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Research Article

Adaptive Fractional Fuzzy Sliding Mode Control for Multivariable Nonlinear Systems

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This paper presents a robust adaptive fuzzy sliding mode control method for a class of uncertain nonlinear systems. The fractional order calculus is employed in the parameter updating stage. The underlying stability analysis as well as parameter update law design is carried out by Lyapunov based technique. In the simulation, two examples including a comparison with the traditional integer order counterpart are given to show the effectiveness of the proposed method. The main contribution of this paper consists in the control performance is better for the fractional order updating law than that of traditional integer order.

1. Introduction

Fuzzy systems have been applied to many control problems effectively because they do not need accurate mathematical model of the controlled system and they can also cooperate with human expert knowledge. It is well known that fuzzy systems and neural networks can uniformly approximate any nonlinear continuous function over some compact set [1]. Based on the universal approximation theorem [1], some adaptive fuzzy control methods [2–5] have been developed for MIMO nonlinear uncertain systems. The stability analysis of the underlying closed loop system has been carried out by means of Lyapunov approach. To deal with the ubiquitous fuzzy approximation errors and external disturbances, these controllers are usually augmented by a robust compensator, which can be a sliding mode control [2, 3, 6–8] or an H_∞ control [3]. And many important results have been given. In [9], fuzzy logic systems are used to obtain an adaptive boundary layer. Erbatur et al. [10] utilize the concept of fuzziness for reducing the adverse effects of chattering of the sliding mode control. The parameter's adaptation law is introduced in [11], where quicker reaching with suppressed oscillations is designed with a comparison with classical sliding mode controller. In [12], a robust control method for uncertain chaotic systems which comprises a nonlinear inversion-based controller with a fuzzy robust compensator is proposed.

Parameter tuning in adaptive control systems is an important part of the overall mechanism alleviating the influence associated with the changes in the parameters and uncertainties of the systems. Many studies can be found in the past two decades, and the domain of adaptation has become a blend of techniques of dynamical systems theory, optimization, soft computing, and heuristics. Now, tuning of parameters based on some set of observations has been facilitated [2, 3, 8, 13–16]. In [16], an in-depth discussion for parameter tuning in continuous as well as discrete time is proposed.

A common feature of all these control methods and the cited research is the fact that the differentiation and integration are performed in integer order. Up to now, with the development of complex engineering applications and natural science, fractional calculus as well as fractional differential equation theory and their applications begin to attract more and more attention from physicists to engineers [17–19]. Particularly for gradient descent rule, which is considered in the integer order in [16], it has been designed in fractional order in [18], where the integer-order integration is replaced with an integration with fractional order of 1.25. On the other hand, sliding mode control (SMC) is an effective technique to robustly control uncertain nonlinear systems [20–22]. The main idea of SMC is to switch the control input to drive the states of the system from the initial states onto some predefined sliding surface. Once the system operates on the sliding surface, it has desired properties such as stability, disturbance rejection capability, and tracking capability. SMC to accommodate fractional order nonlinear systems has not yet attracted much attention, due primarily to the mathematical difficulties in stability analysis. Moreover, there are only limited published results which concern fractional order chaotic systems under SMC.

In the stability analysis of fractional order systems, the Lyapunov function $V(t) = x^T(t)x(t)$ is often used. The αth-order of $V(t)$ can be given as

$$ {}_0D_t^{\alpha} V(t) = \left({}_0D_t^{\alpha} x(t)\right)^T x(t) + x^T(t)\, {}_0D_t^{\alpha} x(t) + 2\Lambda, \quad (1)$$

where $\Lambda = \sum_{k=1}^{\infty}(\Gamma(1+\alpha)/(\Gamma(1+k)\Gamma(1-k+\alpha)))\, {}_0D_t^{k}\tilde{z}\, {}_0D_t^{\alpha-k}\tilde{z}$. In order to obtain the results by using the stability theorems of fractional order systems, in [8, 23], the authors assume that the bounded condition $|\Lambda| \le b|x(t)|$ holds, where b is a positive constant. In this paper, we will prove the condition and establish a fundamental lemma. This lemma is established for stability analysis of fractional order systems, especially for Mittag-Leffler stability [24] analysis of fractional order nonlinear systems.

As a result of the discussion above, the absence of methods designed by fractional differintegration in robust control is visible. The objective of this paper is to fill this gap to the extent that covers the following aspects: (1) better robustness and system uncertainties rejection capabilities than those using conventional integer-order operators; (2) conditions for hitting in finite time; and (3) sliding mode control corporate with fractional order adaptation law. And the aforementioned ideas constitute the major contributions of this paper advancing the subject area to the fractional order adaptation methods.

2. Problem Formulation and Preliminaries

2.1. Problem Formulation. Consider the following MIMO nonlinear dynamic system which can be described by

$$\begin{aligned} y_1^{(r_1)} &= f_1(x) + \sum_{j=1}^{p} g_{1j}(x)\, u_j, \\ &\vdots \\ y_p^{(r_p)} &= f_p(x) + \sum_{j=1}^{p} g_{pj}(x)\, u_j, \end{aligned} \quad (2)$$

where $x = [x_1, \dot{x}_1, \ldots, x_1^{(r_1-1)}, \ldots, x_p, \dot{x}_p, \ldots, x_p^{(r_p-1)}]^T \in R^r$, $r = r_1 + r_2 + \cdots + r_p$ is the system state vector which is assumed to be available for measurement. $u = [u_1, \ldots, u_p]^T \in R^p$ and $y = [x_1, \ldots, x_p]^T \in R^p$ are control input and output vector, respectively. $f_i(x)$, $i = 1, \ldots, p$, are unknown nonlinear functions and $g_{ij}(x)$, $i, j = 1, \ldots, p$ are unknown constant control gains.

