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Research Article

Variational Principles for Bending and Vibration of Partially Composite Timoshenko Beams

Halil Özer

Division of Mechanics, Mechanical Engineering Department, Yıldız Technical University (YTU), Yıldız, Beşiktaş, 34349 Istanbul, Turkey

Correspondence should be addressed to Halil Özer; hozer@yildiz.edu.tr

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Variational principles are established for the partially composite Timoshenko beam using the semi-inverse method. The principles are derived directly from governing differential equations for bending and vibration of the beam considered. It is concluded that the semi-inverse method is a powerful tool for searching for variational principles directly from the governing equations. Comparison between our results and the results reported in literature is given.

1. Introduction

Composite beams composed of different elastic materials have been widely used in many engineering applications. The individual beam components of the composite beam are combined by using the shear connectors. Therefore, the overall behavior of the composite beam depends on the stiffness of connectors. Connector having infinite stiffness eliminates any interlayer shear slip between the individual beam components, which leads to the full interaction connection. However, the stiffness of connector has a finite value and the interlayer slip between the individual components occurs. This type of connection is called partial-interaction connection. Therefore, analysis of the partial-interaction composite beams requires the consideration of the interlayer slip between the beam components. The Euler-Bernoulli beam theory has been extensively used in bending, vibration, and buckling analyses. Ecsedi and Baksa [1] analyzed the static behavior of elastic two-layer beams with interlayer slip and developed closed-form solutions for displacements and interlayer slips. Girhammar and Pan [2] presented general solutions for the deflection and internal actions for partially composite Euler-Bernoulli beams and beam-columns. Ranzi et al. [3] presented an analytical formulation for the analysis of two-layered composite beams with longitudinal and vertical partial-interaction. Their formulation is based on the principle of virtual work expressed in terms of the vertical and axial displacements of the two layers. The model was presented in both its weak and its strong forms. Xu and Wu [4] developed a new plane stress model of composite beams with interlayer slips using the one-dimensional theory. They concluded that the shear force produced by the shear connectors increases with the increase in rigidity of shear connectors.

However, the effect of transverse shear deformation was neglected in the Euler-Bernoulli beam theory. When the beam is thick, the effect of shear deformation becomes significant and cannot be neglected for a valid analysis. The most widely used and fundamentally simpler theory was developed by Timoshenko [5]. Sousa and da Silva [6] studied the behavior of the general case of multilayered composite beams with interlayer slip, under Euler-Bernoulli as well as Timoshenko beam theory (TBT) assumptions. Xu and Wang [7] formulated the principle of virtual work and reciprocal theorem of work for the partial-interaction composite beams using the kinematic assumptions of Timoshenko's beam theory. The variational principles for the frequency of free vibration and critical load of buckling were also

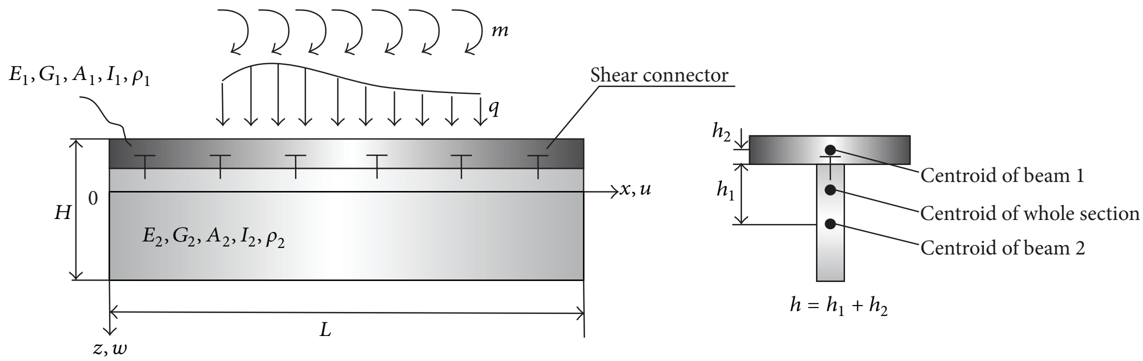

FIGURE 1: A two-layer composite beam.

deduced. Xu and Wang [8] derived the relationships of solutions between single-span Euler-Bernoulli and Timoshenko partial-interaction composite beams.

