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Research Article

Application of Optimal Homotopy Asymptotic Method to Doubly Wave Solutions of the Coupled Drinfel'd-Sokolov-Wilson Equations

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The approximate solution of the doubly periodic wave solutions of the coupled Drinfel'd-Sokolov-Wilson equations has been considered by using the optimal homotopy asymptotic method (OHAM). We obtained the numerical solution of the problem and compared that with the OHAM solution. The obtained solutions show that OHAM is effective, simpler, easier, and explicit and gives a suitable way to control the convergence of the approximate solution.

1. Introduction

The coupled nonlinear partial differential equations (NPDEs) are widely used in applied mathematics, physics, and engineering sciences to offer the description of complex phenomena. Here we consider doubly periodic wave solutions of the coupled Drinfel'd-Sokolov-Wilson equation of the form [1]

$$\begin{gathered}\frac{\partial u(x,t)}{\partial t}+3v(x,t)\frac{\partial v(x,t)}{\partial x}=0,\\ \frac{\partial v(x,t)}{\partial t}+2\frac{\partial^3 v(x,t)}{\partial x^3}\\ +2u(x,t)\frac{\partial v(x,t)}{\partial x}+\frac{\partial u(x,t)}{\partial x}v(x,t)=0,\end{gathered}\tag{1}$$

with

$$u(x,0)=x,\qquad v(x,0)=-x.\tag{2}$$

The exact and explicit solution of the NPDEs in mathematical physics, engineering, and science plays an important role. The exact solution of NPDEs cannot be found easily as all NPDEs have infinitely many solutions. The analytical and exact solution of such problems is either not available in the literature or may be found by using transformation based on the invariance group analysis method [2], the Lie infinitesimal criterion [3], the symbolic computation [4], and the Backlund transformation [5]. All these methods reduced the complex equations into simple equations by using the transformation. In the literature most of the methods like the variational iterative method (VIM) [6], Adomian decomposition method (ADM) [7], differential transform method (DTM) [8], and homotopy perturbation method (HPM) [9] have been used for the solution of weakly NPDEs and few for strongly NPDEs. To tackle the strongly NPDEs the perturbation methods were introduced [10, 11]. These methods contain a small parameter which cannot be found easily. New analytic methods such as the artificial parameters method [12], homotopy analysis method (HAM) [13], and homotopy perturbation method (HPM) [9] were introduced. These methods combined the homotopy with the perturbation techniques. Recently, Vasile Marinca et al. introduced OHAM [14–18] for the solution of nonlinear problems which made the perturbation methods independent of the assumption of small parameters and huge computational work.

The motivation of this paper is to boost OHAM for the solution of coupled NPDEs. In [19–24] OHAM has been proved to be valuable for obtaining an approximate solution of the single partial differential equation (PDE). Before these coupled NPDEs were not solved by OHAM. We have proved that OHAM is useful and reliable for NPDEs, showing its validity and great potential for the solution of transient physical phenomena in science and engineering.

In the succeeding section, the basic idea of OHAM is formulated for the solution of NPDEs. The effectiveness and efficiency of OHAM are shown in Section 3.

2. Fundamental Mathematical Theory of OHAM

Let us see the partial differential equation of the following form:

$$\mathscr{A}(u(x,t)) + f(x,t) = 0, \quad x \in \Omega,$$
$$\mathscr{B}\left(u, \frac{\partial u}{\partial x}\right) = 0, \quad x \in \Gamma, \tag{3}$$

where $\mathscr{A}$ is a differential operator, $u(x,t)$ is an unknown function, x and t denote spatial and temporal independent variables, respectively, Γ is the boundary of Ω, and $f(x,t)$ is a known analytic function. $\mathscr{A}$ can be divided into two parts $\mathscr{L}$ and $\mathscr{N}$ such that

$$\mathscr{A} = \mathscr{L} + \mathscr{N}. \tag{4}$$

$\mathscr{L}$ is the simpler part of the partial differential equation which is easier to solve, and $\mathscr{N}$ contains the remaining part of $\mathscr{A}$.

