(t, x, y) \leqslant \lambda_0 V(t, x) + \sum_{i=1}^{m} \lambda_i V(t - \delta_i(t), y_i) \tag{4.3}$$

for all $t \in [t_{k-1}, t_k)\ (k \in \mathbb{N}),\ x \in \mathbb{R}^n,\ y = (y_1, \dots, y_m) \in \mathbb{R}^{n\times m}$;

(iii) *$V(t_k, x + I_k(x)) \leqslant \gamma V(t_k^-, x)$, $k \in \mathbb{N}$, $x \in \mathbb{R}^n$.*

If $\lambda_0 + \sum_{i=1}^{m} \lambda_i/\gamma + \ln\gamma/\rho < 0$, then the trivial solution of (4.1) *is globally exponentially stable.*

Proof. For $t \in [t_{k-1}, t_k),\ (k \in \mathbb{N}),\ \phi \in PC$, let

$$f(t, \phi) = F(t, \phi(0), \phi(-\delta_1(t)), \dots, \phi(-\delta_m(t))). \tag{4.4}$$

Then system (4.1) becomes system (2.2), and $D^+V(t, \phi(0))$ becomes

$$D^+V(t, \phi(0)) = V_t(t, \phi(0)) + V_x(t, \phi(0))F(t, \phi(0), \phi(-\delta_1(t)), \dots, \phi(-\delta_m(t))). \tag{4.5}$$

If $\lambda_0 + \sum_{i=1}^{m} \lambda_i/\gamma + \ln\gamma/\rho < 0$, then there exists a constant $q > 1/\gamma$, such that

$$\lambda_0 + q\sum_{i=1}^{m}\lambda_i + \frac{\ln\gamma}{\rho} < 0. \tag{4.6}$$

So, if $t \in [t_{k-1}, t_k)$ $(k \in \mathbb{N})$ and $qV(t,\varphi(0)) \geqslant V(t+\theta,\varphi(\theta))$ for all $\theta \in [-\tau, 0]$, then

$$\begin{aligned} D^{+}V(t,\varphi(0)) &\leqslant \lambda_0 V(t,\varphi(0)) + \sum_{i=1}^{m}\lambda_i V(t-\delta_i(t),\varphi(-\delta_i(t))) \\ &\leqslant \left(\lambda_0 + q\sum_{i=1}^{m}\lambda_i\right)V(t,\varphi(0)) = cV(t,\varphi(0)), \end{aligned} \tag{4.7}$$

where $c = \lambda_0 + q\sum_{i=1}^{m}\lambda_i < -\ln\gamma/\rho$. By Theorem 3.1, the trivial solution of (4.1) is globally exponentially stable. □

Corollary 4.2. *Let* $\rho = \sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} < \infty$. *Assume that there exist scalar numbers* $\alpha_i \geqslant 0$ $(1 \leqslant i \leqslant m)$, $0 < \alpha < 1$ *and* η *such that*

$$x^{T}F(t,x,0) \leqslant \eta|x|^2, \tag{4.8}$$

$$\left|F(t,x,y) - F(t,x,0)\right| \leqslant \sum_{i=1}^{m}\alpha_i|y_i|, \tag{4.9}$$

$$|x + I_k(x)| \leqslant \alpha|x| \tag{4.10}$$

for all $t \geqslant t_0$, $x \in \mathbb{R}^n$, $y = (y_1,\ldots,y_m) \in \mathbb{R}^{n\times m}$, $k \in \mathbb{N}$.

If

$$\eta + \frac{\sum_{i=1}^{m}\alpha_i}{\alpha} + \frac{\ln\alpha}{\rho} < 0, \tag{4.11}$$

then the trivial solution of (4.1) *is globally exponentially stable.*

Proof. Let $V(t,x) = x^{T}x = |x|^2$, then we can easily see that condition (i) of Theorem 4.1 holds.

