Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2010, Article ID 903063, 12 pages
doi:10.1155/2010/903063

Research Article

On Regular Elements in an Incline

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Received 17 August 2009; Revised 31 December 2009; Accepted 28 January 2010

Academic Editor: Aloys Krieg

Inclines are additively idempotent semirings in which products are less than (or) equal to either factor. Necessary and sufficient conditions for an element in an incline to be regular are obtained. It is proved that every regular incline is a distributive lattice. The existence of the Moore-Penrose inverse of an element in an incline with involution is discussed. Characterizations of the set of all generalized inverses are presented as a generalization and development of regular elements in a *-regular ring.

1. Introduction

The notion of inclines and their applications are described comprehensively in Cao et al. [1]. Recently, Kim and Roush have surveyed and outlined algebraic properties of inclines and of matrices over inclines [2]. Multiplicative semigroups unlike matrices over a field are not regular; that is, it is not always possible to solve the regularity equation $\mathbf{axa} = \mathbf{a}$. If there exists x, x is called a g-inverse of $\mathbf{a}$ and the element $\mathbf{a}$ is said to be regular. This concept of regularity of elements in a ring goes back to Neumann [3]. If every element in a ring is regular, then it is called a regular ring. Regular rings are important in many branches of mathematics, especially in matrix theory, since the regularity condition is a linear condition that solves linear equations and takes the place of canonical decomposition.

In [4], Hartwig has studied on existence and construction of various g-inverses associated with an element in a *-regular ring, that is, regular ring with an anti-automorphism and developed a technique for computing g-inverses mainly by using star cancellation law. In semirings one of the most important aspects of structure is a collection of equivalence relations called Green's relations and the corresponding equivalence classes. In [2], it is stated that an element is regular if and only if the equivalence $\mathfrak{D}$ class contains an idempotent.

In this paper, we exhibit that Green's equivalence relations on a pair of elements in an incline reduce to the equality of elements. This leads to the characterization of regular element in an incline that is, an element in an incline is regular if and only if it is idempotent and structure of set of all g-inverses of an element in an incline with involution. In Section 2, we present the basic definitions, notations, and required results on inclines. In Section 3, some characterization of regular elements in an incline are obtained as a generalization of regular elements in a *-regular ring studied by Hartwig and as a development of results available in a Fuzzy algebra. The invariance of the product $\mathbf{b}\mathbf{a}^{-}\mathbf{c}$ for elements $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in a regular incline and a g-inverse $\mathbf{a}^{-}$ of $\mathbf{a}$ is discussed. For elements in a regular incline it is proved that equality of right ideals coincides with equality of left ideals. In Section 4, equivalent conditions for the existence of the Moore-Penrose inverse of an element in an incline with involution-T are determined.

Green's equivalence relation reduces to equality of elements. We conclude that the proofs are purely based on incline property without using star cancellation law as in the work of Hartwig [4].

2. Preliminaries

In this section, we give some definitions and notations.

Definition 2.1. An incline is a nonempty set R with binary operations addition and multiplication denoted as $+, \cdot$ defined on $R \cdot R \to R$ such that for all $x, y, z \in R$

$$\begin{gathered} x+y=y+x, \quad x+(y+z)=(x+y)+z, \\ x(y+z)=xy+xz, \quad (y+z)x=yx+zx, \\ x(yz)=(xy)z, \quad x+x=x, \quad x+xy=x, \quad y+xy=y. \end{gathered} \tag{2.1}$$

Definition 2.2. An incline R is said to be commutative if $xy = yx$ for all $x, y \in R$.

Definition 2.3. $(R, \leq)$ is an incline with order relation "$\leq$" defined on R such that for $x, y \in R$, $x \leq y$ if and only if $x + y = y$. If $x \leq y$, then y is said to dominate x.

Property 2.4. For x, y in an incline R, $x + y \geq x$ and $x + y \geq y$.

For $x + y = (x + x) + y = x + (x + y)$, and $x + y = x + (y + y) = (x + y) + y$

Thus $x + y \geq x$ and $x + y \geq y$.

Property 2.5. For x, y in an incline R, $xy \leq x$ and $xy \leq y$.

Throughout let R denote an incline with order relation $\leq$. For an element $a \in R$, $aR = \{ax/x \in R\}$ is the right ideal of $\mathbf{a}$ and $Ra = \{xa/x \in R\}$ is the left ideal of $\mathbf{a}$.