If we denote

$$\begin{aligned} y^{(r)} &= \left[x_1^{(r_1)}, \ldots, x_p^{(r_p)}\right]^T, \\ F(x) &= \left[f_1(x), \ldots, f_p(x)\right]^T, \\ G(x) &= \begin{bmatrix} g_{11}(x) & \cdots & g_{1p}(x) \\ \vdots & \ddots & \vdots \\ g_{p1}(x) & \cdots & g_{pp}(x) \end{bmatrix}, \end{aligned} \quad (3)$$

then, system (2) can be rewritten as

$$y^{(r)} = F(x) + G(x)\, u. \quad (4)$$

The objective of this paper is to construct a control input u such that the output y tracks the specified desired signal $y_d = [y_{d1}, \ldots, y_{dp}] \in R^p$ with all involved signals in the closed loop system keep bounded. To meet the objective, let us make the following assumptions.

Assumption 1. The desired trajectory signal $[y_{d1}, \ldots, y_{d1}^{(r_1)}, \ldots, y_{dp}, \ldots, y_{dp}^{(r_p)}]$ is continuous, bounded, and available for measurement.

Assumption 2. The control gain matrix $G(x)$ is positive definite and satisfies

$$\frac{1}{2}\left\|\dot{G}^{-1}(x)\right\| \le \beta(x), \quad (5)$$

where $\beta(x) > 0$ is an unknown continuous nonlinear function.

Remark 3. Assumption 2 is not restrictive. There are many systems, such as electrical machines and robotic systems which satisfy Assumption 2. And we only assume the existence of the nonlinear function $\beta(x)$ and not its knowledge.

Let us define the tracking error as

$$e_i = y_{di} - y_i, \quad i = 1, \ldots, p, \quad (6)$$

and the sliding mode surface as

$$s_i = \left(\frac{d}{dt} + \lambda_i\right)^{r_i - 1} e_i(t), \quad i = 1, \ldots, p. \quad (7)$$

Equation (7) can be rewritten as

$$\begin{aligned} s_i &= e_i^{(r_i-1)} + (r_i - 1)\lambda_i e_i^{(r_i-2)} + \cdots \\ &\quad + (r_i - 1)\lambda_i^{r_i-2}\dot{e}_i + \lambda_i^{r_i-1} e_i. \end{aligned} \quad (8)$$

Notice that if we select $\lambda_i > 0$, then the roots of the polynomial $H_i(s) = s^{r_1-1} + (r_i-1)\lambda_i s^{r_i-2} + \cdots + (r_i-1)\lambda_i^{r_i-2}\dot{s} + \lambda_i^{r_i-1} = 0$ are all in the left-half complex plane. In other words, the objective of this paper becomes the design of controller to force the filtered tracking error $s_i(t) \to 0$. Differentiating s_i with respect to time we obtain

$$\dot{s}_i = v_i - f_i(x) - \sum_{j=1}^{p} g_{ij}(x)\, u_j, \quad (9)$$

with $v_i = y_{d1}^{(r_i)} + \gamma_{1,r_i-1}e_1^{(r_i-1)} + \cdots + \gamma_{1,1}\dot{e}_i$, $\gamma_{i,j} = ((r_i-1)!/((r_i-j)!(j-1)!))\lambda_i^{r_i-j}$, $i = 1,\ldots,p$ and $j = 1,\ldots,r_i-1$. Denote $s = [s_1,\ldots,s_p]^T$ and $v = [v_1,\ldots,v_p]^T$; then (9) can be rewritten in the following compact form:

$$\dot{s} = v - F(x) - G(x)u. \tag{10}$$

Thereafter, (10) will be used in the construct of the controller as well as the stability analysis.

2.2. Description of the Fuzzy Logic System. The fuzzy logic system that employs singleton fuzzification, sum-product inference, and center-of-sets defuzzification, as shown in Figure 1, can be modeled by

$$\alpha(x) = \frac{\sum_{j=1}^{N}\theta_j\prod_{i=1}^{n}\mu_{F_i^j}(x_i)}{\sum_{j=1}^{N}\left[\prod_{i=1}^{n}\mu_{F_i^j}(x_i)\right]}, \tag{11}$$

where $\alpha(x)$ is the output of the fuzzy system, x is the input vector, $\mu_{F_i^j}(x_i)$ is x_i's membership of the jth rule, and θ_j is the centroid of the jth consequent set. Equation (6) can be rewritten as following equation:

$$\alpha(x) = \theta^T\psi(x), \tag{12}$$

with $= [\theta_1,\ldots,\theta_N]^T$, $\psi(x) = [p_1(x), p_2(x),\ldots,p_N(x)]^T$, and the fuzzy basis function can be written as $p_j(x) = \prod_{i=1}^{n}\mu_{F_i^j}(x_i)/\sum_{j=1}^{N}[\prod_{i=1}^{n}\mu_{F_i^j}(x_i)]$. Suppose there are N rules of the fuzzy system used to approximate the unknown function $\alpha(x)$.

Rule i. if x_1 is F_1^i and $\cdots$ and x_n is F_n^i then $\alpha(x)$ is $B^i, i = 1,2,\ldots,N$.

3. Adaptive Fractional Fuzzy Controller Design

3.1. Ideal Controller. Suppose that the functions $G(x)$ and $F(x)$ are known in advance. From (10) we know

$$G_1(x)\dot{s} = G_1(x)v - G_1(x)F(x) - u, \tag{13}$$

where $G_1(x) = G^{-1}(x)$.

Then we can construct the following ideal controller u^*:

$$u = u^* = G_1(x)v - G_1(x)F(x) + (K + \beta(x)I)s, \tag{14}$$

where $K = \mathrm{diag}[k_1,\ldots,k_p]$ with $k_i > 0$, $i = 1,2,\ldots,p$.

Theorem 4. *Consider system* (2). *If Assumptions 1 and 2 are satisfied. The control input* (14) *can guarantee that all signals in the closed loop system will remain bounded and the tracking errors and their derivatives converge to origin asymptotically; that is,* $e_i^{(j)} \to 0$, $t \to \infty$, $i = 1,\ldots,p$, *and* $j = 1,\ldots,r_j-1$.