Variational formulations provide the basis for a number of approximate and numerical methods. Recently, two significant variational methods are proposed by He; one is the semi-inverse method [9, 10] and the other method is the variational iteration method [11]. The semi-inverse method is used to establish variational principles directly from the governing differential equations. His second method, the variational iteration method, depends on constructing a correction functional by a general Lagrange multiplier. Then, the optimal value of the Lagrange multiplier is identified by using the stationary conditions [12, 13]. However, in the semi-inverse method, the term involving the Lagrange multiplier is replaced by an unknown function F. The semi-inverse method eliminates two important variational crises; one is that the Lagrange multiplier is equal to zero and the other crisis is that making the Lagrangian stationary leads to only some parts of Euler equations. In this study, we will apply the semi-inverse method to establish variational principles directly from the governing differential equations defining the bending and vibration of Timoshenko composite beam with partial-interaction. The variational formulations were obtained by following the rules of the calculus of variations.

2. Timoshenko Composite Beam with Partial-Interaction

Before applying the semi-inverse method, the problem is briefly discussed in Figure 1. Figure 1 shows a partial-interaction composite beam that is composed of two-layer beams with different materials.

In Figure 1, E_i, G_i, A_i, I_i, and ρ_i ($i = 1, 2$) denote the elasticity modulus, shear modulus, cross-sectional area, and moment of inertia of two beam components, respectively. L is the beam length, H is the beam height, and h is the distance between the centroids of two beam sections. q and m denote the distributed load and distributed bending moment, respectively. As seen in Figure 1, shear connectors are used to connect the beam members of the composite beam. Figure 2 shows geometrical relationship among the interlayer slip, rotary angle (the rotation of the cross section), and longitudinal displacements.

In Figure 2, ψ is the rotary angle and u_s is the interlayer slip between two beam layers. u_1 and u_2 are the longitudinal displacements at the centroids of beams 1 and 2, respectively. From Figure 2, the kinematic relationship among the interlayer slip, rotary angle, and longitudinal displacements can be written as follows:

$$u_s = u_2 - u_1 + \psi h. \tag{1}$$

The bending moment, shear force, and interlayer shear force are given, respectively, as [7, 8]:

$$M = -\overline{D}\frac{d\psi}{dx} + \overline{EA}h\frac{du_s}{dx}, \tag{2a}$$

$$Q = C\left(\frac{dw}{dx} - \psi\right), \tag{2b}$$

$$Q_s = k_s u_s, \tag{2c}$$

where w denotes the deflection of the composite beam in the z-direction (see Figure 1) and k_s denotes the rigidity of the shear connectors. The other quantities used in (2a), (2b), and (2c) are defined as follows:

$$\overline{D} = D + h^2\overline{EA}, \tag{3a}$$

$$D = E_1 I_1 + E_2 I_2, \tag{3b}$$

$$\overline{EA} = \frac{E_1 A_1 E_2 A_2}{E_1 A_1 + E_2 A_2}, \tag{3c}$$

$$C = k_1 G_1 A_1 + k_2 G_2 A_2, \tag{3d}$$

in which k_1 and k_2 are the shear correction factors of the Timoshenko beam. $\overline{D}$ is the flexural stiffness of the composite beam in full interaction, D is the flexural stiffness of the composite beam without shear connection, $\overline{EA}$ is the effective axial stiffness, and C is the shear rigidity of the whole cross sections. Deflection of the composite beam is then obtained using the relation below:

$$w = w_0 + w_{\text{slip}} + w_{\text{shear}}, \tag{4}$$

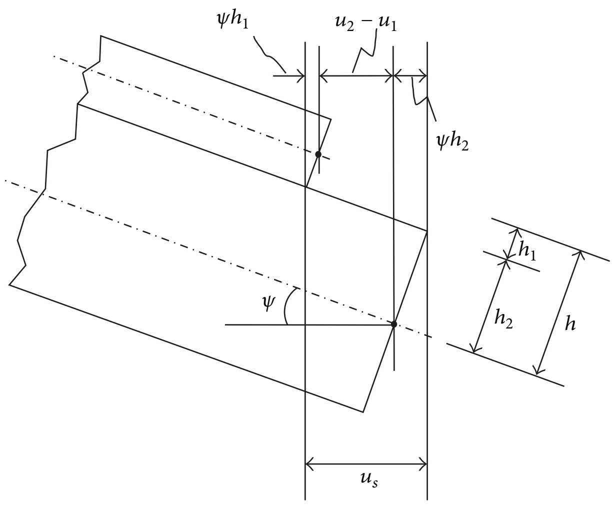

FIGURE 2: Kinematic model of a two-layer partially composite Timoshenko beam [7].

where w_0 is the deflection of the full interaction composite beam. w_{slip} and w_{shear} are the additional deflections due to the interlayer slip and transverse shear deformation, respectively.

In the next sections, using the semi-inverse method, we will illustrate how to establish variational principles directly from the governing differential equations for bending and vibration of the partially composite Timoshenko beams.

3. Derivation of Variational Principle for Bending of the Composite Beam

Consider the governing differential equations for bending of the partial-interaction composite Timoshenko beam under uniformly distributed load and bending moment [7]

$$-\overline{D}\frac{d^2\psi}{dx^2} + \overline{EA}h\frac{d^2}{dx^2}(u_s - \psi h) - C\left(\frac{dw}{dx} - \psi\right) - m = 0, \tag{5a}$$

$$-C\left(\frac{d^2w}{dx^2} - \frac{d\psi}{dx}\right) - q = 0, \tag{5b}$$

$$-\overline{EA}\frac{d^2}{dx^2}(u_s - \psi h) + k_s u_s = 0. \tag{5c}$$

Using the semi-inverse method, a trial variational principle can be constructed as follows [9, 10]:

$$J(\psi, w, u_s) = \int L dx, \tag{6}$$

where L is a trial Lagrangian. There are many approaches for constructing the trial Lagrangian; see [14–17]. We search for such a trial Lagrangian, so that its trial Euler equation gives one of the governing equations, say (5a). Referring to (5a), an energy-like trial Lagrangian can be constructed as follows:

$$L = \frac{1}{2}\overline{D}\left(\frac{d\psi}{dx}\right)^2 + \frac{1}{2}\overline{EA}\left[\frac{d}{dx}(u_s - \psi h)\right]^2 - C\psi\frac{dw}{dx} + \frac{1}{2}\psi^2 - m\psi + F_1(w), \tag{7}$$

where F_1 is an unknown function of w and/or its derivatives. The advantage of the above trial Lagrangian lies in the fact that the stationary condition with respect to ψ results in (5a). Now by making (7) stationary with respect to w, one can get the following trial Euler equation for δw:

$$\delta w : C\frac{d\psi}{dx} + \frac{\delta F_1}{\delta w} = 0, \tag{8}$$

where the operator δ is called a variational operator and δw is the first order variation of w. $\delta F_1/\delta w$ is called He's variational derivative with respect to w, which is defined as

$$\frac{\delta F_1}{\delta w} = \frac{\partial F_1}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial w'}\right), \quad w' = \frac{dw}{dx}. \tag{9}$$

We search for such an F_1 so that (8) is equivalent to (5b). From (9), the unknown function F_1 can be determined as

$$F_1(w) = \frac{1}{2}C\left(\frac{dw}{dx}\right)^2 - qw. \tag{10}$$

By adding the above relation, the trial Lagrangian can be renewed as follows:

$$L = \frac{1}{2}\overline{D}\left(\frac{d\psi}{dx}\right)^2 + \frac{1}{2}\overline{EA}\left[\frac{d}{dx}(u_s - \psi h)\right]^2 - C\psi\frac{dw}{dx} + \frac{1}{2}\psi^2 + \frac{1}{2}C\left(\frac{dw}{dx}\right)^2 - qw - m\psi + F_2(u_s). \tag{11}$$