According to OHAM, one can construct an optimal homotopy $\phi(x,t;p) : \Omega \times [0,1] \rightarrow \mathfrak{R}$ which satisfies

$$H(\phi(x,t;p),p) = (1-p)\{\mathscr{L}(\phi(x,t;p)) + f(x,t)\} - H(p)\{\mathscr{A}(\phi(x,t;p)) + f(x,t)\} = 0, \tag{5}$$

where the auxiliary function $H(p)$ is nonzero for $p \neq 0$ and $H(0) = 0$. Equation (5) is called an optimal homotopy equation. Clearly, we have

$$p = 0 \Longrightarrow H(\phi(x,t;0),0) = \mathscr{L}(\phi(x,t;0)) + f(x,t) = 0,$$
$$p = 1 \Longrightarrow H(\phi(x,t;1),1) = H(1)\{\mathscr{A}(\phi(x,t;p)) + f(x,t)\} = 0. \tag{6}$$

Obviously, when $p = 0$ and $p = 1$ we obtain

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t), \tag{7}$$

respectively. Thus, as p varies from 0 to 1, the solution $\phi(x,t;p)$ approaches from $u_0(x,t)$ to $u(x,t)$, where $u_0(x,t)$ is obtained from (5) for $p = 0$:

$$\mathscr{L}(u_0(x,t)) + f(x,t) = 0, \qquad \mathscr{B}\left(u_0, \frac{\partial u_0}{\partial x}\right) = 0. \tag{8}$$

Next, we choose the auxiliary function $H(p)$ in the form

$$H(p) = pC_1 + p^2C_2 + \cdots. \tag{9}$$

To get an approximate solution, we expand $\phi(x,t;p,C_i)$ by Taylor's series about p in the following manner:

$$\phi(x,t;p,C_i) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t;C_i)\,p^k, \quad i = 1,2,\ldots. \tag{10}$$

Substituting (10) into (5) and equating the coefficient of the like powers of p, we obtain the zeroth order problem, given by (8), the first and second order problems are given by (11)-(12), respectively, and the general governing equations for $u_k(x,t)$ are given by (13):

$$\mathscr{L}(u_1(x,t)) = C_1\mathscr{N}_0(u_0(x,t)), \qquad \mathscr{B}\left(u_1, \frac{\partial u_1}{\partial x}\right) = 0, \tag{11}$$

$$\mathscr{L}(u_2(x,t)) - \mathscr{L}(u_1(x,t)) = C_2\mathscr{N}_0(u_0(x,t)) + C_1[\mathscr{L}(u_1(x,t)) + \mathscr{N}_1(u_0(x,t), u_1(x,t))],$$
$$\mathscr{B}\left(u_2, \frac{\partial u_2}{\partial x}\right) = 0, \tag{12}$$

$$\mathscr{L}(u_k(x,t)) - \mathscr{L}(u_{k-1}(x,t)) = C_k\mathscr{N}_0(u_0(x,t)) + \sum_{i=1}^{k-1} C_i[\mathscr{L}(u_{k-i}(x,t)) + \mathscr{N}_{k-i}(u_0(x,t), u_1(x,t), \ldots, u_{k-i}(x,t))],$$
$$k = 2,3,\ldots, \quad \mathscr{B}\left(u_k, \frac{\partial u_k}{\partial x}\right) = 0, \tag{13}$$

where $\mathscr{N}_{k-i}(u_0(x,t), u_1(x,t), \ldots, u_{k-i}(x,t))$ are the coefficients of p^{k-i} in the expansion of $\mathscr{N}(\phi(x,t;p))$ about the embedding parameter p:

$$\mathscr{N}(\phi(x,t;p,C_i)) = \mathscr{N}_0(u_0(x,t)) + \sum_{k\geq 1} \mathscr{N}_k(u_0, u_1, u_2, \ldots, u_k)\,p^k. \tag{14}$$

It should be underscored that the u_k for $k \geq 0$ are governed by the linear equations with linear boundary conditions that come from the original problem, which can be easily solved.