Proof. Substituting the ideal control input (14) to the tracking error dynamics (13) gives

$$G_1(x)\dot{s} = -\beta(x)Is - Ks. \tag{15}$$

Multiplying s^t to both sides of (15) we have

$$s^TG_1(x)\dot{s} = -s^T\beta(x)Is - s^TKs. \tag{16}$$

Let us define the following Lyapunov function:

$$V = \frac{1}{2}s^TG_1(x)s. \tag{17}$$

Its derivative with respect to time can be given by

$$\dot{V} = s^TG_1(x)\dot{s} + \frac{1}{2}s^T\dot{G}_1(x)s. \tag{18}$$

By using Assumption 2 and (16), we can obtain

$$\begin{aligned}\dot{V} &= s^TG_1(x)\dot{s} + \frac{1}{2}s^T\dot{G}_1(x)s\\ &\le -s^T(K + \beta(x)I)s + \frac{1}{2}\left\|\dot{G}_1(x)\right\|\|s\|^2\\ &\le -s^TKs.\end{aligned} \tag{19}$$

From (19) we can conclude that $s_i \to 0$ as $t \to \infty$. Therefore, the tracking errors and their derivatives converge to origin. And this ends the proof of Theorem 4. □

3.2. Fractional order Adaption Law of Parameters of Fuzzy Systems. Let $0 < \alpha < 1$. The Riemann-Liouville (R-L) definition of the αth-order fractional integration can be given as

$${}_0D_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)\,d\tau, \tag{20}$$

and the αth-order fractional derivative can be given as

$${}_0D_t^{\alpha}f(t) = f^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-\tau)^{-\alpha}f(\tau)\,d\tau, \tag{21}$$

where $\Gamma(\cdot)$ represents the Gamma function $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$. From the above definition, we can get the following properties of the fractional calculus [8, 17]:

$$\begin{gathered}{}_0D_t^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha},\\ {}_0D_t^{-\alpha}\left({}_0D_t^{\alpha}\right)f(t) = f(t) - f^{\alpha-1}(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)}\\ {}_0D_t^{-\alpha}\left({}_0D_t^{\alpha}\right)f(t) = f(t).\end{gathered} \tag{22}$$

In the rest of this section, we assume that the target output of the fuzzy system is known such that the approximation error is available for parameter updating process. Let z and z_d be the output of the fuzzy system (12) and the target output, respectively. Then we have

$$z = \theta^T\psi(x). \tag{23}$$

Let the approximation error of the fuzzy system be

$$\tilde{z} = z - z_d. \tag{24}$$

Now we are ready to give the following results.

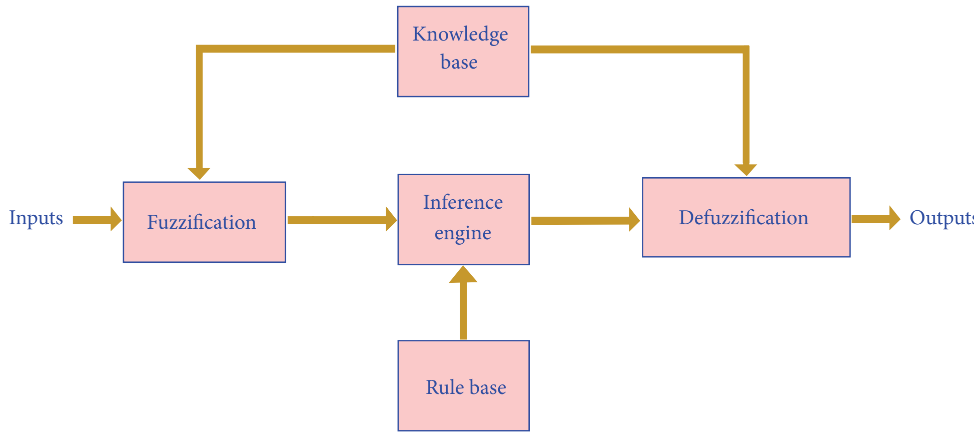

FIGURE 1: Structure of a fuzzy inference system.

Theorem 5. *Suppose the following boundedness conditions hold:*

$$\left|\sum_{k=1}^{\infty}\sum_{i=1}^{N}\binom{\alpha}{k}\,{}_0D_t^{\alpha-k}\theta_i\,{}_0D_t^{k}\psi_i\right| \le a_1, \tag{25}$$

$$\left|\sum_{k=1}^{\infty}\frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)}\,{}_0D_t^{k}\tilde{z}\,{}_0D_t^{\alpha-k}\tilde{z}\right| \le a_2|\tilde{z}|, \tag{26}$$

$$\left|{}_0D_t^{\alpha}z_d\right| \le a_3, \tag{27}$$

where $a_i > 0$, $i = 1,2,3$ are constants. If the adaption law is chosen as

$$ {}_0D_t^{\alpha}\theta = -b\frac{\psi}{\|\psi\|^2}\operatorname{sign}(\tilde{z}), \tag{28}$$

where $\theta = [\theta_1,\ldots,\theta_p]^T$ and $\psi = [\psi_1,\ldots,\psi_p]^T$.

Then the approximation error will converge to zero within some finite time satisfying

$$\frac{b-a_1}{\Gamma(1+\alpha)}t_h^{\alpha} \le \frac{\left|{}_0D_t^{\alpha-1}\tilde{z}(0)\right| + \left|{}_0D_t^{\alpha-1}z_d(0)\right|}{\Gamma(\alpha)}t_h^{\alpha-1} + \left|z_d(t_h)\right| \tag{29}$$

if $b > a_1 + a_2 + a_3$ is satisfied.

Proof. Noting that $s = 0$ corresponds to the fact that the states are on the sliding manifold. While $\tilde{z} = 0$ represents that the control signal eventually results in $s = 0$. As a result, the dynamical conclusions of $\tilde{z} = 0$ are different from $s = 0$.

