International Scholarly Research Network
ISRN Discrete Mathematics
Volume 2012, Article ID 384068, 18 pages
doi:10.5402/2012/384068

Research Article

Bipartite Graphs Related to Mutually Disjoint S-Permutation Matrices

Krasimir Yordzhev

Faculty of Mathematics and Natural Sciences, South-West University, 2700 Blagoevgrad, Bulgaria

Correspondence should be addressed to Krasimir Yordzhev, yordzhev@swu.bg

Received 18 October 2012; Accepted 7 November 2012

Academic Editors: A. Kelarev, W. F. Klostermeyer, T. Prellberg, and W. Wallis

Some numerical characteristics of bipartite graphs in relation to the problem of finding all disjoint pairs of S-permutation matrices in the general $n^2 \times n^2$ case are discussed in this paper. All bipartite graphs of the type $g = \langle R_g \cup C_g, E_g \rangle$, where $|R_g| = |C_g| = 2$ or $|R_g| = |C_g| = 3$, are provided. The cardinality of the sets of mutually disjoint S-permutation matrices in both the 4×4 and 9×9 cases is calculated.

1. Introduction

Let m be a positive integer. By $[m]$ we denote the set

$$[m] = \{1, 2, \ldots, m\}. \tag{1.1}$$

We let $\mathcal{S}_m$ denote the symmetric group of order m, that is, the group of all one-to-one mappings of the set $[m]$ to itself. If $x \in [m]$, $\rho \in \mathcal{S}_m$, then the image of the element x in the mapping ρ we will denote by $\rho(x)$.

A *bipartite graph* is an ordered triple

$$g = \langle R_g, C_g, E_g \rangle, \tag{1.2}$$

where R_g and C_g are nonempty sets such that $R_g \cap C_g = \emptyset$. The elements of $R_g \cup C_g$ will be called *vertices*. The set of edges is $E_g \subseteq R_g \times C_g = \{\langle r, c \rangle \mid r \in R_g,\ c \in C_g\}$. Multiple edges are not allowed in our considerations.

The subject of the present work is bipartite graphs considered up to isomorphism.

We refer to [1] or [2] for more details on graph theory.

Let n and k be two nonnegative integers, and let $0 \leq k \leq n^2$. We denote by $\mathfrak{G}_{n,k}$ the set of all bipartite graphs of the type $g = \langle R_g, C_g, E_g \rangle$, considered up to isomorphism, such that $|R_g| = |C_g| = n$ and $|E_g| = k$.

Let P_{ij}, $1 \leq i, j \leq n$ be n^2 square $n \times n$ matrices, whose entries are elements of the set $[n^2] = \{1, 2, \ldots, n^2\}$. The $n^2 \times n^2$ matrix

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \quad (1.3)$$

is called a *Sudoku matrix*, if every row, every column, and every submatrix P_{ij}, $1 \leq i, j \leq n$ comprise a permutation of the elements of set $[n^2]$, that is, every number $s \in \{1, 2, \ldots, n^2\}$ is found just once in each row, column, and submatrix P_{ij}. Submatrices P_{ij} are called *blocks* of P.

Sudoku is a very popular game, and Sudoku matrices are special cases of Latin squares in the class of gerechte designs [3].

A matrix is called *binary* if all of its elements are equal to 0 or 1. A square binary matrix is called *permutation matrix* if in every row and every column there is just one 1.

Let us denote by Σ_{n^2} the set of all $n^2 \times n^2$ permutation matrices of the following type:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}, \quad (1.4)$$

where for every $s, t \in \{1, 2, \ldots, n\}$, A_{st} is a square $n \times n$ binary submatrix (block) with only one element equal to 1.

The elements of Σ_{n^2} will be called *S-permutation matrices*.

Two Σ_{n^2} matrices $A = (a_{ij})$ and $B = (b_{ij})$, $1 \leq i, j \leq n^2$ will be called *disjoint* if there are not elements a_{ij} and b_{ij} with the same indices such that $a_{ij} = b_{ij} = 1$.

The concept of S-permutation matrix was introduced by Dahl [4] in relation to the popular Sudoku puzzle.

Obviously, a square $n^2 \times n^2$ matrix P with entries from $[n^2] = \{1, 2, \ldots, n^2\}$ is a Sudoku matrix if and only if there are Σ_{n^2} matrices $A_1, A_2, \ldots, A_{n^2}$ pairwise disjoint, such that P can be written in the following way:

$$P = 1 \cdot A_1 + 2 \cdot A_2 + \cdots + n^2 \cdot A_{n^2}. \quad (1.5)$$

In [5] Fontana offers an algorithm which returns a random family of $n^2 \times n^2$ mutually disjoint S-permutation matrices, where $n = 2, 3$. For $n = 3$, he ran the algorithm 1000 times and found 105 different families of nine mutually disjoint S-permutation matrices. Then, applying (1.5), he decided that there are at least $9! \cdot 105 = 38\ 102\ 400$ Sudoku matrices. This number is very small compared with the exact number of 9×9 Sudoku matrices. In [6] it was shown that there are exactly

$$9! \cdot 72^2 \cdot 2^7 \cdot 27\ 704\ 267\ 971 = 6\ 670\ 903\ 752\ 021\ 072\ 936\ 960 \tag{1.6}$$

number of 9×9 Sudoku matrices.

To evaluate the effectiveness of Fontana's algorithm, it is necessary to calculate the probability of two randomly generated matrices being disjoint. As is proved in [4], the number of S-permutation matrices is equal to

$$|\Sigma_{n^2}| = (n!)^{2n}. \tag{1.7}$$

Thus the question of finding a formula for counting disjoint pairs of S-permutation matrices naturally arises. Such a formula is introduced and verified in [7]. In this paper, we demonstrate this formula to compute the number of disjoint pairs of S-permutation matrices in both the 4×4 and 9×9 cases.

2. A Formula for Counting Disjoint Pairs of S-Permutation Matrices

Let $g = \langle R_g, C_g, E_g \rangle \in \mathfrak{G}_{n,k}$ for some natural numbers n and k, and let $v \in V_g = R_g \cup C_g$.