Definition 2.6 (Green's relation [5]). For any two elements a, b in a semigroup S.

(i) $a\mathcal{L}b$ if there exist $x, y \in S$ such that $xa = b$ and $yb = a$.

(ii) $a\mathcal{R}b$ if there exist $x, y \in S$ such that $ax = b$ and $by = a$.

(iii) $a\mathcal{J}b$ if there exist w, x, y, z such that $wax = b$, $ybz = a$.

(iv) $a\mathcal{H}b$ if $a\mathcal{R}b$ and $a\mathcal{L}b$.

(v) $a\mathfrak{D}b$ if there exists $c \in S$ such that $a\mathcal{R}c$ and $c\mathcal{L}b$.

3. Regular Elements in an Incline

In this section, equivalent conditions for regularity of an element in an incline are obtained and it is proved that a regular commutative incline is a distributive lattice. The equality of right (left) ideals of a pair of elements in a regular incline reduces to the equality of elements. This leads to the invariance of the product $\mathbf{b}\mathbf{a}^{-}\mathbf{c}$ for all choice $\mathbf{a}^{-}$ of $\mathbf{a}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in a regular incline. Characterization of the set of all g-inverses of an element in terms of a particular g-inverse is determined.

Just for sake of completeness we will introduce g-inverses of an element in an incline.

Definition 3.1. $\mathbf{a} \in R$ is said to be regular if there exists an element $x \in R$ such that $\mathbf{axa} = \mathbf{a}$. Then x is called a generalized inverse, in short g-inverse or 1-inverse of $\mathbf{a}$ and is denoted as $\mathbf{a}^{-}$. Let $a\{1\}$ denotes the set of all 1-inverses of $\mathbf{a}$.

Definition 3.2. An element $\mathbf{a} \in R$ is called antiregular, if there exists an element $x \in R$ such that $\mathbf{xax} = \mathbf{x}$. Then x is called the 2-inverse of $\mathbf{a}$. $\mathbf{a}\{2\}$ denotes the set of all 2-inverses of $\mathbf{a}$.

Definition 3.3. For $a \in R$ if there exists $x \in R$ such that $\mathbf{axa} = \mathbf{a}$, $\mathbf{xax} = \mathbf{x}$, and $\mathbf{ax} = \mathbf{xa}$, then x is called the Group inverse of $\mathbf{a}$. The Group inverse of $\mathbf{a}$ is a commuting 1-2 inverse of $\mathbf{a}$.

An incline R is said to be regular if every element of R is regular.

Example 3.4. The Fuzzy algebra $\mathcal{F}$ with support $[0,1]$ under the max. min. operation is an incline [2]. Each element in $\mathcal{F}$ is regular as well as idempotent [6, page 212]. Thus $\mathcal{F}$ is a regular incline.

Example 3.5. Let $D = \{a,b,c\}$ and $R = (\mathcal{P}(D), \cup, \cap)$, where $(\mathcal{P}(D))$ the power set of D is an incline. Here for each element $x \in \mathcal{P}(D)$, $x^2 = x \cap x = x$. Hence x is idempotent and x is regular (refer Proposition 3.7). Thus R is a regular incline.

Lemma 3.6. *Let* $\mathbf{a} \in R$ *be regular. Then* $a = ax = xa$ *for all* $x \in a\{1\}$.

Proof. If $\mathbf{a}$ is regular, then by Property 2.5

$$a = axa \le ax \le a. \tag{3.1}$$

Therefore $ax = a$.

Similarly, from $a \le xa \le a$, it follows that $a = xa$. Thus, $a = xa = ax$ for all $x \in a\ \{1\}$. □

Proposition 3.7. *For* $a \in R$, $\mathbf{a}$*is regular if and only if* $\mathbf{a}$ *is idempotent.*

Proof. Let $a \in R$ be regular. Then by Lemma 3.6, $a = ax = xa$ for all $x \in a\{1\}$. $a = axa = (ax)a = a \cdot a = a^2$. Thus $\mathbf{a}$ is idempotent.

Converse is trivial. □

Example 3.8. Let us consider the example $7_2 = ([0,1], \sup(x,y), xy)$ of an incline given in [2]. Here xy is usual multiplication of real numbers. Hence for each $x \in 7_2$, $x^2 \le x$ and x, is not idempotent. Therefore by Proposition 3.7, 7_2 is not a regular incline.