Let us define

$$\Xi = \sum_{k=1}^{\infty}\sum_{i=1}^{N}\binom{\alpha}{k}\,{}_0D_t^{\alpha-k}\theta_i\,{}_0D_t^{k}\psi_i. \tag{30}$$

Consider the following Lypunov function candidate:

$$V_1 = \tilde{z}^2. \tag{31}$$

According to the Leibniz rule of the fractional differentiation, we have

$${}_0D_t^{\alpha}V_1 = \left({}_0D_t^{\alpha}\tilde{z}\right)\tilde{z} + \Lambda, \tag{32}$$

where

$$\Lambda = \sum_{k=1}^{\infty}\frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)}\,{}_0D_t^{k}\tilde{z}\,{}_0D_t^{\alpha-k}\tilde{z}. \tag{33}$$

Then we have

$$\begin{aligned}
\left({}_0D_t^{\alpha}\tilde{z}\right)\tilde{z} &= \left({}_0D_t^{\alpha}z - {}_0D_t^{\alpha}z_d\right)\tilde{z} \\
&= \left({}_0D_t^{\alpha}\left(\sum_{i=1}^{m}\theta_i\psi_i\right) - {}_0D_t^{\alpha}z_d\right)\tilde{z} \\
&= \left(\sum_{k=0}^{\infty}\sum_{i=1}^{N}\binom{\alpha}{k}\,{}_0D_t^{\alpha-k}\theta_i\,{}_0D_t^{k}\psi_i - {}_0D_t^{\alpha}z_d\right)\tilde{z} \\
&= \left(\sum_{i=1}^{N}\psi_i\,{}_0D_t^{\alpha}\theta_i + \sum_{k=1}^{\infty}\sum_{i=1}^{N}\binom{\alpha}{k}\,{}_0D_t^{\alpha-k}\theta_i\,{}_0D_t^{k}\psi_i \right.\\
&\qquad \left. - {}_0D_t^{\alpha}z_d\right)\tilde{z} \\
&= \tilde{z}\psi^T\,{}_0D_t^{\alpha}\theta + \left(\Xi - {}_0D_t^{\alpha}z_d\right)\tilde{z} \\
&= -b\tilde{z}\psi^T\frac{\psi}{\|\psi\|^2}\operatorname{sign}(\tilde{z}) + \left(\Xi - {}_0D_t^{\alpha}z_d\right)\tilde{z} \\
&= -b\operatorname{sign}(\tilde{z})\tilde{z} + \left(\Xi - {}_0D_t^{\alpha}z_d\right)\tilde{z} \\
&\le -b|\tilde{z}| + |\Xi\tilde{z}| + \left|{}_0D_t^{\alpha}z_d\right||\tilde{z}| \\
&\le \left(-b + a_1 + a_3\right)|\tilde{z}|.
\end{aligned} \tag{34}$$

Substituting (34) into (32) and using (26) yields

$${}_0D_t^{\alpha}V_1 \le \left(b - a_1 - a_2 - a_3\right)|\tilde{z}| < 0. \tag{35}$$

Since ${}_0D_t^{\alpha} V_1 < 0$, as the same discuss in [24], we know that the phase space are attracted by $\tilde{z} = 0$.

Now, let us prove that first hitting to the switching surface happens in some finite time t_h. From (34) and the fractional order adaption law (28) we have

$$ {}_0D_t^{\alpha} \tilde{z} = -b \operatorname{sign}(\tilde{z}) + \Xi - {}_0D_t^{\alpha} z_d. \tag{36} $$

Applying the fractional integration operator ${}_0D_{t_h}^{-\alpha}$ on both sides of (36) we can obtain

$$ \begin{aligned} &\tilde{z}(t_h) - {}_0D_{t_h}^{\alpha-1} \tilde{z}(0) \frac{t_h^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad = \frac{-b \operatorname{sign}(\tilde{z}(0))}{\Gamma(\alpha+1)} t_h^{\alpha} + {}_0D_{t_h}^{-\alpha} \left(\Xi - {}_0D_{t_h}^{\alpha} z_d\right). \end{aligned} \tag{37} $$

Multiplying $\operatorname{sign}(\tilde{z}(0))$ to both sides of (37) we have

$$ \begin{aligned} &- {}_0D_{t_h}^{\alpha-1} \tilde{z}(0) \operatorname{sign}(\tilde{z}(0)) \frac{t_h^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad = \frac{-b}{\Gamma(\alpha+1)} t_h^{\alpha} + \operatorname{sign}(\tilde{z}(0)) \, {}_0D_{t_h}^{-\alpha} \Xi \\ &\qquad - \operatorname{sign}(\tilde{z}(0)) \, {}_0D_{t_h}^{-\alpha} \left({}_0D_{t_h}^{\alpha} z_d\right). \end{aligned} \tag{38} $$

Noting that

$$ \begin{aligned} \operatorname{sign}(\tilde{z}(0)) \, {}_0D_{t_h}^{-\alpha} \Xi &\le {}_0D_{t_h}^{-\alpha} |\Xi| \\ &\le {}_0D_{t_h}^{-\alpha} a_1 \\ &= a_1 \frac{t_h^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned} \tag{39} $$

$$ \begin{aligned} &\operatorname{sign}(\tilde{z}(0)) \, {}_0D_{t_h}^{-\alpha} \left({}_0D_{t_h}^{\alpha} z_d\right) \\ &\quad = \operatorname{sign}(\tilde{z}(0)) \left(z_d(t_h) - {}_0D_{t_h}^{\alpha-1} z_d(0) \frac{t_h^{\alpha-1}}{\Gamma(\alpha)}\right), \end{aligned} $$

then we have

$$ \begin{aligned} &- {}_0D_{t_h}^{\alpha-1} \tilde{z}(0) \operatorname{sign}(\tilde{z}(0)) \frac{t_h^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad \le \frac{-b}{\Gamma(\alpha+1)} t_h^{\alpha} + a_1 \frac{t_h^{\alpha-1}}{\Gamma(\alpha+1)} - \operatorname{sign}(\tilde{z}(0)) z_d(t_h) \\ &\qquad + \operatorname{sign}(\tilde{z}(0)) \, {}_0D_{t_h}^{\alpha-1} z_d(0) \frac{t_h^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned} \tag{40} $$

After some straightforward manipulators, we can obtain

$$ \begin{aligned} \frac{b-a_1}{\Gamma(1+\alpha)} t_h^{\alpha} &\le \frac{\left|{}_0D_t^{\alpha-1} \tilde{z}(0)\right| + \left|{}_0D_t^{\alpha-1} z_d(0)\right|}{\Gamma(\alpha)} t_h^{\alpha-1} \\ &\quad + \left|z_d(t_h)\right|. \end{aligned} \tag{41} $$

And this ends the proof of Theorem 5. □

Remark 6. The assumptions (25)–(27) in Theorem 5 are rather stringent. We know that the control system will be stable if these conditions hold; yet, imposing such conditions makes the proposed design valid only in a local region. In the simulation, we present examples to show that the aforementioned region is large enough to achieve a highly satisfactory performance.