By $N(v)$ we denote the set of all vertices of V_g, adjacent with v, that is, $u \in N(v)$ if and only if there is an edge in E_g connecting u and v. If v is an isolated vertex (i.e., there is no edge, incident with v), then by definition $N(v) = \emptyset$ and $\text{degree}(v) = |N(v)| = 0$. If $v \in R_g$, then obviously $N(v) \subseteq C_g$, and if $v \in C_g$, then $N(v) \subseteq R_g$.

Let $g = \langle R_g, C_g, E_g \rangle \in \mathfrak{G}_{n,k}$, and let $u, v \in V_g = R_g \cup C_g$. We will say that u and v are equivalent, and we will write $u \sim v$ if $N(u) = N(v)$. If u and v are isolated, then by definition $u \sim v$ if and only if u and v belong simultaneously to R_g, or C_g. The above introduced relation is obviously an equivalence relation.

By $V_{g/\sim}$ we denote the obtained factor set (the set of the equivalence classes) according to relation $\sim$ and let

$$V_{g/\sim} = \{\Delta_1, \Delta_2, \ldots, \Delta_s\}, \tag{2.1}$$

where $\Delta_i \subseteq R_g$, or $\Delta_i \subseteq C_g$, $i = 1, 2, \ldots s$, $2 \leq s \leq 2n$. We put

$$\delta_i = |\Delta_i|, \quad 1 \leq \delta_i \leq n,\ i = 1, 2, \ldots, s, \tag{2.2}$$

and for every $g \in \mathfrak{G}_{n,k}$ we define multiset (set with repetition)

$$[g] = \{\delta_1, \delta_2, \ldots, \delta_s\}, \tag{2.3}$$

where $\delta_1, \delta_2, \ldots, \delta_s$ are natural numbers, obtained by the above described way.

If $z_1, z_2, \ldots, z_n$ is a permutation of the elements of the set $[n] = \{1, 2, \ldots, n\}$ and we shortly denote ρ this permutation, then in this case we denote by $\rho(i)$ the ith element of this permutation, that is, $\rho(i) = z_i$, $i = 1, 2, \ldots, n$.

The following theorem is proved in [7].

Theorem 2.1 (see [7]). *Let $n \geq 2$ be a positive integer. Then the number D_{n^2} of all disjoint ordered pairs of matrices in Σ_{n^2} is equal to*

$$D_{n^2} = (n!)^{4n} + (n!)^{2(n+1)} \sum_{k=1}^{n^2} (-1)^k \sum_{g \in \mathfrak{G}_{n,k}} \frac{\prod_{v \in R_g \cup C_g} (n - |N(v)|)!}{\prod_{\delta \in [g]} \delta!}. \tag{2.4}$$

The number d_{n^2} of all nonordered pairs of disjoint matrices in Σ_{n^2} is equal to

$$d_{n^2} = \frac{1}{2} D_{n^2}. \tag{2.5}$$

The proof of Theorem 2.1 is described in detail in [7], and here we will miss it.

In order to apply Theorem 2.1 it is necessary to describe all bipartite graphs up to isomorphism $g = \langle R_g, C_g, E_g \rangle$, where $|R_g| = |C_g| = n$.

Let n and k are positive integers, and let $g \in \mathfrak{G}_{n,k}$. We examine the ordered $(n+1)$-tuple

$$\Psi(g) = \langle \psi_0(g), \psi_1(g), \ldots, \psi_n(g) \rangle, \tag{2.6}$$

where $\psi_i(g)$, $i = 0, 1, \ldots, n$ is equal to the number of vertices of g incident with exactly i number of edges. It is obvious that $\sum_{i=1}^{n} i\psi_i(g) = 2k$ is true for all $g \in \mathfrak{G}_{n,k}$. Then formula (2.4) can be presented as

$$D_{n^2} = (n!)^{4n} + (n!)^{2(n+1)} \sum_{k=1}^{n^2} (-1)^k \sum_{g \in \mathfrak{G}_{n,k}} \frac{\prod_{i=0}^{n} [(n-i)!]^{\psi_i(g)}}{\prod_{\delta \in [g]} \delta!}. \tag{2.7}$$

Since $(n-n)! = 0! = 1$ and $[n-(n-1)]! = 1! = 1$, then

$$D_{n^2} = (n!)^{4n} + (n!)^{2(n+1)} \sum_{k=1}^{n^2} (-1)^k \sum_{g \in \mathfrak{G}_{n,k}} \frac{\prod_{i=0}^{n-2} [(n-i)!]^{\psi_i(g)}}{\prod_{\delta \in [g]} \delta!}. \tag{2.8}$$

Consequently, to apply formula (2.8) for each bipartite graph $g \in \mathfrak{G}_{n,k}$ and for the set $\mathfrak{G}_{n,k}$ of bipartite graphs, it is necessary to obtain the following numerical characteristics:

$$\begin{gathered} \omega(g) = \frac{\prod_{i=0}^{n-2} [(n-i)!]^{\psi_i(g)}}{\prod_{\delta \in [g]} \delta!}, \\ \theta(n,k) = \sum_{g \in \mathfrak{G}_{n,k}} \omega(g). \end{gathered} \tag{2.9}$$

Using the numerical characteristics (2.9), we obtain the following variety of Theorem 2.1.

Theorem 2.2. *One has*

$$D_{n^2} = (n!)^{4n} + (n!)^{2(n+1)} \sum_{k=1}^{n^2} (-1)^k \theta(n,k), \tag{2.10}$$

where $\theta(n,k)$ is described using formulas (2.9).

3. Demonstrations in Applying Theorem 2.2

3.1. Counting the Number D_4 of All Ordered Pairs of Disjoint S-Permutation Matrices for $n = 2$

3.1.1. Consider $k = 1$

In $n = 2$ and $k = 1$, $\mathfrak{G}_{2,1}$ consists of a single graph g_1 shown in Figure 1.