For $x \in \mathbb{R}^n$ and $y = (y_1,\ldots,y_m) \in \mathbb{R}^{n\times m}$, from (4.8) and (4.9), we can calculate that

$$\begin{aligned} V_t(t,x) + V_x(t,x)F(t,x,y) &= 2x^{T}F(t,x,y) = 2x^{T}F(t,x,0) + 2x^{T}\left[F(t,x,y) - F(t,x,0)\right] \\ &\leqslant 2\eta|x|^2 + 2\sum_{i=1}^{m}\alpha_i|x|\cdot|y_i|. \end{aligned} \tag{4.12}$$

Let $\gamma = \alpha^2$, by the inequality $ab \leqslant (a^2+b^2)/2$, we have

$$|x|\cdot|y_i| = \left(\gamma^{-1/4}|x|\right)\left(\gamma^{1/4}|y_i|\right) \leqslant \frac{\gamma^{-1/2}|x|^2 + \gamma^{1/2}|y_i|^2}{2}. \tag{4.13}$$

Substituting (4.13) into (4.12), we obtain

$$V_t(t,x) + V_x(t,x)F(t,x,y) \leqslant \left(2\eta + \frac{\sum_{i=1}^m \alpha_i}{\alpha}\right)|x|^2 + \gamma^{1/2}\sum_{i=1}^m \alpha_i |y_i|^2 = \lambda_0 |x|^2 + \sum_{i=1}^m \lambda_i |y_i|^2, \tag{4.14}$$

where $\lambda_0 = 2\eta + \sum_{i=1}^m \alpha_i/\alpha$, $\lambda_i = \gamma^{1/2}\alpha_i$, $i = 1, \dots, m$.

From (4.11), we have

$$\lambda_0 + \frac{\sum_{i=1}^m \lambda_i}{\gamma} + \frac{\ln\gamma}{\rho} < 0. \tag{4.15}$$

The conclusion follows from Theorem 4.1 immediately and the proof is completed. □

Remark 4.3. Let $\varrho = \inf_{k\in N}\{t_k - t_{k-1}\} > 0$. If $\gamma \geqslant 1$, by Theorem 3.2, we can also give some other results on global exponential stability for (4.1). For the details, we omit them here.

Example 4.4. Consider a scalar nonlinear impulsive delay differential equation

$$\begin{aligned} x'(t) &= F(t, x(t), x(t-\delta(t))), \quad t \neq t_k, \\ \Delta x(t_k) &= c_k x(t_k^-), \ t = t_k, \ k \in \mathbb{N}, \\ x(s) &= \phi(s), \quad s \in [-\tau, 0] \end{aligned} \tag{4.16}$$

on $t \geqslant 0$, where $\delta : R^+ \to [0,\tau]$ is a continuous function, $c_k \in \mathbb{R}$, and

$$F(t,x,y) = bx - \frac{1}{10}x^3 - y\cos t, \tag{4.17}$$

with $x = x(t), y = x(t-\delta(t)), b > 0$.

From (4.8)-(4.9), we can see $\eta = b$, $\alpha_1 = 1$. By Corollary 4.2, if there exists a scalar number $0 < \alpha < 1$, such that

$$|1 + c_k| \leqslant \alpha, \qquad b + \frac{1}{\alpha} + \frac{\ln\alpha}{\rho} < 0, \tag{4.18}$$

then the trivial solution of (4.16) is global exponentially stable.

Example 4.5. Consider the following linear impulsive delay system:

$$\begin{aligned} x'(t) &= Ax(t) + Bx(t-\delta(t)), \quad t \neq t_k, \ t \geqslant 0, \\ \Delta x(t_k) &= C_k x(t_k^-), \quad t = t_k, \ k \in \mathbb{N}, \\ x(s) &= \phi(s), \quad -\tau \leqslant s \leqslant 0, \ \phi \in PC, \end{aligned} \tag{4.19}$$

where

$$A = \begin{bmatrix} 0.1 & 0.2 & -0.1 \\ 0.2 & 0.15 & 0.3 \\ 0 & 0.24 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.12 & 0.03 & 0 \\ 0.12 & -0.2 & 0.05 \\ 0 & 0.14 & -0.1 \end{bmatrix}, \quad C_k = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.8 & 0 \\ 0 & 0 & -0.4 \end{bmatrix}, \tag{4.20}$$