Remark 7. The boundedness condition (26) can also be seen in [23, pp. 6927].

In this paper, we will prove the boundedness conditions and establish a fundamental lemma. This lemma is established for stability analysis of fractional order systems, especially for Mittag-Leffler stability [24] analysis of fractional order nonlinear systems. As an example, we will prove the boundedness condition (26).

Lemma 8. *Let*

$$ \Lambda = \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)} \, {}_0D_t^{k} \tilde{z} \, {}_0D_t^{\alpha-k} \tilde{z}. \tag{42} $$

Then there exists some positive constant $a_2 > 0$ such that

$$ \left| \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)} \, {}_0D_t^{k} \tilde{z} \, {}_0D_t^{\alpha-k} \tilde{z} \right| \le a_2 |\tilde{z}|. \tag{43} $$

Proof. Since ${}_0D_t^{k} \tilde{z}_0$ exists, it is obvious that ${}_0D_t^{k} \tilde{z}_0$ are bounded. As a result, there exists M such that

$$ \left\| {}_0D_t^{k} \tilde{z}_0 \right\| \le M. \tag{44} $$

On the other hand, because $0 < \alpha < 1$, we have

$$ \left\| {}_0D_t^{\alpha-k} \tilde{z} \right\| \le \overline{K} \|\tilde{z}\|. \tag{45} $$

It is known that the Gamma function is nonzero everywhere along the real line, and there is in fact no complex number z for which $\Gamma(z) = 0$. As a result the reciprocal Gamma function $1/\Gamma(z)$ is an entire function. There exists a lower bound L such that $0 < L \le |\Gamma(1-\alpha+k)|$ for $k = 1, 2, 3, \ldots$.

Since $\Gamma(k)/(\Gamma(k+1)) = 1/k$ and the infinite series $\sum_{k=1}^{\infty} 1/(\Gamma(k+1))$ is convergence, there exists an upper bound $H > 0$ such that

$$ 0 < \sum_{k=1}^{\infty} \frac{1}{\Gamma(k+1)} < H. \tag{46} $$

From above discussion, we can obtain the following inequality:

$$ \left| \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)} \, {}_0D_t^{k} \tilde{z} \, {}_0D_t^{\alpha-k} \tilde{z} \right| \le a_2 |\tilde{z}| \tag{47} $$

in which

$$ a_2 = \frac{\Gamma(1+\alpha) M \overline{K} H}{L}. \tag{48} $$

□

3.3. Controller Design. Since the nonlinear functions $F(x)$ and $G(x)$ are uncertain, the ideal controller in Section 3.1 cannot be used directly. Firstly, let us rearrange the ideal controller (14) as

$$u^* = G_1(x)v - G_1(x)F(x) + (K + \beta(x)I)s = \gamma(x) + Ks, \quad (49)$$

where $\gamma(x) = G_1(x)v - G_1(x)F(x) + \beta(x)Is = [\gamma_1(x), \ldots, \gamma_p(x)]^T$ represents the system uncertainty. Then from the discussions in Section 3.2, the unknown function $\gamma_j(x)$ can be approximated by the fuzzy logic system as

$$\gamma_j(x) = \theta_j^T \psi(x), \quad j = 1, 2, \ldots, p. \quad (50)$$

Let us denote

$$\gamma(x) = \theta\psi(x) = [\theta_1\psi(x), \ldots, \theta_p\psi(x)]^T, \quad (51)$$

then the controller can be designed as

$$u = \theta^T\psi(x) + Ks + K_1 \operatorname{sign}(s), \quad (52)$$

where $K_1 \operatorname{sign}(s)$ is a robust term used to cancel the approximation error of the fuzzy systems.

Remark 9. Noting that there are p fuzzy logic systems are employed to approximate the unknown function $\gamma(x)$. And for every fuzzy system, the parameters are updated by the adaption law (28); that is, there are p fractional adaption laws which are used in this paper.

From above discussions, now we are ready to give the following theorem.

Theorem 10. *Consider system* (2). *Suppose that Assumptions 1 and 2 are satisfied. If the control input is defined by* (52) *with fraction adaption laws* (28), *then all signals in the closed loop system will remain bounded and the tracking errors and their derivatives converge to origin asymptotically; that is,* $e_i^{(j)} \to 0$, $t \to \infty$, $i = 1, \ldots, p$, $j = 1, \ldots, r_j - 1$.

From Theorems 4 and 5, we can easily get Theorem 5. Here we omit the proof of Theorem 10.

4. Simulation Results

Two nonlinear systems are utilized to show the effectiveness of the proposed hybrid control scheme.

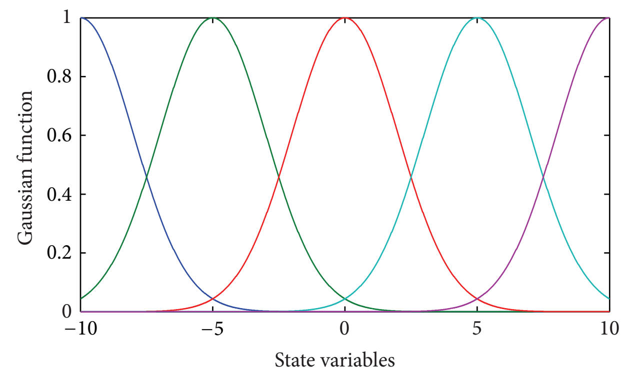

Figure 2: Membership functions in fuzzy system.