For graph $g_1 \in \mathfrak{G}_{2,1}$ we have

$$[g_1] = \{1,1,1,1\},$$
$$\Psi(g_1) = \langle \psi_0(g_1), \psi_1(g_1), \psi_2(g_1) \rangle = \langle 2,2,0 \rangle. \tag{3.1}$$

Then we get

$$\omega(g_1) = \frac{[(2-0)!]^2}{1!\ 1!\ 1!\ 1!} = 4, \tag{3.2}$$

and therefore

$$\theta(2,1) = \sum_{g \in \mathfrak{G}_{2,1}} \omega(g) = 4. \tag{3.3}$$

3.1.2. Consider $k = 2$

The set $\mathfrak{G}_{2,2}$ consists of three graphs g_2, g_3, and g_4 depicted in Figure 2.

For graph $g_2 \in \mathfrak{G}_{2,2}$ we have

$$[g_2] = \{1,1,1,1\},$$
$$\Psi(g_2) = \langle \psi_0(g_2), \psi_1(g_2), \psi_2(g_2) \rangle = \langle 0,4,0 \rangle,$$
$$\omega(g_1) = \frac{[(2-0)!]^0}{1!\ 1!\ 1!\ 1!} = 1. \tag{3.4}$$

For graphs $g_3 \in \mathfrak{G}_{2,2}$ and $g_4 \in \mathfrak{G}_{2,2}$ we have

$$[g_3] = [g_4] = \{2,1,1\},$$
$$\Psi(g_3) = \Psi(g_4) = \langle 1,2,1 \rangle,$$
$$\omega(g_3) = \omega(g_4) = \frac{[(2-0)!]^1}{2!\ 1!\ 1!} = 1. \tag{3.5}$$

Then for the set $\mathfrak{G}_{2,2}$ we get

$$\theta(2,2) = \sum_{g \in \mathfrak{G}_{2,2}} \omega(g) = 1 + 1 + 1 = 3. \tag{3.6}$$

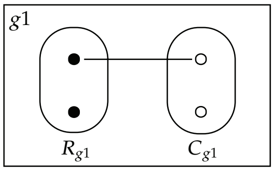

Figure 1: $n = 2, k = 1$.

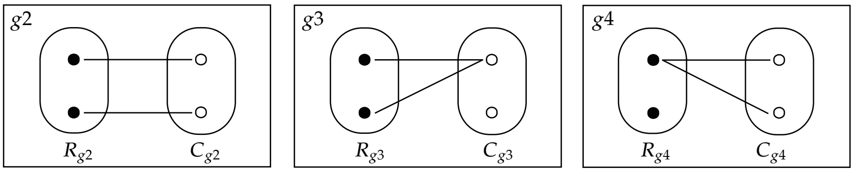

Figure 2: $n = 2, k = 2$.

3.1.3. Consider $k = 3$

In $n = 2$ and $k = 3$, $\mathfrak{G}_{2,3}$ consists of a single graph g_5 shown in Figure 3.

For graph $g_5 \in \mathfrak{G}_{2,3}$ we have

$$\begin{gathered}[g_5] = \{1,1,1,1\}, \\ \Psi(g_5) = \langle \psi_0(g_5), \psi_1(g_5), \psi_2(g_5) \rangle = \langle 0,2,2 \rangle.\end{gathered} \tag{3.7}$$

Then we get

$$\omega(g_5) = \frac{[(2-0)!]^0}{1! \ \ 1! \ \ 1! \ \ 1!} = 1 \tag{3.8}$$

and therefore

$$\theta(2,3) = \sum_{g \in \mathfrak{G}_{2,3}} \omega(g) = 1. \tag{3.9}$$

3.1.4. Consider $k = 4$

When $n = 2$ and $k = 4$, there is only one graph, and this is the complete bipartite graph g_6 which is shown in Figure 4.

For graph $g_6 \in \mathfrak{G}_{2,4}$ we have

$$\begin{gathered}[g_6] = \{2,2\}, \\ \Psi(g_6) = \langle \psi_0(g_6), \psi_1(g_6), \psi_2(g_6) \rangle = \langle 0,0,4 \rangle.\end{gathered} \tag{3.10}$$

Then we get

$$\omega(g_6) = \frac{[(2-0)!]^0}{2! \ \ 2!} = \frac{1}{4}, \tag{3.11}$$

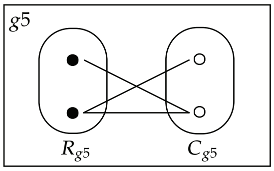

Figure 3: $n = 2, k = 3$.

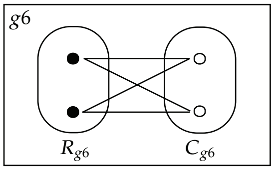

Figure 4: $n = 2, k = 4$.

and therefore

$$\theta(2,4) = \sum_{g\in\mathfrak{G}_{2,1}} \omega(g) = \frac{1}{4}. \tag{3.12}$$

Having in mind the formulas (2.10), (3.3), (3.6), (3.9), and (3.12) for the number D_4 of all ordered pairs disjoint S-permutation matrices in $n = 2$ we finally get

$$\begin{aligned} D_4 &= (2!)^8 + (2!)^6[-\theta(2,1) + \theta(2,2) - \theta(2,3) + \theta(2,4)] \\ &= 256 + 64\left(-4 + 3 - 1 + \frac{1}{4}\right) = 144. \end{aligned} \tag{3.13}$$

The number d_4 of all nonordered pairs disjoint matrices from Σ_4 is equal to

$$d_4 = \frac{1}{2} D_4 = 72. \tag{3.14}$$

3.2. Counting the Number D_9 of All Ordered Pairs of Disjoint S-Permutation Matrices for $n = 3$

3.2.1. *Consider* $k = 1$

Graph g_7, which is displayed in Figure 5, is the only bipartite graph belonging to the set $\mathfrak{G}_{3,1}$.