It has been observed that the convergence of the series equation (10) depends upon the auxiliary constants $C_1, C_2, \ldots$. If it is convergent at $p = 1$, one has

$$\tilde{u}(x,t;C_i) = u_0(x,t) + \sum_{k\geq 1} u_k(x,t;C_i). \tag{15}$$

Substituting (15) into (1), it results the following expression for the residual:

$$R\left(x,t;C_i\right)=\mathscr{L}\left(\tilde{u}\left(x,t;C_i\right)\right)+f\left(x,t\right)+\mathscr{N}\left(\tilde{u}\left(x,t;C_i\right)\right). \tag{16}$$

In actual computation $k=1,2,3,\ldots,m$.

If $R(x,t;C_i)=0$, then $\tilde{u}(x,t;C_i)$ is the exact solution of the problem. Generally it does not happen, especially in nonlinear problems.

For the determinations of auxiliary constants, C_i, $i=1,2,\ldots,m$, there are different methods like Galerkin's method, the Ritz method, the least squares method, and, the collocation method. One can apply the method of least squares as under

$$J\left(C_i\right)=\int_0^t\int_\Omega R^2\left(x,t;C_i\right)dx\,dt, \tag{17}$$

$$\frac{\partial J}{\partial C_1}=\frac{\partial J}{\partial C_2}=\cdots=\frac{\partial J}{\partial C_m}=0. \tag{18}$$

The mth order approximation can be obtained by these constants. The constants C_i can also be determined by another method as under

$$R\left(h_1;C_i\right)=R\left(h_2;C_i\right)=\cdots=R\left(h_m;C_i\right)=0,\qquad i=1,2,\ldots,m, \tag{19}$$

at any time t, where $h_i\in\Omega$.

The more general auxiliary function $H(p)$ is useful for convergence, which depends upon constants $C_1,C_2,\ldots$, can be optimally identified by (18), and is useful in error minimization.

3. Application of OHAM to Doubly Periodic Wave Solutions of the Coupled Drinfel'd-Sokolov-Wilson Equation

To demonstrate the effectiveness of the extended formulation of OHAM for coupled nonlinear partial differential equations (NPDEs), we consider the doubly periodic wave solutions of the coupled Drinfel'd-Sokolov-Wilson equations (1) with the boundary condition (2).

Applying the method formulated in Section 2 leads to the following:

$$\begin{gathered}
\mathscr{L}\left(u\left(x,t\right)\right)=\frac{\partial u\left(x,t\right)}{\partial t},\qquad \mathscr{N}\left(v\left(x,t\right)\right)=3v\left(x,t\right)\frac{\partial v\left(x,t\right)}{\partial x},\\
\mathscr{L}\left(v\left(x,t\right)\right)=\frac{\partial v\left(x,t\right)}{\partial t},\\
\mathscr{N}\left(u,v\right)=2\frac{\partial^3 v\left(x,t\right)}{\partial x^3}+2u\left(x,t\right)\frac{\partial v\left(x,t\right)}{\partial x}+\frac{\partial u\left(x,t\right)}{\partial x}v\left(x,t\right),\\
\left(1-p\right)\frac{\partial u\left(x,t\right)}{\partial t}-H_1\left(p\right)\left[\frac{\partial u\left(x,t\right)}{\partial t}+3v\left(x,t\right)\frac{\partial v\left(x,t\right)}{\partial t}\right]=0,\\
\left(1-p\right)\frac{\partial v\left(x,t\right)}{\partial t}-H_2\left(p\right)\left[\frac{\partial u\left(x,t\right)}{\partial t}+2\frac{\partial^3 v\left(x,t\right)}{\partial x^3}+2u\left(x,t\right)\frac{\partial v\left(x,t\right)}{\partial x}+\frac{\partial u\left(x,t\right)}{\partial x}v\left(x,t\right)\right]=0.
\end{gathered} \tag{20}$$

We consider

$$\begin{gathered}
u=u_0+pu_1+p^2u_2,\\
v=v_0+pv_1+p^2v_2,\\
H_1\left(p\right)=pC_{11}+p^2C_{12},\\
H_2\left(p\right)=pC_{21}+p^2C_{22}.
\end{gathered} \tag{21}$$