4.1. Example 1. Firstly, let us use the following MIMO system:

$$\begin{aligned} \dot{x}_{11} &= x_{12}, \\ \dot{x}_{12} &= x_{21} - 0.3\sin(x_{11}x_{12}) + x_{12}^2 \\ &\quad + (4 + \cos(x_{11}))u_1 + (1 + \sin^2(x_{21}))u_2, \\ \dot{x}_{21} &= x_{22}, \\ \dot{x}_{22} &= x_{22}^3 + e^{x_{11}} - 1 + x_{12}^2 \\ &\quad + (1 + \sin^2(x_{21}))u_1 + (1 + \sin^2(x_{22}))u_2, \\ y_1 &= x_{11}, \qquad y_2 = x_{21}. \end{aligned} \quad (53)$$

System (53) can be written as the following compact form:

$$\ddot{x} = F(x) + G(x)u, \quad (54)$$

where $F(x) = \begin{bmatrix} x_{21} - 0.3\sin(x_{11}x_{12}) + x_{12}^2 \\ x_{22}^3 + e^{x_{11}} - 1 + x_{12}^2 \end{bmatrix}$, $G(x) = \begin{bmatrix} 4 + \cos(x_{11}) & 1 + \sin^2(x_{21}) \\ 1 + \sin^2(x_{21}) & 1 + \sin^2(x_{22}) \end{bmatrix}$.

The fuzzy systems have $[x_{11}, x_{12}, x_{21}, x_{22}]^T$ as input. For each variable of these fuzzy systems, we define five Gaussian membership functions distributed on the interval $[-10, 10]$. The Gaussian membership functions are shown in Figure 2. So there are $5^4 = 625$ rules that are used in the simulation. The initial value of θ_j, $j = 1, 2, 3, 4$, is chosen as 0.

The design parameters are chosen as $\lambda = 3$, $b = 10$, $K = \operatorname{diag}[5, 5]$, and $K_1 = K$. The initial value of the system and the desired signal are chosen as $x(0) = [1, -1, 2, -2]^T$ and $y_d = [0, 0]^T$, respectively.

Figure 3 shows the simulation results of the proposed method. Figures 3(a) and 3(b) indicate that the tracking errors are bounded and converge to zero rapidly. Figure 3(c) shows the boundedness of control inputs. Figure 3(d) gives the time response of the sliding surface. From these results, we can conclude that the sliding surface and the tracking errors converge in the vicinity of the origin. To show the change and boundedness of the fuzzy system parameters, time responses of $\|\theta_1\|$ and $\|\theta_2\|$ are depicted in Figure 4.