For graph $g_7 \in \mathfrak{G}_{3,1}$ we have

$$\begin{gathered} [g_7] = \{1,1,2,2\}, \\ \Psi(g_7) = \langle \psi_0(g_7), \psi_1(g_7), \psi_2(g_7), \psi_3(g_7), \psi_4(g_8) \rangle = \langle 4,2,0,0 \rangle. \end{gathered} \tag{3.15}$$

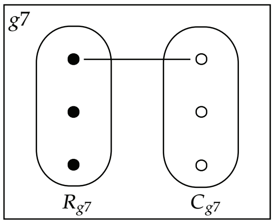

Figure 5: $n = 3$, $k = 1$.

Then we get

$$\omega(g_7) = \frac{[(3-0)!]^4[(3-1)!]^2}{1!\ \ 1!\ \ 2!\ \ 2!} = \frac{6^4 \cdot 2^2}{1 \cdot 1 \cdot 2 \cdot 2} = 1296, \tag{3.16}$$

and therefore

$$\theta(3,1) = \sum_{g \in \mathfrak{G}_{3,1}} \omega(g) = 1296. \tag{3.17}$$

3.2.2. Consider $k = 2$

In this case $\mathfrak{G}_{3,2} = \{g_8, g_9, g_{10}\}$. The graphs g_8, g_9, and g_{10} are shown in Figure 6.

For graph $g_8 \in \mathfrak{G}_{3,2}$ we have

$$\begin{gathered}[g_8] = \{1,1,1,1,1,1\}, \\ \Psi(g_8) = \langle \varphi_0(g_8), \varphi_1(g_8), \varphi_2(g_8), \varphi_3(g_8), \varphi_4(g_8) \rangle = \langle 2,4,0,0 \rangle, \\ \omega(g_8) = \frac{[(3-0)!]^2[(3-1)!]^4}{1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!} = 6^2 \cdot 2^4 = 576.\end{gathered} \tag{3.18}$$

For graphs $g_9 \in \mathfrak{G}_{3,2}$ and $g_{10} \in \mathfrak{G}_{3,2}$ we have

$$\begin{gathered}[g_9] = [g_{10}] = \{1,1,2,2\}, \\ \Psi(g_9) = \Psi(g_{10}) = \langle 3,2,1,0 \rangle, \\ \omega(g_9) = \omega(g_{10}) = \frac{[(3-0)!]^3[(3-1)!]^2}{1!\ \ 1!\ \ 2!\ \ 2!} = \frac{6^3 \cdot 2^2}{1 \cdot 1 \cdot 2 \cdot 2} = 216.\end{gathered} \tag{3.19}$$

Then for the set $\mathfrak{G}_{3,2}$ we get

$$\theta(3,2) = \sum_{g \in \mathfrak{G}_{3,2}} \omega(g) = 576 + 216 + 216 = 1008. \tag{3.20}$$

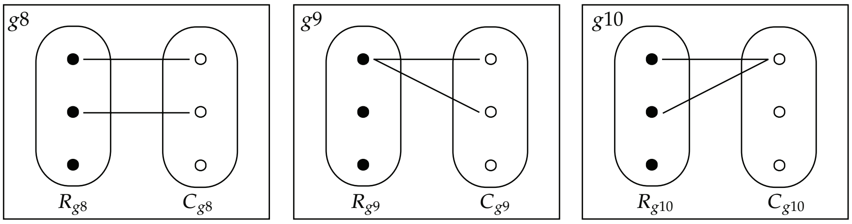

Figure 6: $n = 3, k = 2$.

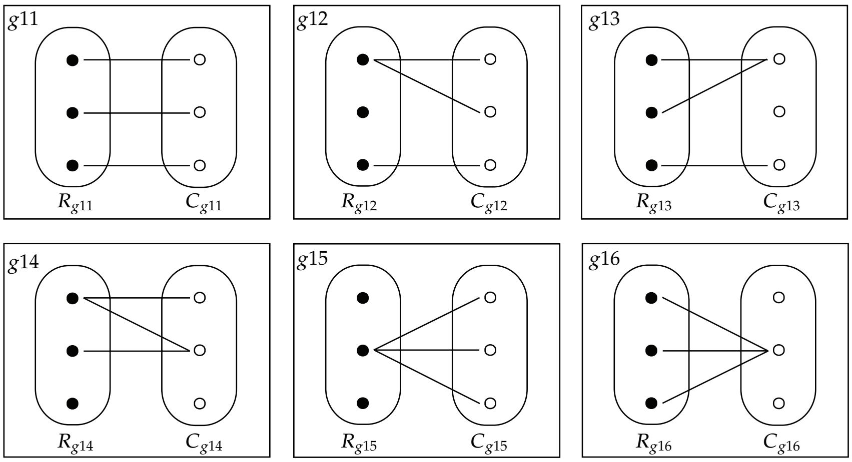

Figure 7: $n = 3, k = 3$.

3.2.3. Consider $k = 3$

When $n = 3$ and $k = 3$, the set $\mathfrak{G}_{3,3} = \{g_{11}, g_{12}, g_{13}, g_{14}, g_{15}, g_{16}\}$ consists of six bipartite graphs, which are shown in Figure 7.