Proposition 3.9. *If* $\mathbf{a}$ *is regular, then* $\mathbf{a}$ *is the smallest* g*-inverse of* $\mathbf{a}$, *that is,* $a \le x$ *for all* $x \in a\{1\}$.

Proof. Let **a** be regular, then by Proposition 3.7, $a \in a\{1\}$. By Lemma 3.6 $a = ax$ for all $x \in a\{1\}$. Hence by Property 2.5 $a \leq x$. Thus **a** is the smallest g-inverse of **a**. □

It is well known that [7] every distributive lattice is an incline, but an incline need not be a distributive lattice. Now we shall show that regular commutative incline is a distributive lattice in the following.

Proposition 3.10 (see [1]). *A commutative incline is a distributive lattice as (semiring) if and only if $x^2 = x$ for all $x \in X$.*

Lemma 3.11 (see [7]). *DL is a distributive lattice. (DL is the set of all idempotent elements in an incline L.)*

Proposition 3.12. *Let R be a commutative incline, R is regular $\Leftrightarrow$ R is a distributive lattice.*

Proof. Let R is commutative incline.

R is regular: $\Leftrightarrow$ every element in R is idempotent (by Proposition 3.7),

$\Leftrightarrow DR = R$, where DR is the set of all idempotent elements of R,

$\Rightarrow R = DR$ is distributive lattice (by [7, Lemma 2.1]).

Conversely, if R is a distributive lattice then by Proposition 111 in [1] every element of R is idempotent, again by Proposition 3.7R is a regular incline. □

Next we shall see some characterization of regular elements in an incline.

Theorem 3.13. *For $\mathbf{a} \in R$, the following are equivalent:*

(i) **a** *is regular,*

(ii) **a** *is idempotent,*

(iii) $a\{1,2\} = \{a\}$,

(iv) *group inverse of* **a** *exists and coincides with* **a**,

(v) $a = va^2$ *for some* $v \in a\{1\}$,

(vi) $a = a^2u$ *for some* $u \in a\{1\}$.

In either case v, u, vau are all g-inverses of **a** *and vau is invariant for all choice of $u, v \in a\{1\}$. vau is the smallest g-inverse of* **a**.

Proof. (i)$\Leftrightarrow$(ii) This is precisely Proposition 3.7.

To prove the theorem it is enough to prove the following implications:

(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(i); (i)$\Rightarrow$(v)$\Rightarrow$(ii) and (i)$\Rightarrow$(vi)$\Rightarrow$(ii).

(ii)$\Rightarrow$(iii) If **a** is idempotent, then $a \in a\{1\}$. For any $x \in a\{1,2\}$ we have $x = xax$ and by

Lemma 3.6 we get $x = (xa)x = ax = a$. Therefore $a\{1,2\} = \{a\}$.

Thus (iii) holds.

(iii)$\Rightarrow$(iv) If $a\{1,2\} = \{a\}$ then **a** is the only commuting 1-2 inverse of **a**.

Therefore by Definition 3.3 the Group inverse of **a** exists and coincides with **a**.

(iv)$\Rightarrow$(i) This is trivial.

(i)$\Rightarrow$(v) Let **a** be regular, then by Lemma 3.6, for some $v \in a\{1\}$,

$$a = (av)a = (va)a = va^2. \tag{3.2}$$

Thus (v) holds.

(v)⇒(ii) Let $a = va^2$ for some $v \in a\{1\}$. By Property 2.5,

$$
\begin{aligned}
a &= va^2 \leq va \leq a \\
&\Longrightarrow a = va = va^2 \\
a &= va^2 = (va)a = a^2.
\end{aligned}
\tag{3.3}
$$

Therefore **a** is idempotent

Thus (ii) holds.

(i)⇒(vi)⇒(ii) can be proved along the same lines and hence omitted.

Now if $a = va^2$ holds then we can show that $v \in a\{1\}$

$$a = va^2 \leq va \leq a. \tag{3.4}$$

Therefore $a = va^2 = va$

$$a^2 = a \cdot a = va^2 = a, \tag{3.5}$$

and $ava = a^2 = a$

Thus $v \in a\{1\}$.

In a similar manner we can show $u \in a\{1\}$.

Now consider, $x = vau$, where $u, v \in a\{1\}$. It can be verified that

$$x = vau \in a\{1,2\} = \{a\}. \tag{3.6}$$

Hence, $a = vau$ for all $v, u \in a\{1\}$.