3.1. Zeroth Order System. We have

$$\begin{gathered}
\frac{\partial u_0\left(x,t\right)}{\partial t}=0,\\
\frac{\partial v_0\left(x,t\right)}{\partial t}=0,
\end{gathered} \tag{22}$$

with initial conditions

$$u_0\left(x,0\right)=x,\qquad v_0\left(x,0\right)=-x. \tag{23}$$

Its solution

$$u_0\left(x,t\right)=x,\qquad v_0\left(x,t\right)=-x. \tag{24}$$

3.2. First Order System. We have

$$\frac{\partial u_1(x,t)}{\partial t} = (1+C_{11})\frac{\partial u_0(x,t)}{\partial t} + 3C_{11}v_0(x,t)\frac{\partial v_0(x,t)}{\partial x},$$
$$\frac{\partial v_1(x,t)}{\partial t} = \left\{\frac{\partial v_0(x,t)}{\partial t} + C_{21}v_0(x,t)\left(1+\frac{\partial u_0(x,t)}{\partial t}\right) + 2C_{21}u_0(x,t)\frac{\partial v_0(x,t)}{\partial x} + 2C_{21}\frac{\partial^3 v_0(x,t)}{\partial x^3}\right\},$$
$$u_1(x,0)=0, \qquad v_1(x,0)=0. \tag{25}$$

Its solutions

$$u_1(x,t) = 3txC_{11},$$
$$v_1(x,t) = -t(x+4xC_{21}). \tag{26}$$

3.3. Second Order System. We have

$$\frac{\partial u_2(x,t)}{\partial t} = \left\{(1+C_{11})\frac{\partial u_1(x,t)}{\partial t} + C_{21}\frac{\partial u_0(x,t)}{\partial x} + 3(C_{12}v_0(x,t)+C_{11}v_1(x,t))\frac{\partial v_0(x,t)}{\partial x} + 3C_{11}v_0(x,t)\frac{\partial v_1(x,t)}{\partial x}\right\},$$
$$\frac{\partial v_2(x,t)}{\partial t} = \left\{\frac{\partial v_1(x,t)}{\partial t} + C_{21}v_1(x,t) + C_{22}v_0(x,t) + (C_{22}v_0(x,t)+C_{21}v_1(x,t)) \times \frac{\partial u_0(x,t)}{\partial x} + 2(C_{22}u_0(x,t)+C_{21}u_1(x,t))\frac{\partial v_0(x,t)}{\partial x} + 2C_{21}u_0(x,t)\frac{\partial v_1(x,t)}{\partial x} + 2C_{22}\frac{\partial^3 v_0(x,t)}{\partial x^3} + 2C_{21}\frac{\partial^3 v_1(x,t)}{\partial x^3}\right\}, \tag{27}$$

with

$$u_2(x,0)=0, \qquad v_2(x,0)=0. \tag{28}$$

Its solutions

$$u_2(x,t) = \frac{3}{2}\left[C_{11}(1+2x)t^2 + 2C_{11}^2xt(1+4t) + C_{12}xt\right],$$
$$v_2(x,t) = \frac{1}{2}\left[-2xt - 4C_{21}xt(2+t) - 9C_{11}C_{21}xt^2 - 16C_{21}^2xt^2 - 8C_{22}xt\right]. \tag{29}$$

Adding (26), (29), and (28), we obtain

$$u(x,t) = x + 3txC_{11} + \frac{3}{2}\left[C_{11}(1+2x)t^2 + 2C_{11}^2xt(1+4t) + C_{12}xt\right],$$
$$v(x,t) = -x - t(x+4xC_{21}) + \frac{1}{2}\left[-2xt - 4C_{21}xt(2+t) - 9C_{11}C_{21}xt^2 - 16C_{21}^2xt^2 - 8C_{22}xt\right]. \tag{30}$$

For the calculations of the constants C_{11}, C_{12}, C_{21}, and C_{22} using (30) in (1) and applying the procedure mentioned in (16)–(19), we get

$$C_{11}=0, \qquad C_{12}=-0.6282197208603675,$$
$$C_{21}=0.3177122963169145,$$
$$C_{22}=-1.689990604397596,$$
$$u(x,t) = x - 1.88466xt,$$
$$v(x,t) = -x - 2.27085xt + \frac{1}{2}\left(8.97823xt - 2.88591xt^2\right). \tag{31}$$