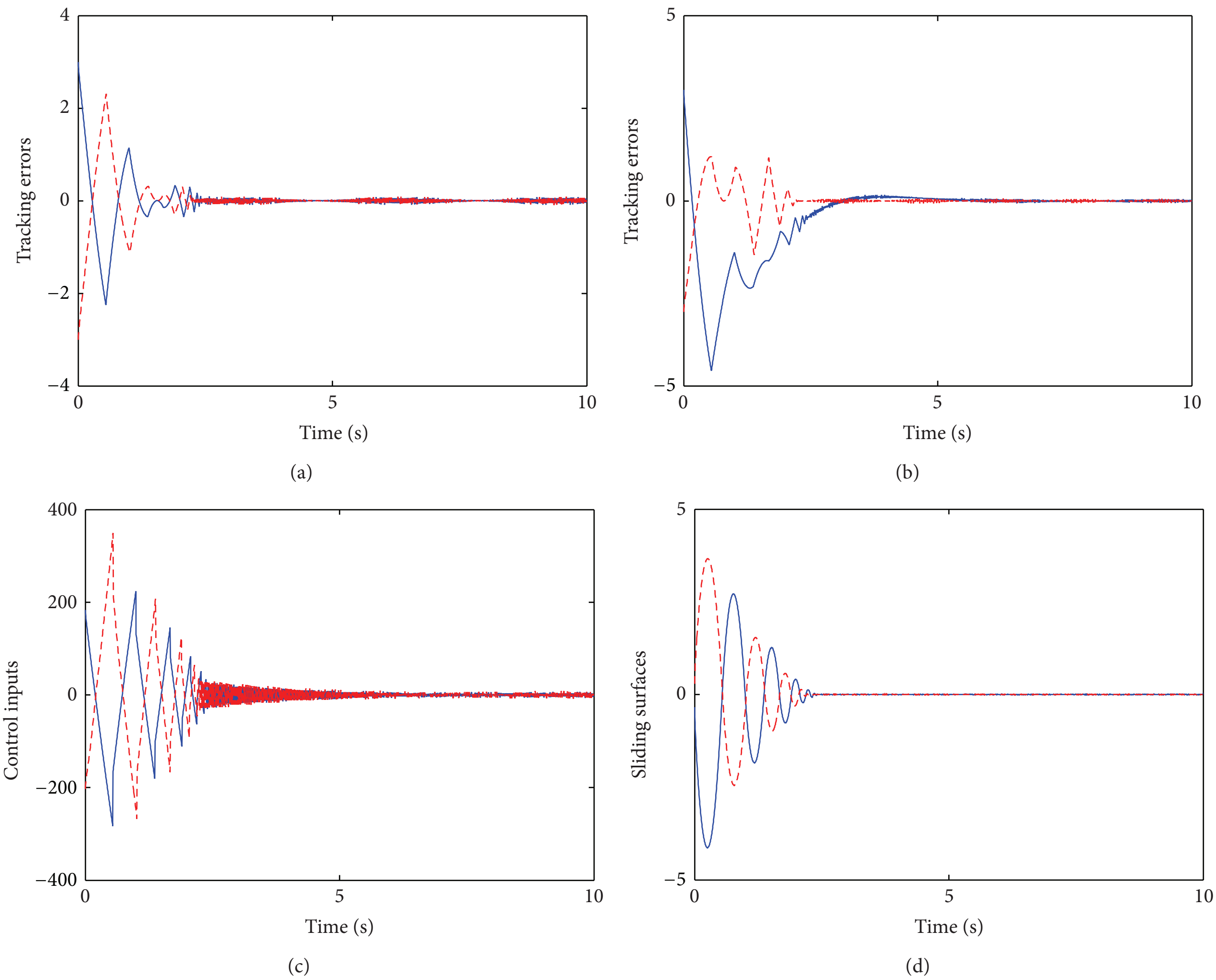

FIGURE 3: Simulation results for Example 1: (a) tracking errors: e_1 (solid line) and $\dot{e}_1$ (dashed line). (b) Tracking errors: e_2 (solid line) and $\dot{e}_2$ (dashed line). (c) Control inputs: u_1 (solid line) and u_2 (dashed line). (d) Sliding surface: s_1 (solid line) and s_2 (dashed line).

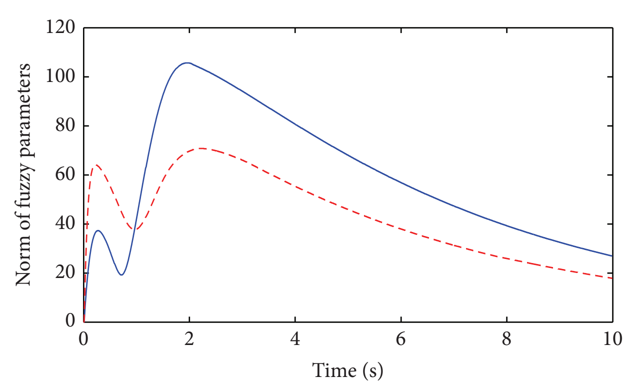

FIGURE 4: Time response of the fuzzy system parameters: $\|\theta_1\|$ (solid line) and $\|\theta_2\|$ (dashed line).

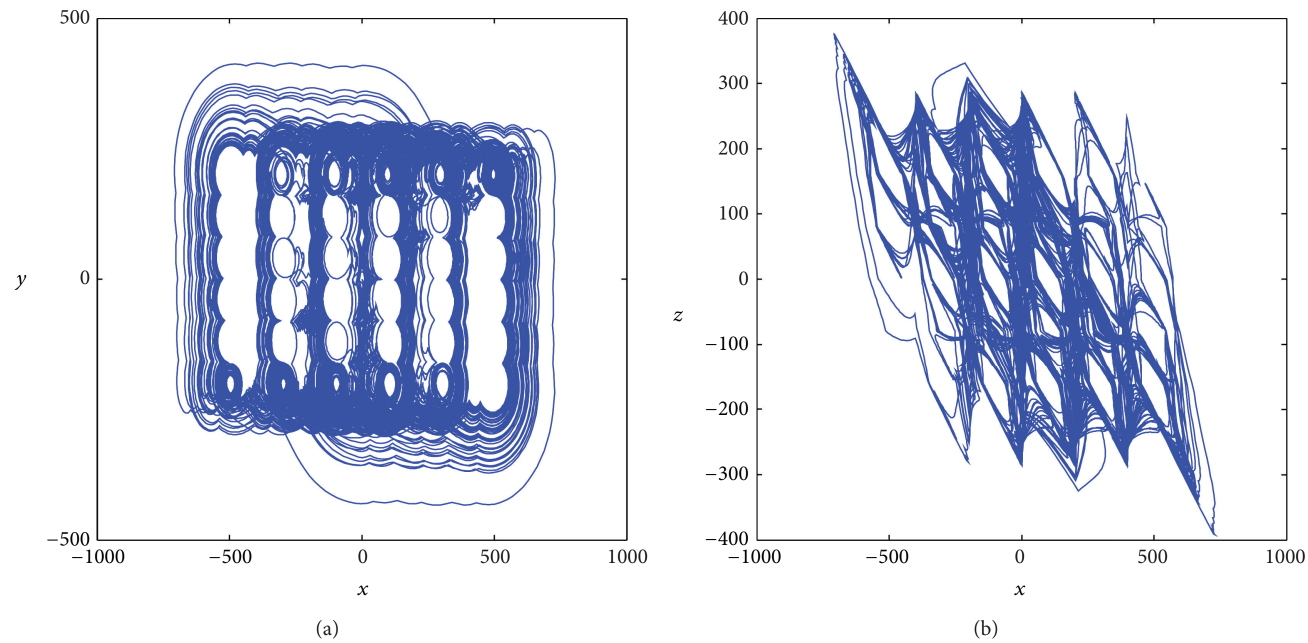

FIGURE 5: 3D multiscroll attractor: (a) $x-y$ plane and (b) $x-z$ plane.

4.2. Example 2. In this Section, a 3D saturated multiscroll chaotic system will be used. The comparison between our control method and the control method proposed in [12] will be made.

A 3D saturated multiscroll chaotic system can be described by [12, 25]

$$\begin{aligned}\dot{x}_1 &= x_2 - \frac{d_2 s(x_2; k_2, h_2, p_2, q_2)}{b} + u_1 = f_1(x) + u_1,\\ \dot{x}_2 &= x_3 - \frac{d_3 s(x_3; k_3, h_3, p_3, q_3)}{c} + u_2 = f_2(x) + u_2,\\ \dot{x}_3 &= -ax_1 - bx_2 - cx_3 + \sum_{i=1}^{3} d_i s(x_i; k_i, h_i, p_i, q_i) + u_3\\ &= f_3(x) + u_3,\end{aligned}$$

where the saturated function series $s(x; k, h, p, q)$ is defined as

$$s(x; k, h, p, q) = \begin{cases} 2q+1 & \text{if } x > qh+1\\ k(x-ih)+2ik & \text{if } |x-ih| \le 1,\ -p \le i \le q\\ (2i+1)k & \text{if } ih+1 < x < (i+1)h-1,\\ & -p \le i < q\\ -(2p+1)k & \text{if } x < -ph-1. \end{cases} \quad (55)$$

When $a = d_1 = 0.7$, $b = c = d_2 = d_3 = 0.8$, $k_1 = 100$, $h_1 = 200$, $k_2 = k_3 = 40$, $h_2 = h_3 = 80$, and $p_i = q_i = 2$, the unforced system (34) has a 3D $6\times6\times6$-grid scroll chaotic attractor, as shown in Figure 5.