Thus vau is invariant for all choice of g-inverse of **a**. By Proposition 3.9, $a = vau$ is the smallest g-inverse of **a**. □

Remark 3.14. If **a** is regular, then (i) $aR = a^2R$ and (ii) $Ra = Ra^2$ automatically holds. The converse holds for an incline with unit.

Remark 3.15. Let us illustrate the relation between various inverses associated with an element in an incline in the following.

Let $R = \{0, a, b, c, d, 1\}$ be a lattice ordered by the following Hasse graph. Define $\cdot$: $R \times R \to R$ by $x \cdot y = d$ for all $x, y \in \{1, b, c, d\}$ and 0 otherwise. Then $(R, \vee, \cdot)$ is an incline which is not a distributive lattice.

In this incline R, the only two elements 0, d are regular which satisfies the Theorem 3.13.

(1) $d \cdot x \cdot d = d$ for each $x \in \{b, c, d, 1\}$.

Hence $d\{1\} = \{b, c, d, 1\}$ and $0\{1\} = R$.

(2) Since $d \in R$, $x \cdot d \cdot x = x$ for $x = 0$, and $x = d$.

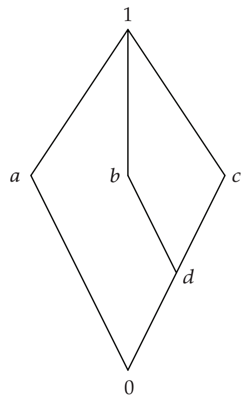

Figure 1

Hence d is antiregular

$$d\{2\} = \{0, d\}. \tag{3.7}$$

(3) $d\{1\} \neq d\{2\}$ and $d\{1,2\} = d\{1\} \cap d\{2\} \;=\; d$.

Theorem 3.16. *Let R be a regular incline. For $a, b, c \in R$ the following hold:*

(i)

$$\begin{aligned} b = ya &\Longleftrightarrow b = ba \Longleftrightarrow Rb \subseteq Ra, \\ c = ax &\Longleftrightarrow c = ac \Longleftrightarrow cR \subseteq aR. \end{aligned} \tag{3.8}$$

(ii) *x is a 1-inverse of* **a**

$$\begin{aligned} &\Longleftrightarrow (ax)^2 = ax, \quad axR = aR, \\ &\Longleftrightarrow (xa)^2 = xa, \quad Rxa = Ra. \end{aligned} \tag{3.9}$$

(iii) *x is a 2-inverse of* **a**

$$\begin{aligned} &\Longleftrightarrow (xa)^2 = xa, \quad xaR = xR, \\ &\Longleftrightarrow (ax)^2 = ax, \quad Rax = Rx. \end{aligned} \tag{3.10}$$

(iv) *$qaR \subseteq qapR$ and $Ra \subseteq Rqa$ implies $pq = a^-$.*

(v) *If $c = ax$ and $b = ya$ then ba^-c is invariant under all choice of 1-inverse of* **a**.

(vi)

$$pwq = a- \Longleftrightarrow \begin{cases} qapwR = qaR \text{ and } w = (qap)^-, \\ Rqa = Ra. \end{cases} \tag{3.11}$$

Proof. Whenever two symmetric results are involved we shall prove the first leaving the second.

(i) Let $b = ya$, since **a** is regular.

$$ba = ba^{-}a = yaa^{-}a = ya = b. \tag{3.12}$$

Thus $b = ya \Rightarrow ba = b$.

Let $ba = b$ then for $z \in Rb$

$$\begin{aligned} z &= xb \text{ for some } x \in R \\ &= (xb)a \in Ra. \end{aligned} \tag{3.13}$$

Thus $b = ba \Rightarrow Rb \subseteq Ra$.

Since b is regular, by Lemma 3.6, $b = xb \in Rb$, since $Rb \subseteq Ra$, $b = xb = ya$. Thus $Rb \subseteq Ra \Rightarrow b = ya$.

Hence (i) holds.

(ii) Let $x \in a\{1\}$ then by Lemma 3.6 and Proposition 3.7 we have

$$(ax)^2 = ax = (xa) = a. \tag{3.14}$$

Hence, $axR = aR$ and $Rxa = Ra$.