4. Results and Discussions

The formulation presented in Section 2 provides highly accurate solutions for the problems demonstrated in Section 3. We have used Mathematica 7 for most of our computational work. In Tables 1 and 3, we have presented absolute errors for $u(x,t)$ and $v(x,t)$ at a spatial domain $[0, 0.4]$ for $t = 0.01$, $t = 0.015$, $t = 0.1$, $t = 0.2$, $t = 0.3$, and $t = 0.4$, while in Tables 2 and 4 the convergence of the OHAM solution is given, through first and second order absolute errors at time $t = 0.1$ and $0 \leq x \leq 1$. Here we observe that the OHAM solution converges rapidly with increasing order of approximation. From Tables 1–4 and Figures 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 it is evident that the OHAM results are nearly identical to the numerical results. Here the results are very consistent with the increasing time.

5. Conclusion

In this paper, we have seen the effectiveness of OHAM [16–20] in doubly periodic wave solutions of the coupled

TABLE 1: Absolute error of OHAM solution of $u(x,t)$ corresponding to the numerical solution.

x	$t = 0.01$	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$
0.1	4.10634×10^{-6}	4.10654×10^{-5}	5.52825×10^{-6}	3.36769×10^{-6}	1.28783×10^{-6}
0.2	1.83012×10^{-6}	7.75201×10^{-5}	2.39901×10^{-6}	4.39072×10^{-6}	1.49213×10^{-6}
0.3	4.78801×10^{-6}	4.78851×10^{-5}	7.84255×10^{-6}	1.46108×10^{-6}	1.06140×10^{-6}
0.4	4.90183×10^{-6}	2.39950×10^{-6}	6.03249×10^{-6}	7.54685×10^{-6}	1.23535×10^{-6}
0.5	6.43592×10^{-6}	6.58079×10^{-6}	2.84924×10^{-6}	1.11220×10^{-6}	3.14244×10^{-6}
0.6	7.97001×10^{-6}	8.11025×10^{-6}	2.85555×10^{-6}	7.02914×10^{-6}	9.25554×10^{-6}
0.7	4.95914×10^{-6}	4.95414×10^{-6}	6.48274×10^{-6}	9.22908×10^{-7}	4.49099×10^{-7}
0.8	6.03249×10^{-7}	1.49270×10^{-6}	3.29949×10^{-6}	5.25885×10^{-7}	6.18914×10^{-7}
0.9	5.59132×10^{-7}	6.83350×10^{-6}	2.27285×10^{-7}	7.90433×10^{-7}	1.41393×10^{-7}
1.0	2.84925×10^{-7}	2.17437×10^{-7}	6.93298×10^{-7}	1.00563×10^{-7}	4.59537×10^{-8}

TABLE 2: Comparison of first order and second order errors of $u(x,t)$ corresponding to the numerical solution at time $t = 0.1$, and $0 \leq x \leq 1$.

x	First order absolute error	Second order absolute error
0.1	1.60818×10^{-2}	4.10654×10^{-5}
0.2	3.39636×10^{-2}	7.75201×10^{-5}
0.3	5.18454×10^{-2}	4.78851×10^{-5}
0.4	6.97272×10^{-2}	2.39950×10^{-6}
0.5	8.76090×10^{-2}	6.58079×10^{-6}
0.6	1.05491×10^{-1}	8.11025×10^{-6}
0.7	1.23373×10^{-1}	4.95414×10^{-6}
0.8	1.41254×10^{-1}	1.49270×10^{-6}
0.9	1.59136×10^{-1}	6.83350×10^{-6}
1.0	1.77018×10^{-1}	2.17437×10^{-7}

TABLE 3: Absolute error of OHAM solution of $v(x,t)$ corresponding to the numerical solution.

