Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 616981, 8 pages
doi:10.1155/2011/616981

*Research Article*

# Graphs Based on BCK/BCI-Algebras

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Received 2 October 2010; Accepted 13 December 2010

Academic Editor: Eun Hwan Roh



The associated graphs of BCK/BCI-algebras will be studied. To do so, the notions of ($l$-prime) quasi-ideals and zero divisors are first introduced and related properties are investigated. The concept of associative graph of a BCK/BCI-algebra is introduced, and several examples are displayed.

## 1. Introduction

Many authors studied the graph theory in connection with (commutative) semigroups and (commutative and noncommutative) rings as we can refer to references. For example, Beck [1] associated to any commutative ring $R$ its zero-divisor graph $G(R)$ whose vertices are the zero-divisors of $R$ (including 0), with two vertices $a, b$ joined by an edge in case $ab = 0$. Also, DeMeyer et al. [2] defined the zero-divisor graph of a commutative semigroup $S$ with zero ($0x = 0\ \forall x \in S$).

In this paper, motivated by these works, we study the associated graphs of BCK/BCI-algebras. We first introduce the notions of ($l$-prime) quasi-ideals and zero divisors and investigated related properties. We introduce the concept of associative graph of a BCK/BCI-algebra and provide several examples. We give conditions for a proper (quasi-) ideal of a BCK/BCI-algebra to be $l$-prime. We show that the associative graph of a BCK-algebra is a connected graph in which every nonzero vertex is adjacent to 0, but the associative graph of a BCI-algebra is not connected by providing an example.

## 2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following axioms:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra $X$ satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then $X$ is called a *BCK-algebra*. Any BCK/BCI-algebra $X$ satisfies the following conditions:

(a1) $(\forall x \in X)\ (x * 0 = x)$,

(a2) $(\forall x, y, z \in X)\ (x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0)$,

(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,

(a4) $(\forall x, y, z \in X)\ (((x * z) * (y * z)) * (x * y) = 0)$.

We can define a partial ordering $\leq$ on a BCK/BCI-algebra $X$ by $x \leq y$ if and only if $x * y = 0$.

A subset $A$ of a BCK/BCI-algebra $X$ is called an *ideal* of $X$ if it satisfies the following conditions:

(b1) $0 \in A$,

(b2) $(\forall x, y \in X)\ (x * y \in A, y \in A \Rightarrow x \in A)$.

We refer the reader to the books [3, 4] for further information regarding BCK/BCI-algebras.

## 3. Associated Graphs

In what follows, let $X$ denote a BCK/BCI-algebra unless otherwise specified.

For any subset $A$ of $X$, we will use the notations $r(A)$ and $l(A)$ to denote the sets

$$r(A) := \{x \in X \mid a * x = 0, \forall a \in A\},$$
$$l(A) := \{x \in X \mid x * a = 0, \forall a \in A\}. \tag{3.1}$$

**Proposition 3.1.** *Let $A$ and $B$ be subsets of $X$, then*

(1) *$A \subseteq l(r(A))$ and $A \subseteq r(l(A))$,*

(2) *If $A \subseteq B$, then $l(B) \subseteq l(A)$ and $r(B) \subseteq r(A)$,*

(3) *$l(A) = l(r(l(A)))$ and $r(A) = r(l(r(A)))$.*

*Proof.* Let $a \in A$ and $x \in l(A)$, then $x * a = 0$, and so $a \in r(l(A))$. This says that $A \subseteq r(l(A))$. Dually, $A \subseteq l(r(A))$. Hence, (1) is valid.

Assume that $A \subseteq B$ and let $x \in l(B)$, then $x * b = 0$ for all $b \in B$, which implies from $A \subseteq B$ that $x * b = 0$ for all $b \in A$. Thus, $x \in l(A)$, which shows that $l(B) \subseteq l(A)$. Similarly, we have $r(B) \subseteq r(A)$. Thus, (2) holds.

Using (1) and (2), we have $l(r(l(A))) \subseteq l(A)$ and $r(l(r(A))) \subseteq r(A)$. If we apply (1) to $l(A)$ and $r(A)$, then $l(A) \subseteq l(r(l(A)))$ and $r(A) \subseteq r(l(r(A)))$. Hence, $l(A) = l(r(l(A)))$ and $r(A) = r(l(r(A)))$. □

**Table 1:** $*$-operation.

| $*$ | 0 | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | 0 | 0 | 0 |
| $b$ | $b$ | $a$ | 0 | $a$ | 0 |
| $c$ | $c$ | $a$ | $a$ | 0 | 0 |
| $d$ | $d$ | $b$ | $a$ | $b$ | 0 |