For graph $g_{11} \in \mathfrak{G}_{3,3}$ we have

$$\begin{aligned}
[g_{11}] &= \{1,1,1,1,1,1\}, \\
\Psi(g_{11}) &= \langle 0,6,0,0\rangle, \\
\omega(g_{11}) &= \frac{[(3-0)!]^0[(3-1)!]^6}{1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!} = 6^0 \cdot 2^6 = 64.
\end{aligned} \tag{3.21}$$

For graphs $g_{12}, g_{13} \in \mathfrak{G}_{3,3}$ we have

$$\begin{aligned}
[g_{12}] = [g_{13}] &= \{1,1,1,1,2\}, \\
\Psi(g_{12}) = \Psi(g_{13}) &= \langle 1,4,1,0\rangle, \\
\omega(g_{12}) = \omega(g_{13}) &= \frac{[(3-0)!]^1[(3-1)!]^4}{1!\ \ 1!\ \ 1!\ \ 1!\ \ 2!} = \frac{6^1 \cdot 2^4}{2} = 48.
\end{aligned} \tag{3.22}$$

For graph $g_{14} \in \mathfrak{G}_{3,3}$ we have

$$\begin{gathered}[g_{14}] = \{1,1,1,1,1,1\}, \\ \Psi(g_{14}) = \langle 2,2,2,0\rangle, \\ \omega(g_{14}) = \frac{[(3-0)!]^2[(3-1)!]^2}{1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!} = 6^2 \cdot 2^2 = 144.\end{gathered} \tag{3.23}$$

For graphs $g_{15}, g_{16} \in \mathfrak{G}_{3,3}$ we have

$$\begin{gathered}[g_{15}] = [g_{16}] = \{1,2,3\}, \\ \Psi(g_{15}) = \Psi(g_{16}) = \langle 2,3,0,1\rangle, \\ \omega(g_{15}) = \omega(g_{16}) = \frac{[(3-0)!]^2[(3-1)!]^3}{1!\ \ 2!\ \ 3!} = \frac{6^2 \cdot 2^3}{2 \cdot 6} = 24.\end{gathered} \tag{3.24}$$

Then for the set $\mathfrak{G}_{3,3}$ we get

$$\theta(3,3) = \sum_{g \in \mathfrak{G}_{3,3}} \omega(g) = 64 + 48 + 48 + 144 + 24 + 24 = 352. \tag{3.25}$$

3.2.4. *Consider $k = 4$*

When $n = 3$ and $k = 4$, the set $\mathfrak{G}_{3,4} = \{g_{17}, g_{18}, g_{19}, g_{20}, g_{21}, g_{22}, g_{23}\}$ consists of seven bipartite graphs, which are shown in Figure 8.

For graph $g_{17} \in \mathfrak{G}_{3,4}$ we have

$$\begin{gathered}[g_{17}] = \{1,1,2,2\}, \\ \Psi(g_{17}) = \langle 2,0,4,0\rangle, \\ \omega(g_{17}) = \frac{[(3-0)!]^2[(3-1)!]^0}{1!\ \ 1!\ \ 2!\ \ 2!} = \frac{6^2 \cdot 2^0}{2^2} = 9.\end{gathered} \tag{3.26}$$

For graph $g_{18} \in \mathfrak{G}_{3,4}$ we have

$$\begin{gathered}[g_{18}] = \{1,1,2,2\}, \\ \Psi(g_{18}) = \langle 0,4,2,0\rangle, \\ \omega(g_{18}) = \frac{[(3-0)!]^0[(3-1)!]^4}{1!\ \ 1!\ \ 2!\ \ 2!} = \frac{6^0 \cdot 2^4}{2^2} = 4.\end{gathered} \tag{3.27}$$

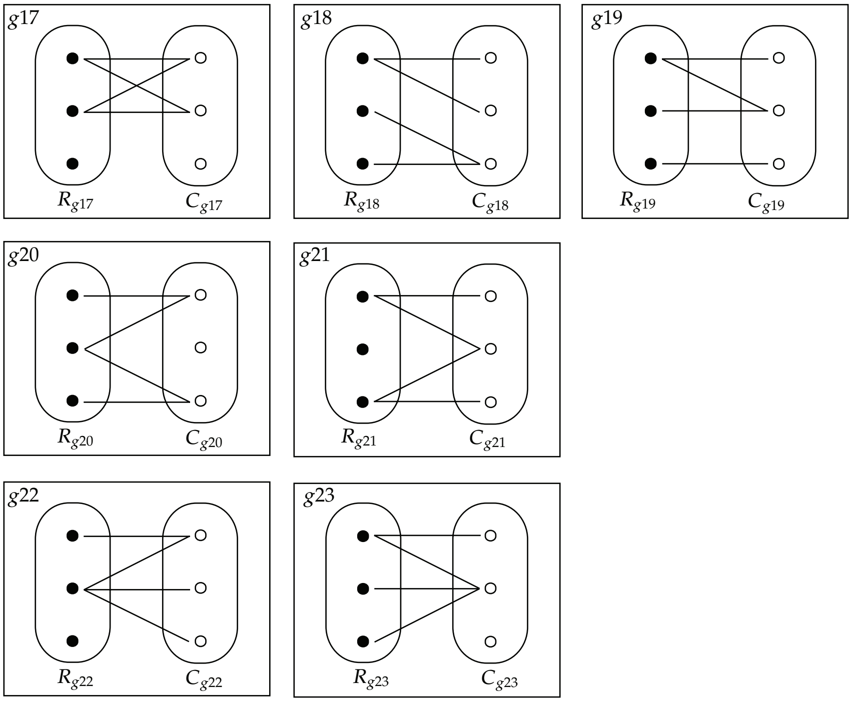

Figure 8: $n = 3$, $k = 4$.

For graph $g_{19} \in \mathfrak{G}_{3,4}$ we have

$$\begin{gathered}[g_{19}] = \{1,1,1,1,1,1\}, \\ \Psi(g_{19}) = \langle 0,4,2,0 \rangle, \\ \omega(g_{19}) = \frac{[(3-0)!]^0[(3-1)!]^4}{1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!} = 6^0 \cdot 2^4 = 16.\end{gathered} \tag{3.28}$$

For graphs $g_{20} \in \mathfrak{G}_{3,4}$ and $g_{21} \in \mathfrak{G}_{3,4}$ we have

$$\begin{gathered}[g_{20}] = [g_{21}] = \{1,1,1,1,1,1\}, \\ \Psi(g_{20}) = \Psi(g_{21}) = \langle 1,2,3,0 \rangle, \\ \omega(g_{20}) = \omega(g_{21}) = \frac{[(3-0)!]^1[(3-1)!]^2}{1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!} = 6^1 \cdot 2^2 = 24.\end{gathered} \tag{3.29}$$