x	$t = 0.01$	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$
0.1	-8.38255×10^{-5}	-7.99845×10^{-6}	-8.50767×10^{-5}	-1.17209×10^{-4}	-1.25237×10^{-3}
0.2	-4.70697×10^{-5}	-8.90496×10^{-6}	-1.58744×10^{-5}	-2.24198×10^{-4}	-3.79536×10^{-3}
0.3	-1.19722×10^{-5}	-1.34145×10^{-6}	-9.61422×10^{-5}	-1.02833×10^{-4}	-6.03144×10^{-3}
0.4	-1.94189×10^{-5}	-9.51779×10^{-6}	-1.93986×10^{-5}	-1.02833×10^{-4}	-8.46213×10^{-3}
0.5	-4.28514×10^{-5}	-1.08137×10^{-5}	-2.03181×10^{-5}	-1.01524×10^{-4}	-1.40224×10^{-3}
0.6	-1.15699×10^{-5}	-7.64659×10^{-5}	-1.78514×10^{-5}	-9.06291×10^{-3}	-6.36149×10^{-3}
0.7	-7.91541×10^{-4}	-9.20329×10^{-5}	-2.25841×10^{-4}	-6.88889×10^{-3}	-1.54670×10^{-2}
0.8	-1.50929×10^{-4}	-6.25928×10^{-5}	-9.04686×10^{-4}	-8.30865×10^{-2}	-9.92225×10^{-2}
0.9	-1.12584×10^{-4}	-1.06202×10^{-4}	-2.26639×10^{-4}	-7.63087×10^{-2}	-8.63452×10^{-1}
1.0	-7.34287×10^{-4}	-2.96571×10^{-4}	-5.29813×10^{-3}	-1.67205×10^{-2}	-1.34788×10^{-1}

TABLE 4: Comparison of first order and second order errors of $v(x,t)$ corresponding to the numerical solution at time $t = 0.1$, and $0 \leq x \leq 1$.

x	First order absolute error	Second order absolute error
0.1	5.67234×10^{-2}	2.25631×10^{-6}
0.2	1.13447×10^{-2}	1.58864×10^{-6}
0.3	1.70170×10^{-1}	2.38296×10^{-6}
0.4	2.26894×10^{-1}	3.17721×10^{-6}
0.5	2.83617×10^{-1}	2.91721×10^{-5}
0.6	3.40341×10^{-1}	1.25812×10^{-5}
0.7	3.97064×10^{-1}	2.50126×10^{-5}
0.8	4.53787×10^{-1}	1.28317×10^{-4}
0.9	5.10510×10^{-1}	1.29810×10^{-4}
1.0	5.67234×10^{-1}	2.94185×10^{-4}

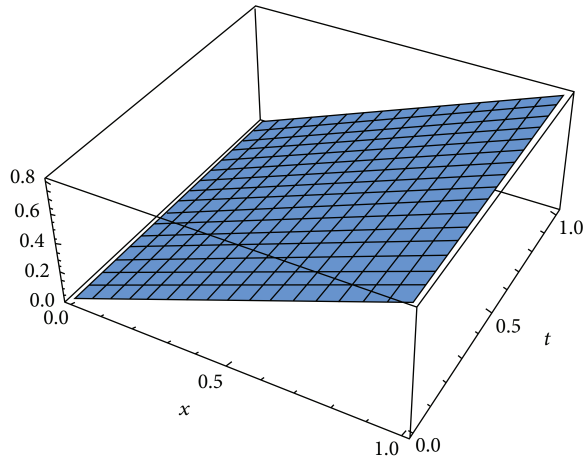

FIGURE 1: 3D approximate solution of $u(x,t)$ for $t = 0.1$.

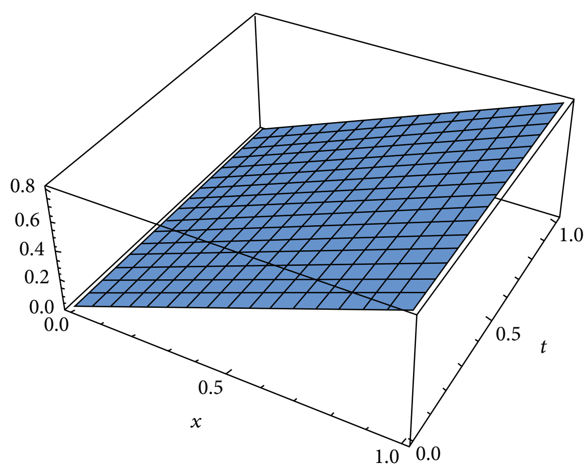

FIGURE 2: 3D numerical solution of $u(x,t)$ for $t = 0.1$.