Conversely, let $axR = aR$ and $(ax)^2 = ax$.
Then, $aR \subseteq axR \Rightarrow axa = a$ (by (i))

$$\Longrightarrow x \in a\{1\}. \tag{3.15}$$

(iii) Interchange x and a in (ii) then (iii) holds.

(iv) Let $qaR \subseteq qapR$ and $Ra \subseteq Rqa$

$$qaR \subseteq qapR \Rightarrow qapqa = qa \ \text{(by (i))} \tag{3.16}$$

$Ra \subseteq Rqa \Rightarrow aqa = a$, that is, **a** is regular with $q \in a\{1\}$.
Now,

$$\begin{gathered} qapqa = qa, \\ (aqa)pqa = aqa, \\ apqa = a. \end{gathered} \tag{3.17}$$

Therefore $pq \in a\{1\}$.

Thus (iv) holds.

(v) Let $c = ax$ and $b = ya$ for some $x, y \in R$

$$ba^{-}c = y(aa^{-}a)x = yax. \tag{3.18}$$

Which is independent of a^{-} and $ba^{-}c$ is invariant for all choice of a^{-} of **a**.

(vi) Let $pwq = a^{-} \Rightarrow a = a(pwq)a$ (By Definition 3.1).

From the statement (ii), we have
$Ra = Rpwqa \subseteq Rqa \subseteq Ra$

$$\Rightarrow Ra = Rqa. \tag{3.19}$$

Therefore $q \in a\{1\}$ (by (ii)).
Now,

$$\begin{aligned}
apwqa = a &\Longrightarrow qapwaqa = qa, \\
qaR &\subseteq qapwR \subseteq qaR, \\
qaR &= qapwR, \\
qapwqa &= qa, \\
(aqa)pwqa &= aqa, \\
a(pwq)a &= a, \\
&\Longrightarrow pwq \in a\{1\}.
\end{aligned} \tag{3.20}$$

Thus (vi) holds. □

Corollary 3.17. *For a, b in a regular incline one has the following:*

$$Ra = Rb \Longleftrightarrow aR = bR \Longleftrightarrow a = b. \tag{3.21}$$

Proof. Since $Ra = Rb, Ra \subseteq Rb, \text{and } Rb \subseteq Ra$.
By Theorem 3.16(i) we have

$$\begin{aligned}
Ra \subseteq Rb \Rightarrow a = ab \Rightarrow a \leq b \quad &(\text{by Property 2.5}) \\
Rb \subseteq Ra \Rightarrow b = ba \Rightarrow b \leq a \quad &(\text{by Property 2.5}).
\end{aligned} \tag{3.22}$$

Therefore $a = b$. In a similar manner we can show $aR = bR \Rightarrow a = b$.
On the other hand $a = b$ automatically implies $Ra = Rb$ and $aR = bR$. □

Remark 3.18. We note that Corollary 3.17 fails for regular matrices over an incline.

Let us consider $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = PB$, where $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Since $P^2 = I_2, PA = B$. Here A and B are regular.

By Proposition 2.4 in [8, page 297], $R(A) \subseteq R(B)$ and $R(B) \subseteq R(A)$.

Hence $R(A) = R(B)$ but $A \neq B$.

It is well known that [9, page 26] if a^- is a particular g-inverse of **a** in a ring with unit, then the general solution of the equation $axa = a$ is given by $a^- + h - a^-ahaa^-$, where h is arbitrary. Here we shall generalize this for incline.

Theorem 3.19. *Let* **a** $\in R$ *and* a^- *be any particular 1-inverse of a then* $a_g{}^-\{1\} = \{a^- + h/h$ *is arbitrary element in* $R\}$ *is the set of all g-inverses of* **a** *dominating* a^-. *Furthermore,* $a\{1\} = \cup a_g{}^-\{1\}$, *union over all g-inverses of* **a**.

Proof. Let $\mathcal{A}$ denote the set $\{a^- + h/h$ is arbitrary element in $R\}$. Suppose that x is arbitrary element of $a_g{}^-\{1\}$ then $x \geq a^-$ which implies $x + k \geq a^- + k$ for $k \in R$ and by Property 2.4 we have $a^- + k \geq a^-$.

Therefore $x + k \geq a^- + k \geq a^-$

Pre- and postmultiplication by **a** we get

$$\begin{gathered} axa + aka \geq a(a^- + k)a \geq aa^-a, \\ a + aka \geq a(a^- + k)a \geq a \end{gathered} \tag{3.23}$$

(By Definition 3.1).