**Table 2:** $*$-operation.

| $*$ | 0 | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | $c$ | $c$ | $a$ |
| 1 | 1 | 0 | $c$ | $c$ | $a$ |
| $a$ | $a$ | $a$ | 0 | 0 | $c$ |
| $b$ | $b$ | $a$ | 1 | 0 | $c$ |
| $c$ | $c$ | $c$ | $a$ | $a$ | 0 |

*Definition 3.2.* A nonempty subset $I$ of $X$ is called a *quasi-ideal* of $X$ if it satisfies

$$(\forall x \in X) \quad (\forall y \in I) \quad (x * y = 0 \Longrightarrow x \in I). \tag{3.2}$$

*Example 3.3.* Let $X = \{0, a, b, c, d\}$ be a set with the $*$-operation given by Table 1, then $(X; *, 0)$ is a BCK-algebra (see [4]). The set $I := \{0, a, b\}$ is a quasi-ideal of $X$.

Obviously, every quasi-ideal $I$ of a BCK-algebra $X$ contains the zero element 0. The following example shows that there exists a quasi-ideal $I$ of a BCI-algebra $X$ such that $0 \notin I$.

*Example 3.4.* Let $X = \{0, 1, a, b, c\}$ be a set with the $*$-operation given by Table 2, then $(X; *, 0)$ is a BCI-algebra (see [3]). The set $I := \{0, 1, a\}$ is a quasi-ideal of $X$ containing the zero element 0, but the set $J := \{a, b, c\}$ is a quasi-ideal of $X$ which does not contain the zero element 0.

Obviously, every ideal of $X$ is a quasi-ideal of $X$, but the converse is not true. In fact, the quasi-ideal $I := \{0, a, b\}$ in Example 3.3 is not an ideal of $X$. Also, quasi-ideals $I$ and $J$ in Example 3.4 are not ideals of $X$.

*Definition 3.5.* A (quasi-) ideal $I$ of $X$ is said to be *l-prime* if it satisfies

(i) $I$ is proper, that is, $I \neq X$,

(ii) $(\forall x, y \in X)$ $(l(\{x, y\}) \subseteq I \Rightarrow x \in I$ or $y \in I)$.

*Example 3.6.* Consider the BCK-algebra $X = \{0, a, b, c, d\}$ with the operation $*$ which is given by the Table 3, then the set $I = \{0, a, c\}$ is an $l$-prime ideal of $X$.

**Theorem 3.7.** *A proper (quasi-) ideal $I$ of $X$ is l-prime if and only if it satisfies*

$$l(\{x_1, \ldots, x_n\}) \subseteq I \Longrightarrow (\exists i \in \{1, \ldots, n\}) \quad (x_i \in I), \tag{3.3}$$

*for all $x_1, \ldots, x_n \in X$.*

**Table 3:** $*$-operation.

| $*$ | 0 | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | $a$ | 0 | 0 |
| $b$ | $b$ | $b$ | 0 | $b$ | 0 |
| $c$ | $c$ | $a$ | $c$ | 0 | $a$ |
| $d$ | $d$ | $d$ | $d$ | $d$ | 0 |

**Table 4:** $*$-operation.

| $*$ | 0 | 1 | 2 | $a$ | $b$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $a$ | $a$ |
| 1 | 1 | 0 | 1 | $b$ | $a$ |
| 2 | 2 | 2 | 0 | $a$ | $a$ |
| $a$ | $a$ | $a$ | $a$ | 0 | 0 |
| $b$ | $b$ | $a$ | $b$ | 1 | 0 |

*Proof.* Assume that $I$ is an $l$-prime (quasi-) ideal of $X$. We proceed by induction on $n$. If $n = 2$, then the result is true. Suppose that the statement holds for $n-1$. Let $x_1, \ldots, x_n \in X$ be such that $l(\{x_1, \ldots, x_{n-1}, x_n\}) \subseteq I$. If $y \in l(\{x_1, \ldots, x_{n-1}\})$, then $l(\{y, x_n\}) \subseteq l(\{x_1, \ldots, x_{n-1}, x_n\}) \subseteq I$. Assume that $x_n \notin I$, then $y \in I$ by the $l$-primeness of $I$, which shows that $l(\{x_1, \ldots, x_{n-1}\}) \subseteq I$. Using the induction hypothesis, we conclude that $x_i \in I$ for some $i \in \{1, \ldots, n-1\}$. The converse is clear. □

For any $x \in X$, we will use the notation $Z_x$ to denote the set of all elements $y \in X$ such that $l(\{x, y\}) = \{0\}$, that is,

$$Z_x := \{y \in X \mid l(\{x, y\}) = \{0\}\}, \tag{3.4}$$

which is called the set of zero divisors of $x$.