For graphs $g_{22} \in \mathfrak{G}_{3,4}$ and $g_{23} \in \mathfrak{G}_{3,4}$ we have

$$\begin{gathered}[g_{22}] = [g_{23}] = \{1,1,1,1,2\}, \\ \Psi(g_{22}) = \Psi(g_{23}) = \langle 1,3,1,1 \rangle, \\ \omega(g_{22}) = \omega(g_{23}) = \frac{[(3-0)!]^1[(3-1)!]^3}{1!\ \ 1!\ \ 1!\ \ 1!\ \ 2!} = \frac{6^1 \cdot 2^3}{2} = 24.\end{gathered} \tag{3.30}$$

Then we get

$$\theta(3,4) = \sum_{g\in\mathfrak{G}_{3,4}} \omega(g) = 9+4+16+24+24+24+24 = 125. \tag{3.31}$$

3.2.5. Consider $k = 5$

When $n = 3$ and $k = 5$, the set $\mathfrak{G}_{3,5}$ consists of seven bipartite graphs $g_{24} \div g_{30}$, which are shown in Figure 9.

For graph $g_{24} \in \mathfrak{G}_{3,5}$ we have

$$\begin{gathered} [g_{24}] = \{1,1,2,2\}, \\ \Psi(g_{24}) = \langle 0,4,0,2\rangle, \\ \omega(g_{18}) = \frac{[(3-0)!]^0[(3-1)!]^4}{1!\ \ 1!\ \ 2!\ \ 2!} = \frac{6^0\cdot 2^4}{2^2} = 4. \end{gathered} \tag{3.32}$$

For graph $g_{25} \in \mathfrak{G}_{3,5}$ we have

$$\begin{gathered} [g_{25}] = \{1,1,2,2\}, \\ \Psi(g_{25}) = \langle 0,2,4,0\rangle, \\ \omega(g_{18}) = \frac{[(3-0)!]^0[(3-1)!]^2}{1!\ \ 1!\ \ 2!\ \ 2!} = \frac{6^0\cdot 2^2}{2^2} = 1. \end{gathered} \tag{3.33}$$

For graph $g_{26} \in \mathfrak{G}_{3,5}$ we have

$$\begin{gathered} [g_{26}] = \{1,1,1,1,1,1\}, \\ \Psi(g_{26}) = \langle 0,2,4,0\rangle, \\ \omega(g_{26}) = \frac{[(3-0)!]^0[(3-1)!]^2}{1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!} = 6^0\cdot 2^2 = 4. \end{gathered} \tag{3.34}$$

For graphs $g_{27} \in \mathfrak{G}_{3,5}$ and $g_{28} \in \mathfrak{G}_{3,5}$ we have

$$\begin{gathered} [g_{27}] = [g_{28}] = \{1,1,1,1,1,1\}, \\ \Psi(g_{27}) = \Psi(g_{28}) = \langle 0,3,2,1\rangle, \\ \omega(g_{27}) = \omega(g_{28}) = \frac{[(3-0)!]^0[(3-1)!]^3}{1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!} = 6^0\cdot 2^3 = 8. \end{gathered} \tag{3.35}$$

For graphs $g_{29} \in \mathfrak{G}_{3,5}$ and $g_{30} \in \mathfrak{G}_{3,5}$ we have

$$\begin{gathered} [g_{29}] = [g_{30}] = \{1,1,1,1,2\}, \\ \Psi(g_{29}) = \Psi(g_{30}) = \langle 1,1,3,1\rangle, \\ \omega(g_{29}) = \omega(g_{30}) = \frac{[(3-0)!]^1[(3-1)!]^1}{1!\ \ 1!\ \ 1!\ \ 1!\ \ 2!} = \frac{6^1\cdot 2^1}{2} = 6. \end{gathered} \tag{3.36}$$

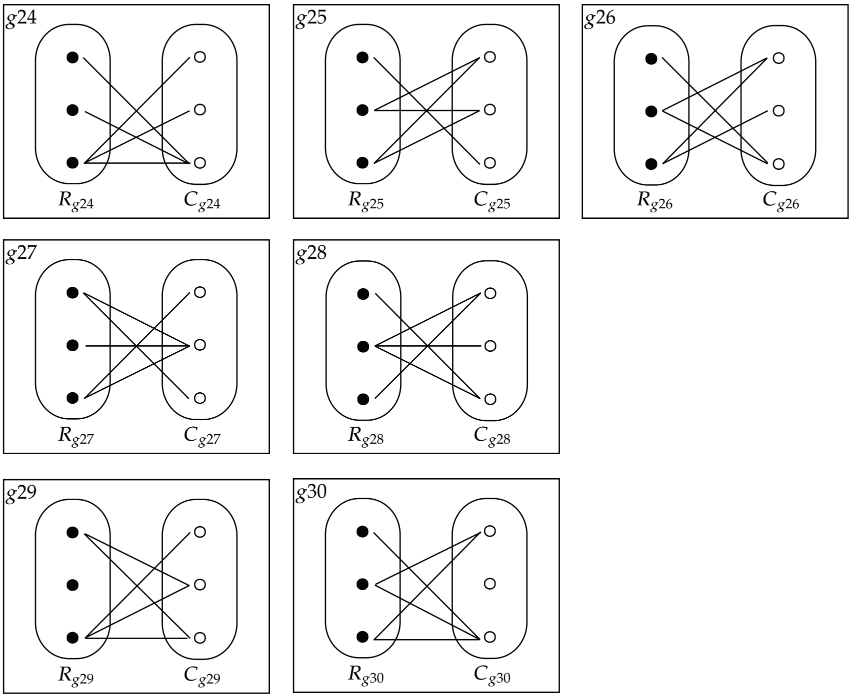

Figure 9: $n = 3$, $k = 5$.

Then we get

$$\theta(3,5) = \sum_{g \in \mathfrak{G}_{3,5}} \omega(g) = 4 + 1 + 4 + 8 + 8 + 6 + 6 = 37. \tag{3.37}$$

3.2.6. *Consider* $k = 6$

When $n = 3$ and $k = 6$, the set $\mathfrak{G}_{3,6} = \{g_{31}, g_{32}, g_{33}, g_{34}, g_{35}, g_{36}\}$ consists of six bipartite graphs, which are shown in Figure 10.