By Property 2.5 $aka \leq a$, hence $a + aka = a$.

$$\begin{gathered} a \geq a(a^- + k)a \geq a, \\ a(a^- + k)a = a. \end{gathered} \tag{3.24}$$

Therefore $(a^- + k) \in a\{1\}$.

Thus for each $x \in a_g{}^-\{1\}$ there exists an element in $\mathcal{A}$. Hence $a_g{}^-\{1\} \subseteq \mathcal{A}$.

On the other hand for any $y \in \mathcal{A}, y = a^- + h \geq a^-$ by Property 2.4.

From Definition 3.1 and Property 2.5, we get

$$aya = a(a^- + h)a = a + aha = a. \tag{3.25}$$

Hence $y \in a_g{}^-\{1\}$, which implies $\mathcal{A} \subseteq a_g{}^-\{1\}$. Therefore $\mathcal{A} = a_g{}^-\{1\}$

$$\begin{aligned} a\{1\} &= \text{ set of all g-inverses of } \mathbf{a} \\ &= \bigcup a_g{}^-\{1\}, \text{union over all g-inverses of } \mathbf{a}. \end{aligned} \tag{3.26}$$

□

4. Projection on an Incline with Involution-T

In this section, the existence of the Moore-Penrose inverse of an element in an incline with involution-T is discussed as a generalization of that for elements is a *-regular ring and for elements in a Fuzzy algebra studied by Hartwig [4], Kim and Roush [8] and Meenakshi [6], respectively. Characterization of the set of all $\{1,3\}$, $\{1,4\}$ inverses and a formula for Moore-Penrose are obtained analogous to those of the result established for fuzzy matrices in [6, 8].

An involution-T of an incline R is an involutary anti-automorphism, that is, $(a^T)^T = a, (a+b)^T = a^T + b^T, (ab)^T = b^T a^T, a^T = 0$ if and only if $a = 0$ for all $a, b \in R$.

Definition 4.1. An element $a \in R$ is said to be a projection if $a^T = a = a^2$, that is **a** is symmetric and idempotent.

Definition 4.2. For a in an incline R with involution-T, we say that $x \in R$ is a 3-inverse of **a** if $(ax)^T = ax$, and we say that $y \in R$ is a 4-inverse of **a** if $(ya)^T = ya$.

Definition 4.3. An element $x \in R$ is said to be Moore-Penrose inverse of a, if x satisfies the following: (i) $axa = a$, (ii) $xax = x$, (iii) $(ax)^T = ax$, and (iv) $(xa)^T = xa$, denoted as $a^\dagger$.

In [2] it is stated that for an element **a** in an incline with involution-T, $a^\dagger$ exists if and only if $\mathbf{a}\mathscr{H}\mathbf{a}\mathbf{a}^T\mathbf{a}$. Here we derive equivalent condition for the existence of $a^\dagger$ in terms of the weaker relation $\mathbf{a}\mathcal{L}\mathbf{a}\mathbf{a}^T\mathbf{a}$.

First we shall show that Green's equivalence relation on an incline R reduces to equality of elements in R.

Lemma 4.4. *For $a, b \in R$ the following hold:*

(i)$a\mathcal{L}b \Rightarrow a = b$, (ii)$a\mathcal{R}b \Rightarrow a = b$.

Converse holds for elements in a regular incline or incline with unit.

Proof. (i) If $a\mathcal{L}b$ then by Definition 2.6 there exist $x, y \in R$ such that $xa = b$ and $yb = a$. By Property 2.5 we have $xa = b \Rightarrow b \le a$ and $yb = a \Rightarrow a \le b$.

Therefore, $a\mathcal{L}b \Rightarrow a = b$.

(ii) This can be proved in a similar manner and hence omitted.