It can be easily proved that the stationary condition of the above Lagrangian with respect to w satisfies (5b). In (11), F_2 is a newly introduced undetermined function of u_s and/or its derivatives and is free from the variables ψ and w. Making the new trial Lagrangian (11) stationary with respect to u_s results in the relation below:

$$\delta u_s : -\overline{EA}\frac{d^2}{dx^2}(u_s - \psi h) + \frac{\delta F_2}{\delta u_s} = 0, \tag{12}$$

which is the last trial Euler equation. The second term on the left is the variational derivative with respect to u_s and reads

$$\frac{\delta F_2}{\delta u_s} = \frac{\partial F_2}{\partial u_s} - \frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial u_s'}\right), \quad u_s' = \frac{du_s}{dx}, \tag{13}$$

from which the unknown F_2 can be determined as

$$F_2(u_s) = \frac{1}{2}k_s u_s^2. \tag{14}$$

Substituting F_2 into (11) and rearranging lead to the necessary variational principle as

$$J(\psi, w, u_s) = \int_0^L \left\{ \frac{1}{2}\overline{D}\left(\frac{d\psi}{dx}\right)^2 + \frac{1}{2}\overline{EA}\left[\frac{d}{dx}(u_s - \psi h)\right]^2 + \frac{1}{2}C\left(\frac{dw}{dx} - \psi\right)^2 + \frac{1}{2}k_s u_s^2 - qw - m\psi \right\} dx, \quad (15)$$

which is the total potential energy of partial-interaction composite Timoshenko beam subjected to uniformly distributed load and bending moment (see [7]) and yields the minimum potential energy principle by letting $\delta J = 0$.

Proof. Making the above functional (15) stationary with respect to ψ, w, and u_s, the Euler equations turn out to be (5a)–(5c), respectively. □

The Ritz method can be used to obtain an approximate analytical solution of the problem. We can write the one-term trial functions which satisfy the boundary conditions as

$$w = w_0 f_1(x), \qquad \psi = \psi_0 f_2(x), \qquad u_s = u_0 f_3(x), \quad (16)$$

where w_0, ψ_0, and u_0 are unknown constants, which can be determined from the following stationary conditions:

$$\frac{\partial J}{\partial w_0} = 0, \qquad \frac{\partial J}{\partial \psi_0} = 0, \qquad \frac{\partial J}{\partial u_0} = 0. \quad (17)$$

By solving the system of (17) simultaneously, the unknown constants can then be obtained.

4. Derivation of Variational Principle for Free Vibration of the Composite Beam

Differential equations of motion for partial-interaction composite members under uniformly distributed load and bending moment can be written as [7]:

$$-\overline{D}\frac{\partial^2\psi}{\partial x^2} + \overline{EA}h\frac{\partial^2}{\partial x^2}(u_s - \psi h) - C\left(\frac{\partial w}{\partial x} - \psi\right) + \rho I_0\frac{\partial^2\psi}{\partial t^2} - m = 0, \quad (18a)$$

$$-C\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\psi}{\partial x}\right) + \rho A_0\frac{\partial^2 w}{\partial t^2} - q = 0, \quad (18b)$$

$$-\overline{EA}\frac{\partial^2}{\partial x^2}(u_s - \psi h) + k_s u_s = 0, \quad (18c)$$

where t denotes time, $\rho A_0 = \rho_1 A_1 + \rho_2 A_2$, and $\rho I_0 = \rho_1 I_1 + \rho_2 I_2$. We can construct the following trial variational principle using the semi-inverse method [9, 10]:

$$J(\psi, w, u_s) = \iint L\,dx\,dt. \quad (19)$$

Similarly, referring to (18a) and making some modifications so that the stationary condition with respect to ψ can identify (18a) lead us to the following trial Lagrangian:

$$L = \frac{1}{2}\overline{D}\left(\frac{\partial\psi}{\partial x}\right)^2 + \frac{1}{2}\overline{EA}\left[\frac{\partial}{\partial x}(u_s - \psi h)\right]^2 - C\psi\frac{\partial w}{\partial x} + \frac{1}{2}\psi^2 - \frac{1}{2}\rho I_0\left(\frac{\partial\psi}{\partial t}\right)^2 - m\psi + F_3(w), \quad (20)$$

with F_3 being an unknown function of w and/or its derivatives. As can be seen easily, the stationary condition of the above Lagrangian with respect to ψ results in (18a). Now making (20) stationary with respect to w, we obtain the following trial Euler equation:

$$\delta w : C\frac{\partial\psi}{\partial x} + \frac{\delta F_3}{\delta w} = 0, \quad (21)$$

where $\delta F_3/\delta w$ is defined as

$$\frac{\delta F_3}{\delta w} = \frac{\partial F_3}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial w'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F_3}{\partial \dot{w}}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F_3}{\partial w''}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial F_3}{\partial \ddot{w}}\right) + \cdots, \quad \dot{w} = \frac{\partial w}{\partial t}, \; w' = \frac{\partial w}{\partial x}. \quad (22)$$

From the above relation, we can identify F_3 in the form

$$F_3(w) = \frac{1}{2}C\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\rho A_0\left(\frac{\partial w}{\partial t}\right)^2 - qw. \quad (23)$$

Then, the Lagrangian (20) is further updated as follows:

$$L = \frac{1}{2}\overline{D}\left(\frac{\partial\psi}{\partial x}\right)^2 + \frac{1}{2}\overline{EA}\left[\frac{\partial}{\partial x}(u_s - \psi h)\right]^2 - C\psi\frac{\partial w}{\partial x} + \frac{1}{2}\psi^2 - m\psi - \frac{1}{2}\rho I_0\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{2}C\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\rho A_0\left(\frac{\partial w}{\partial t}\right)^2 - qw + F_4(u_s). \quad (24)$$

It is obvious that making the renewed trial functional stationary with respect to w satisfies (18b). In (24), F_4 is a new undetermined function of u_s and/or its derivatives. It must be noted that (18c) has the same form as (5c). Therefore, by following the same steps as before (see (12)-(13)), it is easily seen that $F_4 = F_2$. Finally, we can easily arrive at the required variational principle:

$$J(\psi, w, u_s) = \iint \left\{ \frac{1}{2}\overline{D}\left(\frac{\partial\psi}{\partial x}\right)^2 + \frac{1}{2}\overline{EA}\left[\frac{\partial}{\partial x}(u_s - \psi h)\right]^2 + \frac{1}{2}C\left(\frac{\partial w}{\partial x} - \psi\right)^2 + \frac{1}{2}k_s u_s^2 - \frac{1}{2}\left[\rho I_0\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{2}\rho A_0\left(\frac{\partial w}{\partial t}\right)^2\right] - qw - m\psi \right\} dx\,dt. \quad (25)$$

The above functional is the same as that reported in [7] and yields Hamilton's principle by letting $\delta J = 0$. The fifth term inside the braces is the kinetic energyof the beam componentsand reads

$$T = \frac{1}{2}\int_0^L \left[\rho I_0\left(\frac{\partial \psi}{\partial t}\right)^2 + \frac{1}{2}\rho A_0\left(\frac{\partial w}{\partial t}\right)^2\right] dx. \qquad (26)$$

Proof. Making the above functional (25) stationary with respect to ψ, w, and u_s, the Euler equations correspond to (18a)–(18c), respectively. □

By following the same procedures performed for beam bending, the approximate solutions are obtained conveniently for beam vibrating by the Ritz method.