For graph $g_{31} \in \mathfrak{G}_{3,6}$ we have

$$\begin{aligned} [g_{31}] &= \{1,1,1,1,1,1\}, \\ \Psi(g_{31}) &= \langle 0,0,6,0 \rangle, \\ \omega(g_{31}) &= \frac{[(3-0)!]^0 [(3-1)!]^0}{1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!} = 1. \end{aligned} \tag{3.38}$$

For graphs $g_{32} \in \mathfrak{G}_{3,6}$ and $g_{33} \in \mathfrak{G}_{3,6}$ we have

$$\begin{aligned} [g_{32}] = [g_{33}] &= \{1,1,1,1,2\}, \\ \Psi(g_{32}) = \Psi(g_{33}) &= \langle 0,1,4,1 \rangle, \\ \omega(g_{32}) = \omega(g_{33}) &= \frac{[(3-0)!]^0 [(3-1)!]^1}{1!\ \ 1!\ \ 1!\ \ 1!\ \ 2!} = \frac{6^0 \cdot 2^1}{2} = 1. \end{aligned} \tag{3.39}$$

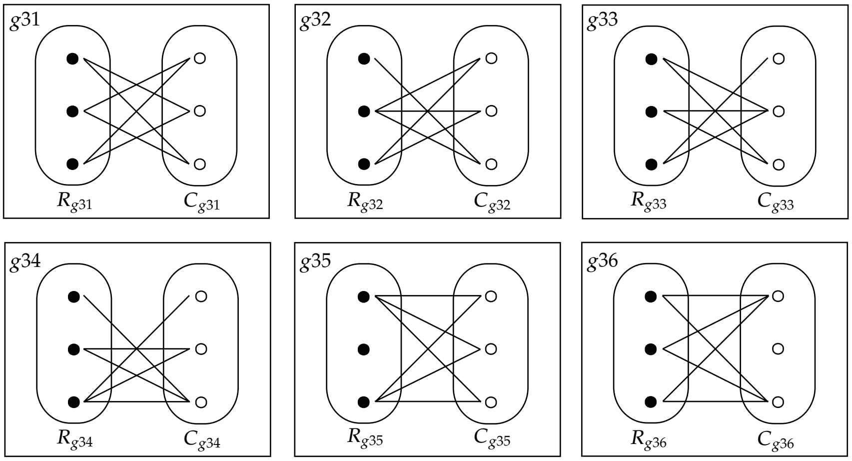

Figure 10: $n = 3$, $k = 6$.

For graph $g_{34} \in \mathfrak{G}_{3,6}$ we have

$$\begin{gathered}
[g_{34}] = \{1,1,1,1,1,1\}, \\
\Psi(g_{34}) = \langle 0,2,2,2 \rangle, \\
\omega(g_{34}) = \frac{[(3-0)!]^0[(3-1)!]^2}{1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!\ \ 1!} = \frac{6^0 \cdot 2^2}{1} = 4.
\end{gathered} \tag{3.40}$$

For graphs $g_{35} \in \mathfrak{G}_{3,6}$ and $g_{36} \in \mathfrak{G}_{3,6}$ we have

$$\begin{gathered}
[g_{35}] = [g_{36}] = \{1,2,3\}, \\
\Psi(g_{35}) = \Psi(g_{36}) = \langle 1,0,3,2 \rangle, \\
\omega(g_{35}) = \omega(g_{36}) = \frac{[(3-0)!]^1[(3-1)!]^0}{1!\ \ 2!\ \ 3!} = \frac{6^1 \cdot 2^0}{2 \cdot 6} = \frac{1}{2}.
\end{gathered} \tag{3.41}$$

Then for the set $\mathfrak{G}_{3,6}$ we get

$$\theta(3,6) = \sum_{g \in \mathfrak{G}_{3,6}} \omega(g) = 1 + 1 + 1 + 4 + \frac{1}{2} + \frac{1}{2} = 8. \tag{3.42}$$

3.2.7. Consider $k = 7$

When $n = 3$ and $k = 7$ the set $\mathfrak{G}_{3,7} = \{g_{37}, g_{38}, g_{39}\}$ consists of three bipartite graphs, which are shown in Figure 11.

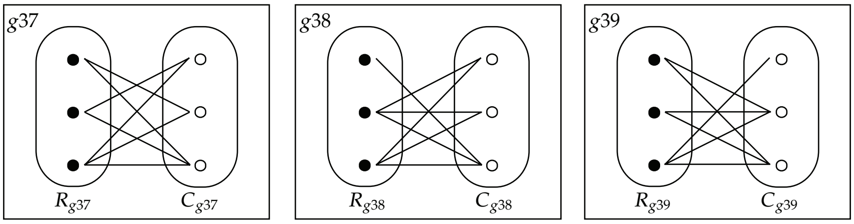

Figure 11: $n = 3$, $k = 7$.

For graph $g_{37} \in \mathfrak{G}_{3,7}$ it is true that

$$\begin{gathered}
[g_{37}] = \{1,1,1,1,1,1\}, \\
\Psi(g_{37}) = \langle 0,0,4,2 \rangle, \\
\omega(g_{37}) = \frac{[(3-0)!]^0[(3-1)!]^0}{1!\ 1!\ 1!\ 1!\ 1!\ 1!} = \frac{6^0 \cdot 2^0}{1} = 1.
\end{gathered} \tag{3.43}$$

For graphs $g_{38} \in \mathfrak{G}_{3,7}$ and $g_{39} \in \mathfrak{G}_{3,7}$ we get

$$\begin{gathered}
[g_{38}] = [g_{39}] = \{1,1,2,2\}, \\
\Psi(g_{38}) = \Psi(g_{39}) = \langle 0,1,2,3 \rangle, \\
\omega(g_{38}) = \omega(g_{39}) = \frac{[(3-0)!]^0[(3-1)!]^1}{1!\ 1!\ 2!\ 2!} = \frac{6^0 \cdot 2^1}{2^2} = \frac{1}{2}.
\end{gathered} \tag{3.44}$$

Then for the set $\mathfrak{G}_{3,7}$ we get

$$\theta(3,7) = \sum_{g \in \mathfrak{G}_{3,7}} \omega(g) = 1 + \frac{1}{2} + \frac{1}{2} = 2. \tag{3.45}$$

3.2.8. Consider $k = 8$

Graph g_{40}, which is displayed in Figure 12, is the only bipartite graph belonging to the set $\mathfrak{G}_{3,8}$ in the case $n = 3$ and $k = 8$.