The converse holds for a regular incline. For, if a, b are regular, then by Lemma 3.6 $a = b = yb = by$ and $b = a = xa = ax$ for some $x, y \in a\{1\}$. Hence $a = b \Rightarrow a\mathcal{L}b$ and $a\mathcal{R}b$. $a\mathcal{L}b$ and $a\mathcal{R}b$ trivially hold for incline with unit. □

Theorem 4.5. *Let R be an incline with involution-T. For* **a** $\in R$ *the following are equivalent:*

(i) **a** *is a projection,*

(ii) **a** *has 1-3 inverse,*

(iii) **a** *has 1-4 inverse,*

(iv) $a^\dagger$ *exists and equals* **a**,

(v) $a^T ax = a^T$ *has a solution in R,*

(vi) $xaa^T = a^T$ *has a solution in R,*

(vii) *a is regular and* $a^T \in a\{1\}$,

(viii) $a\mathcal{L}aa^T a$,

(ix) $a\mathcal{R}aa^T a$,

Proof. (i)⇔(ii) Let **a** be a projection, by Definition 4.1 **a** is symmetric idempotent. **a** is regular follows from Proposition 3.7. Thus **a** has 1-inverse x (say) and by Lemma 3.6 $a = ax = xa$. Since **a** is symmetric, $a = a^T$. Therefore x is a 1-3 inverse of **a**.

Thus **a** has 1–3 inverses. Coverersly if **a** has 1–3 inverses, then again by Lemma 3.6 there exists $x \in R$, such that $a = ax = xa$ and $ax = (ax)^T$. Hence **a** is symmetric idempotent. Thus (i) holds.

(i)⇔(iii) This can be proved along the same lines as that of (i)⇔(iii), hence omitted.

(i)⇔(iv) This equivalence can be proved directly by verifying that **a** satisfy the four equations in Definition 4.3.

(ii)⇔(v) Let **a** has 1–3 inverses, x (say) then

$$\begin{gathered} a^T ax = a^T(ax) = a^T(ax)^T = a^T x^T a^T = (axa)^T = a^T, \\ a^T ax = a^T, \end{gathered} \tag{4.1}$$

(by Definition 4.2).

Conversely, if $a^T ax = a^T$, then $a^T \le a^T a \le a^T \Rightarrow a^T = a^T a$ and therefore $a = a^T a$ and **a** is symmetric. Hence the given condition $a^T ax = a^T$ reduces to a $x = a^T$

Now, $axa = (ax)a = a^T a = a$ and $ax = a^T = a = (ax)^T x \in a\{1,3\}$. Thus **a** has 1–3 inverses.

(iii)⇔(vi) This can be proved in the same manner and hence omitted.

(vii)⇔(i) **a** is regular and $a^T \in a\{1\}$

⇔ **a** is regular and $a = aa^T a$

⇔ **a** is idempotent and $a = a^T a = aa^T$ (by Proposition 3.7 and Lemma 3.6)

⇔ **a** is symmetric and idempotent

⇔ **a** is a projection.

(vii)⇔(viii)⇔(ix) follow from Lemma 4.4. □

Remark 4.6. It is well known that [4] for an element **a** in a *-regular ring if $a^\dagger$ exists then $a^\dagger = a^{(1,4)} a\, a^{(1,3)}$. We observe that for an element a in an incline with involution-T if a is regular, then by Lemma 3.6 it follows that $a\{1,3\} = a\{1,4\}$. If $a^\dagger$ exists it is unique and given by $a^\dagger = a^{(1,3)} a\, a^{(1,3)}$.

Remark 4.7. Let us consider the incline R in Remark 3.15 under the identity involution-T on R. Here each element in R is symmetric and the 3-inverse of the element d is R and 4-inverse also the same.

Hence $0, d$ are the only projections in R

$$\begin{gathered} d^\dagger = d = d\{1,2\} \ne d\{1,3\} = d\{1,4\} = d\{1\} = \{b, c, d, 1\}, \\ b \cdot d \cdot c = d = d^\dagger \quad \forall b, c \in d\{1,4\}. \end{gathered} \tag{4.2}$$

Theorem 4.8. *Let R be an incline with involution-T. For any element $a \in R$ and $x \in a\{1,3\}$ given, then $a_g\{1,3\} = \{x + h/h$ is arbitrary element in $R\}$ is the set of all $\{1,3\}$ inverses of* **a** *dominating x.*

Proof. This can be proved along the same lines as that of Theorem 3.19 and hence omitted. □

5. Conclusion

The main results in the present paper are the generalization of the available results shown in the reference for elements in a *-regular ring [4] and for elements in a Fuzzy algebra [8]. We have proved the results by using Property 2.5 without using star cancellation law.

In [2] it is stated that an element is regular if and only if $\mathfrak{D}$ class contains an idempotent. By Lemma 4.4 the $\mathfrak{D}$ class $\{b/b\mathfrak{D}a\} = \{a\}$ and by Proposition 3.1 **a** is regular if and only if **a** is idempotent.

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