**Lemma 3.8.** *If X is a BCK-algebra, then $l(\{x, 0\}) = \{0\}$ for all $x \in X$.*

*Proof.* Let $x \in X$ and $a \in l(\{x, 0\})$, then $a * x = 0 = a * 0 = a$, and so $l(\{x, 0\}) = \{0\}$ for all $x \in X$. □

If $X$ is a BCI-algebra, then Lemma 3.8 does not necessarily hold. In fact, let $X = \{0, 1, 2, a, b\}$ be a set with the $*$-operation given by Table 4, then $(X; *, 0)$ is a BCI-algebra (see [4]). Note that $l(\{x, 0\}) = \{0\}$ for all $x \in \{1, 2\}$ and $l(\{x, 0\}) = \emptyset$ for all $x \in \{a, b\}$.

**Corollary 3.9.** *If X is a BCI-algebra, then $l(\{x, 0\}) = \{0\}$ for all $x \in X$ with $l(\{x, 0\}) \neq \emptyset$.*

**Lemma 3.10.** *If X is a BCI-algebra, then $l(\{x, 0\}) = \{0\}$ for all $x \in X_+$, where $X_+$ is the BCK-part of X.*

*Proof.* Straightforward. □

**Lemma 3.11.** *For any elements a and b of a BCK-algebra X, if $a * b = 0$, then $l(\{a\}) \subseteq l(\{b\})$ and $Z_b \subseteq Z_a$.*

**Table 5:** $*$-operation.

| $*$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | $a$ | $a$ |
| $b$ | $b$ | $b$ | 0 | $b$ |
| $c$ | $c$ | $c$ | $c$ | 0 |

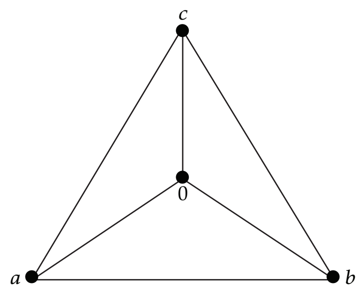


**Figure 1:** Associated graph $\Gamma(X)$ of $X$.

*Proof.* Assume that $a * b = 0$. Let $x \in l(\{a\})$, then $x * a = 0$, and so

$$0 = (x * b) * (x * a) = (x * b) * 0 = x * b. \tag{3.5}$$

Thus, $x \in l(\{b\})$, which shows that $l(\{a\}) \subseteq l(\{b\})$. Obviously, $Z_b \subseteq Z_a$. □

**Theorem 3.12.** *For any element $x$ of a BCK-algebra $X$, the set of zero divisors of $x$ is a quasi-ideal of $X$ containing the zero element 0. Moreover, if $Z_x$ is maximal in $\{Z_a \mid a \in X, Z_a \neq X\}$, then $Z_x$ is $l$-prime.*

*Proof.* By Lemma 3.8, we have $0 \in Z_x$. Let $a \in X$ and $b \in Z_x$ be such that $a * b = 0$. Using Lemma 3.11, we have

$$l(\{x, a\}) = l(\{x\}) \cap l(\{a\}) \subseteq l(\{x\}) \cap l(\{b\}) = l(\{x, b\}) = \{0\}, \tag{3.6}$$

and so $l(\{x, a\}) = \{0\}$. Hence, $a \in Z_x$. Therefore, $Z_x$ is a quasi-ideal of $X$. Let $a, b \in X$ be such that $l(\{a, b\}) \subseteq Z_x$ and $a \notin Z_x$, then $l(\{a, b, x\}) = \{0\}$. Let $0 \neq y \in l(\{a, x\})$ be an arbitrarily element, then $l(\{b, y\}) \subseteq l(\{a, b, x\}) = \{0\}$, and so $l(\{b, y\}) = \{0\}$, that is, $b \in Z_y$. Since $y \in l(\{a, x\})$, we have $y * x = 0$. It follows from Lemma 3.11 that $Z_x \subseteq Z_y \neq X$ so from the maximality of $Z_x$ it follows that $Z_x = Z_y$. Hence, $b \in Z_x$, which shows that $Z_x$ is $l$-prime. □

*Definition 3.13.* By the *associated graph* of a BCK/BCI-algebra $X$, denoted $\Gamma(X)$, we mean the graph whose vertices are just the elements of $X$, and for distinct $x, y \in \Gamma(X)$, there is an edge connecting $x$ and $y$, denoted by $x - y$ if and only if $l(\{x, y\}) = \{0\}$.

*Example 3.14.* Let $X = \{0, a, b, c\}$ be a set with the $*$-operation given by Table 5, then $X$ is a BCK-algebra (see [4]). The associated graph $\Gamma(X)$ of $X$ is given by the Figure 1.