For graph $g_{40} \in \mathfrak{G}_{3,8}$ it is true that

$$\begin{gathered}
[g_{40}] = \{1,1,2,2\}, \\
\Psi(g_{40}) = \langle 0,0,2,4 \rangle, \\
\omega(g_{40}) = \frac{[(3-0)!]^0[(3-1)!]^0}{1!\ 1!\ 2!\ 2!} = \frac{6^0 \cdot 2^0}{2^2} = \frac{1}{4}.
\end{gathered} \tag{3.46}$$

Therefore,

$$\theta(3,8) = \sum_{g \in \mathfrak{G}_{3,8}} \omega(g) = \frac{1}{4}. \tag{3.47}$$

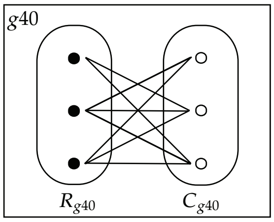

Figure 12: $n = 3$, $k = 8$.

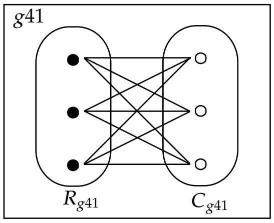

Figure 13: $n = 3$, $k = 9$.

3.2.9. Consider $k = 9$

When $n = 3$ and $k = 9$ there is only one graph, and this is the complete bipartite graph g_{41} which is shown in Figure 13.

For graph g_{41} it is true that

$$\begin{gathered} [g_{41}] = \{3, 3\}, \\ \Psi(g_{41}) = \langle 0, 0, 0, 6 \rangle, \\ \omega(g_{41}) = \frac{[(3-0)!]^0 [(3-1)!]^0}{3!\ \ 3!} = \frac{6^0 \cdot 2^0}{6^2} = \frac{1}{36}. \end{gathered} \tag{3.48}$$

Therefore

$$\theta(3, 9) = \sum_{g \in \mathfrak{G}_{3,9}} \omega(g) = \frac{1}{36}. \tag{3.49}$$

Having in mind the formula (2.10) and formulas (3.17) ÷ (3.49) for the number D_9 of all ordered pairs disjoint S-permutation matrices in $n = 3$ we finally get

$$\begin{aligned} D_9 &= (3!)^{12} + (3!)^8 \left[\sum_{k=1}^{9} (-1)^k \theta(n, k) \right] \\ &= 2\ \ 176\ \ 782\ \ 336 + 1\ \ 679\ \ 616 \left(-1296 + 1008 - 352 + 125 - 37 + 8 - 2 + \frac{1}{4} - \frac{1}{36} \right) \\ &= 1\ \ 260\ \ 085\ \ 248. \end{aligned} \tag{3.50}$$

The number d_9 of all nonordered pairs disjoint matrices from Σ_9 is equal to

$$d_9 = \frac{1}{2} D_9 = 630\ \ 042\ \ 624. \tag{3.51}$$

3.3. On a Combinatorial Problem of Graph Theory Related to the Number of Sudoku Matrices

Problem. Let $n \geq 2$ be a natural number, and let G be a simple graph having $(n!)^{2n}$ vertices. Let each vertex of G be identified with an element of the set Σ_{n^2} of all $n^2 \times n^2$ S-permutation matrices. Two vertices are connected by an edge if and only if the corresponding matrices are disjoint. The problem is to find the number of all complete subgraphs of G having n^2 vertices.

Note that the number of edges in graph G is equal to d_{n^2} and can be calculated using formulas (2.4) and (2.5) (resp., formulas (2.9), (2.10), and (2.5)).

Denote by z_n the solution of the Problem **??**, and let σ_n be the number of all $n^2 \times n^2$ Sudoku matrices. Then according to formula (1.5) and the method of construction of the graph G, it follows that the next equality is valid:

$$z_n = \frac{\sigma_n}{(n^2)!}. \tag{3.52}$$

We do not know a general formula for finding the number of all $n^2 \times n^2$ Sudoku matrices for each natural number $n \geq 2$, and we consider that this is an open combinatorial problem. Only some special cases are known. For example in $n = 2$ it is known that $\sigma_2 = 288$ [8]. Then according to formula (3.52) we get

$$z_2 = \frac{\sigma_2}{4!} = \frac{288}{24} = 12. \tag{3.53}$$

In [6] it has been shown that in $n = 3$ there are exactly,

$$\begin{aligned} \sigma_3 &= 6\ \ 670\ \ 903\ \ 752\ \ 021\ \ 072\ \ 936\ \ 960 \\ &= 9! \times 72^2 \times 2^7 \times 27\ \ 704\ \ 267\ \ 971 \\ &= 2^{20} \times 3^8 \times 5^1 \times 7^1 \times 27\ \ 704\ \ 267\ \ 971^1 \sim 6.671 \times 10^{21}, \end{aligned} \tag{3.54}$$

a number of Sudoku matrices. Then according to formula (3.52) we get

$$z_3 = \frac{\sigma_3}{9!} = \frac{6\ \ 670\ \ 903\ \ 752\ \ 021\ \ 072\ \ 936\ \ 960}{362\ \ 880} = 18\ \ 383\ \ 222\ \ 420\ \ 692\ \ 992. \tag{3.55}$$

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