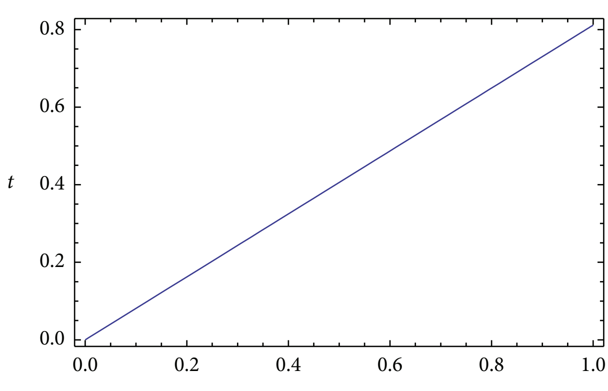

FIGURE 3: 2D approximate solution of $u(x,t)$ for $t = 0.1$.

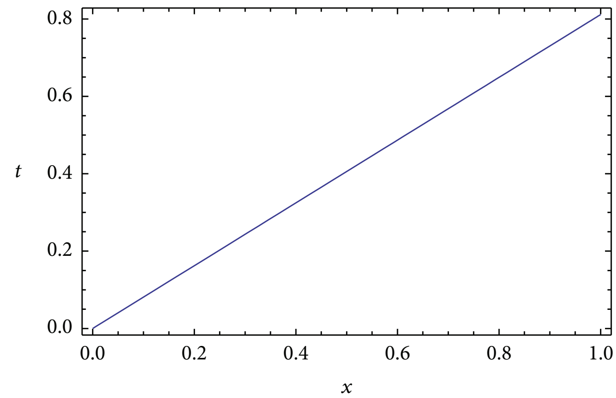

FIGURE 4: 2D numerical solution for $t = 0.1$.

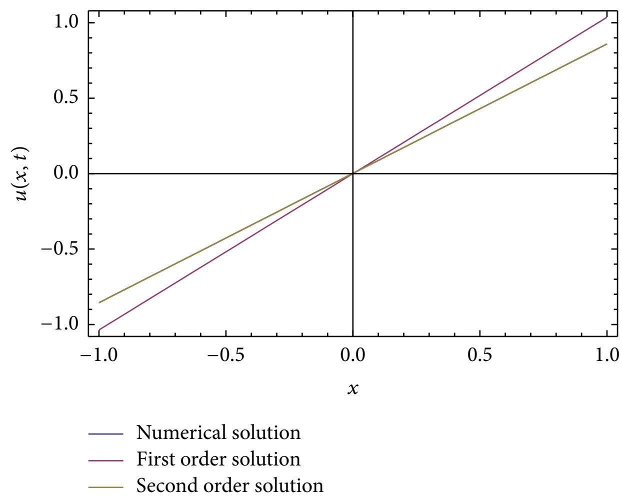

FIGURE 5: 2D numerical, zeroth, first, and second order solutions of $u(x,t)$ for $t = 0.5$.

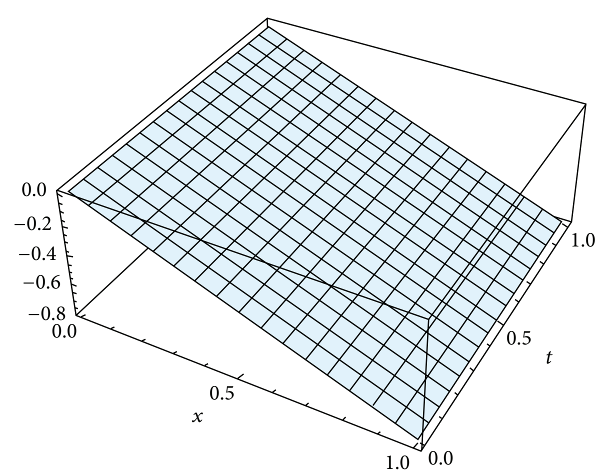

FIGURE 6: 3D approximate solution of $v(x,t)$ for $t = 0.1$.