The initial values are chosen as $x_1(0) = 5$, $x_2(0) = -5$, $x_3(0) = -4$, $\theta_i(0) = \overline{0}$, and $\hat{\overline{\varepsilon}}_i(0) = 0$, $i = 1, 2, 3$. The desired trajectories are $x_{d1} = x_{d2} = x_{d3} = \sin t$.

The design parameters are chosen as $\lambda = 3$, $b = 10$, $K = \text{diag}[0.5, 0.5, 0.5]$, and $K_1 = K$. Note that k_i are chosen the same as in [12]. *The discontinuous function* sign(·) *has been replaced by smooth function* arctan(20·) *to cancel the chattering phenomenon.*

Figure 6 shows the simulation results of the proposed scheme. From the simulation results, we can see that the tracking significantly decreased by using the proposed method. Compared with the control scheme in [12], the proposed controller can achieve a better performance in the presence of disturbances and system uncertainties.

5. Conclusions

This paper proposes a fractional order integration method for updating the parameters of fuzzy systems. It is shown that the proposed controller is applicable to MIMO nonlinear systems. According to the results in this paper, the fractional order updating law outperforms the updating mechanisms exploiting integer-order operators. To demonstrate the effectiveness of fractional order operators in the fuzzy system parameters updating, this paper investigates a wide range of applications from the domain of adaptive control. Specifically, the adaptive fuzzy sliding mode control method is focused on in this paper.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

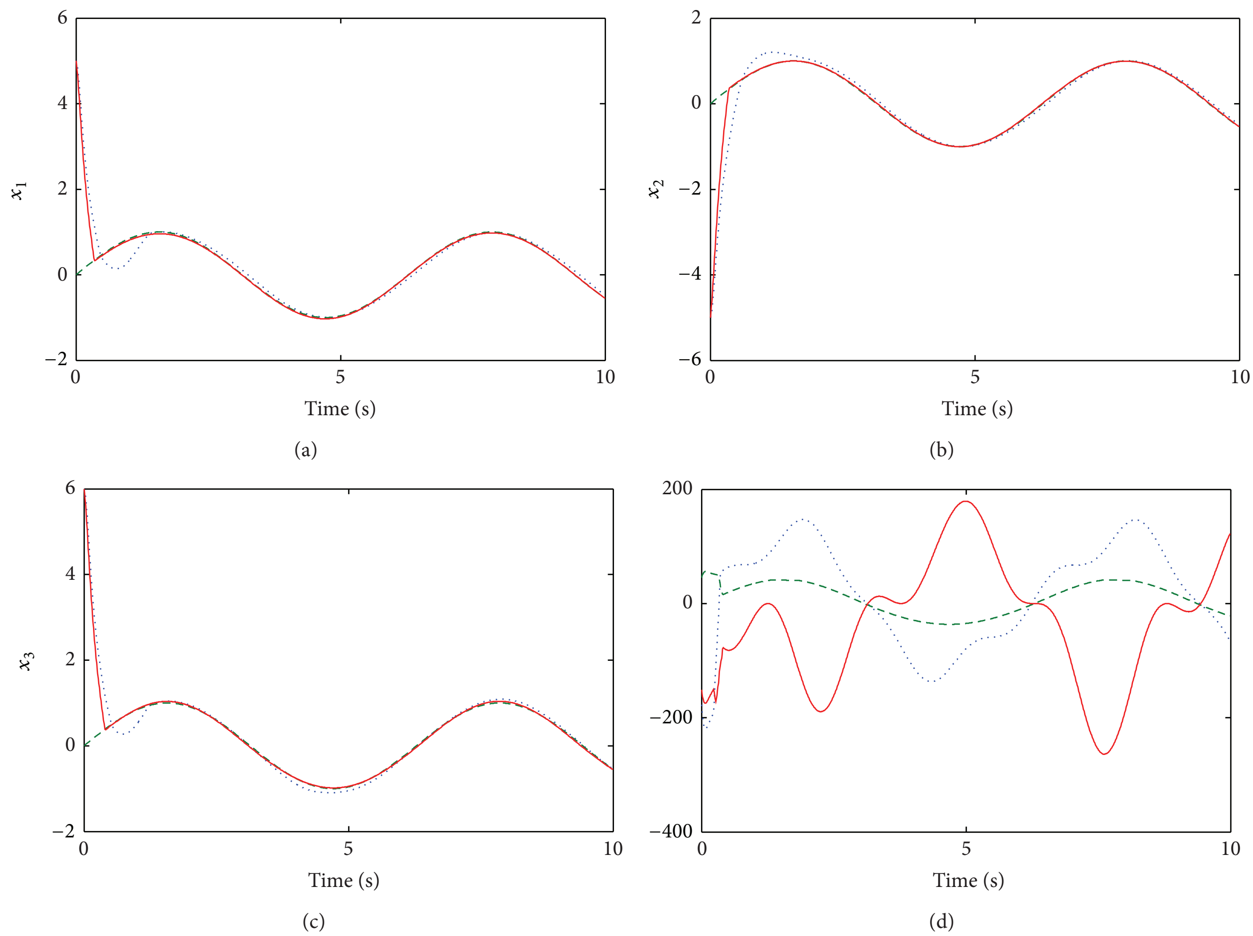

Figure 6: Comparison between our control method and the control method proposed in [12]. (a), (b), (c) Tracking of x_1, x_2, and x_3: control method proposed in [12] (dotted line), desired curve (dashed line), and our control method (solid line). (d) The control inputs: u_1 (dotted line), u_2 (dashed line), and u_3 (solid line).

Acknowledgments

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