**Table 6:** $*$-operation.

| $*$ | 0 | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | $a$ | 0 | $a$ |
| $b$ | $b$ | $b$ | 0 | $b$ | 0 |
| $c$ | $c$ | $a$ | $c$ | 0 | $c$ |
| $d$ | $d$ | $d$ | $d$ | $d$ | 0 |

**Table 7:** $*$-operation.

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 | 1 |
| 2 | 2 | 1 | 0 | 2 | 2 |
| 3 | 3 | 3 | 3 | 0 | 3 |
| 4 | 4 | 4 | 4 | 4 | 0 |

*Example 3.15.* Let $X = \{0, a, b, c, d\}$ be a set with the $*$-operation given by Table 6, then $X$ is a BCK-algebra (see [4]). By Lemma 3.8, each nonzero point is adjacent to 0. Note that $l(\{a, b\}) = l(\{a, d\}) = l(\{b, c\}) = l(\{c, d\}) = \{0\}$, $l(\{a, c\}) = \{0, a\}$, and $l(\{b, d\}) = \{0, b\}$. Hence the associated graph $\Gamma(X)$ of $X$ is given by the Figure 2.

*Example 3.16.* Let $X = \{0, 1, 2, 3, 4\}$ be a set with the $*$-operation given by Table 7, then $X$ is a BCK-algebra (see [4]). By Lemma 3.8, each nonzero point is adjacent to 0. Note that $l(\{1, 2\}) = \{0, 1\}$, that is, 1 is not adjacent to 2 and $l(\{1, 3\}) = l(\{1, 4\}) = l(\{2, 3\}) = l(\{2, 4\}) = l(\{3, 4\}) = \{0\}$. Hence, the associated graph $\Gamma(X)$ of X is given by Figure 3.

*Example 3.17.* Consider a BCI-algebra $X = \{0, 1, 2, a, b\}$ with the $*$-operation given by Table 4, then

$$l(\{1, a\}) = l(\{1, b\}) = l(\{2, a\}) = l(\{2, b\}) = \emptyset, \tag{3.7}$$

$l(\{a, b\}) = \{a\}$, and $l(\{1, 2\}) = \{0\}$. Since $X_+ = \{0, 1, 2\}$, we know from Lemma 3.10 that two points 1 and 2 are adjacent to 0. The associated graph $\Gamma(X)$ of $X$ is given by Figure 4.

**Theorem 3.18.** *Let $\Gamma(X)$ be the associated graph of a BCK-algebra X. For any $x, y \in \Gamma(X)$, if $Z_x$ and $Z_y$ are distinct l-prime quasi-ideals of X, then there is an edge connecting $x$ and $y$.*

*Proof.* It is sufficient to show that $l(\{x, y\}) = \{0\}$. If $l(\{x, y\}) \neq \{0\}$, then $x \notin Z_y$ and $y \notin Z_x$. For any $a \in Z_x$, we have $l(\{x, a\}) = \{0\} \subseteq Z_y$. Since $Z_y$ is $l$-prime, it follows that $a \in Z_y$ so that $Z_x \subseteq Z_y$. Similarly, $Z_y \subseteq Z_x$. Hence, $Z_x = Z_y$, which is a contradiction. Therefore, $x$ is adjacent to $y$. □

**Theorem 3.19.** *The associated graph of a BCK-algebra is connected in which every nonzero vertex is adjacent to 0.*

*Proof.* It follows from Lemma 3.8. □

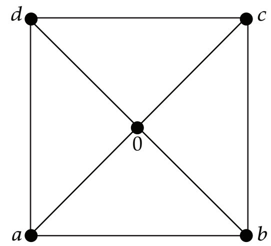


**Figure 2:** Associated graph $\Gamma(X)$ of $X$.

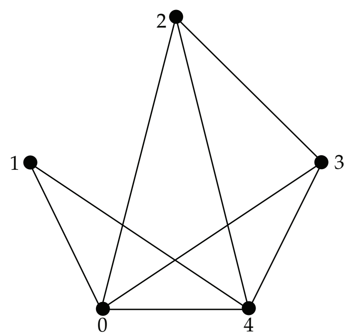


**Figure 3:** Associated graph $\Gamma(X)$ of $X$.

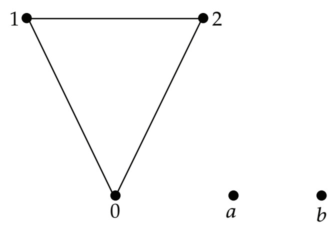


**Figure 4:** Associated graph $\Gamma(X)$ of $X$.

Example 3.17 shows that the associated graph of a proper BCI-algebra may not be connected.

## 4. Conclusions

We have introduced the associative graph of a BCK/BCI-algebra with several examples. We have shown that the associative graph of a BCK-algebra is connected, but the associative graph of a BCI-algebra is not connected.

Our future work is to study how to induce BCK/BCI-algebras from the given graph (with some additional conditions).

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