$\delta : \mathbb{R}^+ \to [0, \tau]$ is a continuous function. Since $\lambda_{\min}(A) = -0.2315$ and $\lambda_{\max}(A) = 0.4388$, we can not use the results in [5, 14] to determine the exponential stability. From Corollary 4.2, we can choose $\eta = \lambda_{\max}((A + A^T)/2) = 0.4409$, $\alpha_1 = \|B\| = 0.2905$, $\alpha = \|I + C_k\| = 0.6$, by (4.11), we obtain that if $\sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} < -\alpha \ln \alpha/(\alpha_1 + \alpha\lambda_1) = 0.5521$, then trivial solution of (4.19) is globally exponentially stable.

Remark 4.6. For (4.19), the authors in [15] chose $\delta(t) = (1/40)(1 + e^{-t})$, $\tau = 0.05$, $\sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} \leqslant 0.2$ such that $\tau \leqslant t_k - t_{k-1} \leqslant 0.2$. However, if $\tau > \sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\}$, the results in [15, 16] fail to determine the global exponential stability.

5. Conclusion

In this paper, some new Razumikhin-type theorems on global exponential stability for IFDEs are obtained by employing Lyapunov-Razumikhin technique. Some applications to IDDEs are also given. It should be mentioned that our results may allow us to develop an effective impulsive control strategy to stabilize an underlying delay dynamical system even if it may be unstable in practice, which is particularly meaningful for applications in engineering and technology. Two examples are also given to demonstrate the effectiveness of the theoretical results.

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References

[1] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*, vol. 99 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1993.
[2] V. B. Kolmanovskiĭ and V. R. Nosov, *Stability of Functional Differential Equations*, vol. 180 of *Mathematics in Science and Engineering*, Academic Press, London, UK, 1986.
[3] A. M. Samoĭlenko and N. A. Perestyuk, *Impulsive Differential Equations*, vol. 14 of *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*, World Scientific, Singapore, 1995.
[4] X. Liu and G. Ballinger, "Uniform asymptotic stability of impulsive delay differential equations," *Computers & Mathematics with Applications*, vol. 41, no. 7-8, pp. 903–915, 2001.
[5] X. Liu, "Stability of impulsive control systems with time delay," *Mathematical and Computer Modelling*, vol. 39, no. 4-5, pp. 511–519, 2004.
[6] Y. Zhang and J. Sun, "Strict stability of impulsive functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 301, no. 1, pp. 237–248, 2005.
[7] K. Liu and X. Fu, "Stability of functional differential equations with impulses," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 2, pp. 830–841, 2007.

[8] F. Chen and X. Wen, "Asymptotic stability for impulsive functional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 2, pp. 1149–1160, 2007.
[9] R. Liang and J. Shen, "Uniform stability theorems for delay differential equations with impulses," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 62–74, 2007.
[10] Z. Chen and X. Fu, "New Razumikhin-type theorems on the stability for impulsive functional differential systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 9, pp. 2040–2052, 2007.
[11] Y. Zhang and J. Sun, "Stability of impulsive functional differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 12, pp. 3665–3678, 2008.
[12] Z. Luo and J. Shen, "Stability of impulsive functional differential equations via the Liapunov functional," *Applied Mathematics Letters*, vol. 22, no. 2, pp. 163–169, 2009.
[13] K. Liu and G. Yang, "The improvement of Razumikhin type theorems for impulsive functional differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 9, pp. 3104–3109, 2009.
[14] Q. Wang and X. Liu, "Exponential stability for impulsive delay differential equations by Razumikhin method," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 2, pp. 462–473, 2005.
[15] Q. Wang and X. Liu, "Impulsive stabilization of delay differential systems via the Lyapunov-Razumikhin method," *Applied Mathematics Letters*, vol. 20, no. 8, pp. 839–845, 2007.
[16] X. Liu and Q. Wang, "The method of Lyapunov functionals and exponential stability of impulsive systems with time delay," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 7, pp. 1465–1484, 2007.