5. Conclusion

We used the semi-inverse method to establish a set of variational principles directly from governing differential equations. By following the rules of the calculus of variations, we obtained necessary variational principles for bending and vibration of the Timoshenko composite beam with partial-interaction. The obtained variational principles have been compared with those reported in literature and proved to be correct. It is concluded that the semi-inverse method is a powerful tool for searching for variational principles directly from the governing equations. Moreover, introducing an unknown function instead of a Lagrange multiplier, additional variational principles can also be written by constraining the trial Lagrangian with the different boundary conditions, which may facilitate the implementation of complicated boundary conditions. The direct variational method such as the Ritz method can be used to obtain the approximate solutions of the problem.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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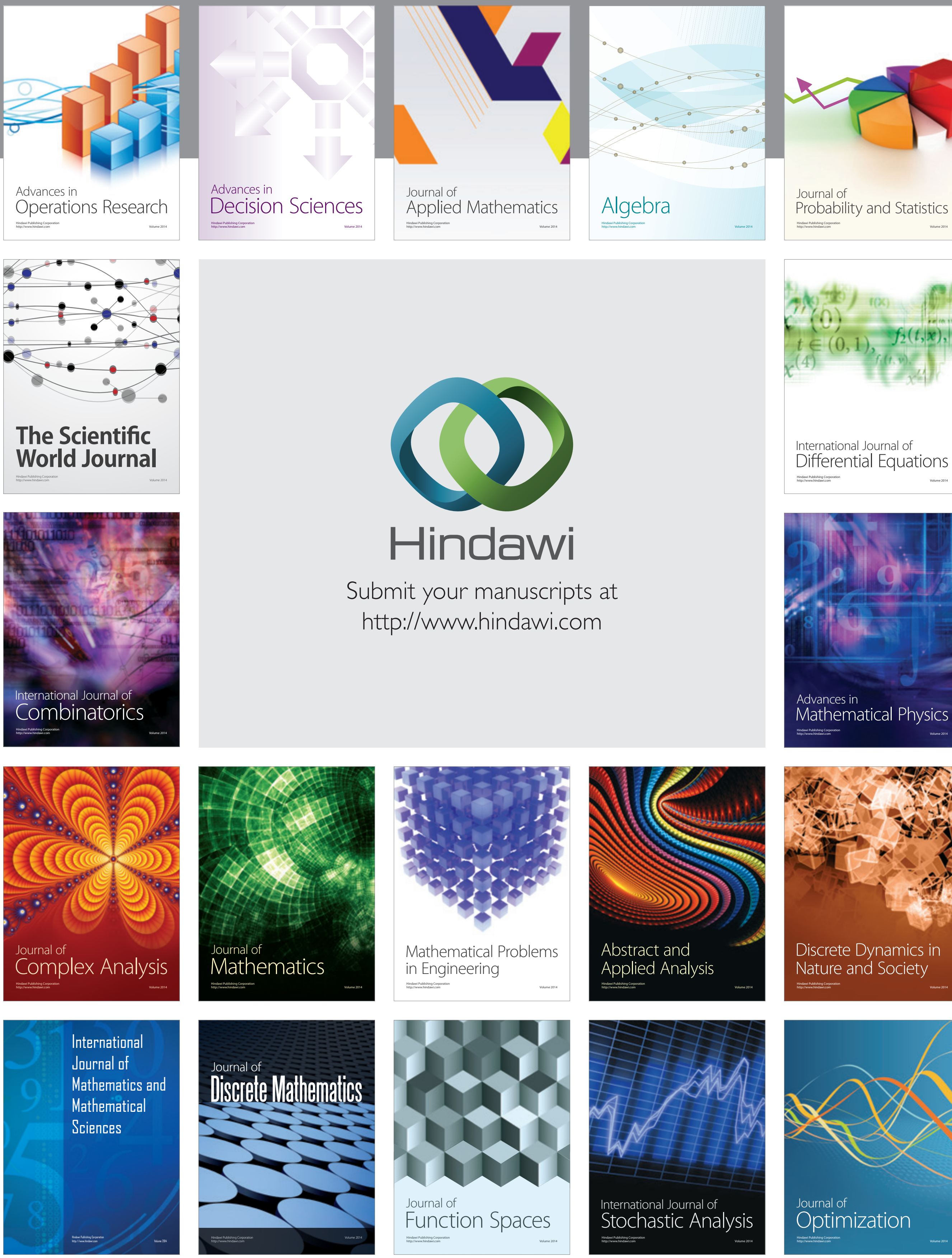
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