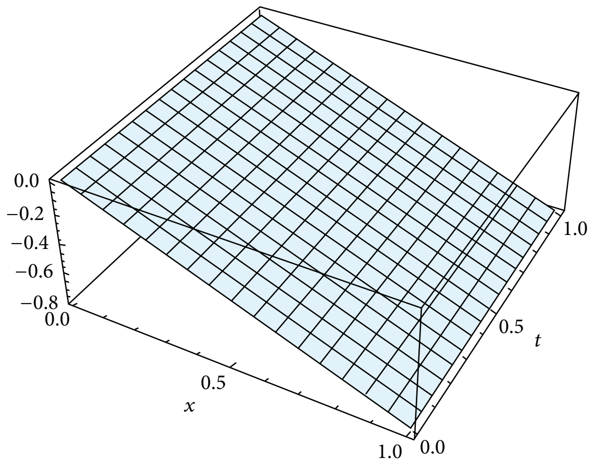

Figure 7: 3D exact solution of $v(x,t)$ for $t = 0.1$.

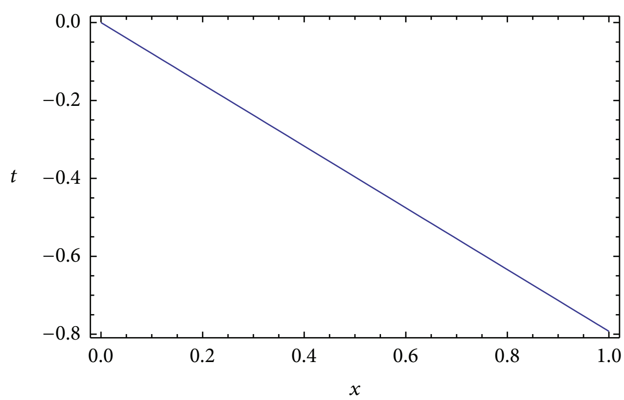

Figure 8: 2D approximate solution of $v(x,t)$ for $t = 0.1$.

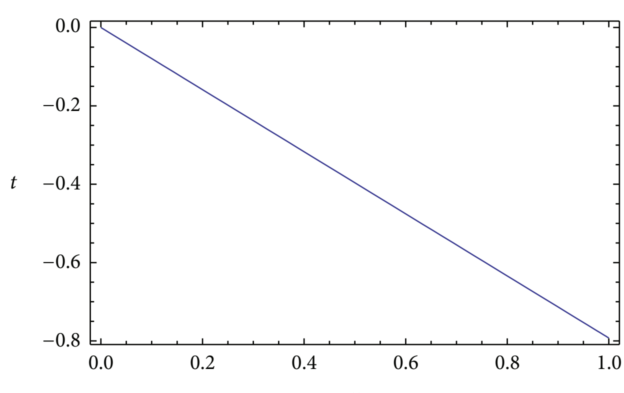

Figure 9: 2D numerical solution of $v(x,t)$ for $t = 0.1$.

Drinfel'd-Sokolov-Wilson equation. By applying the basic idea of OHAM to doubly periodic wave solutions of the coupled Drinfel'd-Sokolov-Wilson equation, we found it simpler in applicability, more convenient to control convergence, and involving less computational overhead. Therefore, OHAM

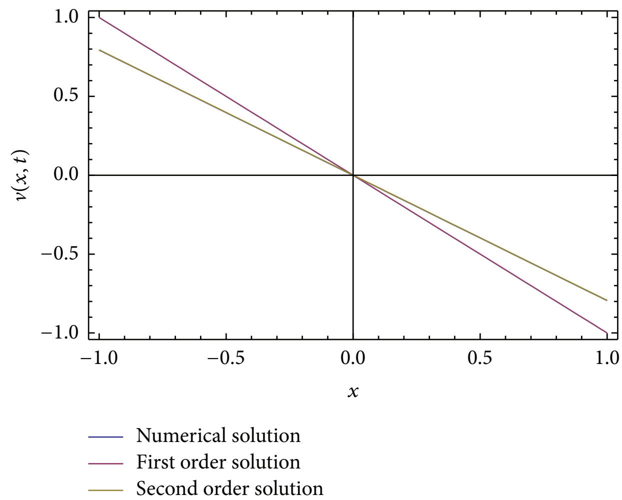

Figure 10: 2D numerical, zeroth, first, and second order solution of $v(x,t)$ for $t = 0.5$.

shows its validity and great potential for the solution of time dependant problems in science and engineering.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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