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Research Article

The Diagonally Dominant Degree and Disc Separation for the Schur Complement of Ostrowski Matrix

Jianxing Zhao,[1] Feng Wang,[1,2] and Yaotang Li[1]

[1] *School of Mathematics and Statistics, Yunnan University, Kunming, Yunnan 650091, China*
[2] *Department of Mathematics, Heze University, Heze, Shandong 274015, China*

Correspondence should be addressed to Yaotang Li; liyaotang@ynu.edu.cn

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By applying the properties of Schur complement and some inequality techniques, some new estimates of diagonally and doubly diagonally dominant degree of the Schur complement of Ostrowski matrix are obtained, which improve the main results of Liu and Zhang (2005) and Liu et al. (2012). As an application, we present new inclusion regions for eigenvalues of the Schur complement of Ostrowski matrix. In addition, a new upper bound for the infinity norm on the inverse of the Schur complement of Ostrowski matrix is given. Finally, we give numerical examples to illustrate the theory results.

1. Introduction

Let $\mathbb{C}^{n\times n}$ denote the set of all $n\times n$ complex matrices, $N=\{1,2,\ldots,n\}$, and $A=(a_{ij})\in\mathbb{C}^{n\times n}(n\ge 2)$. Denote

$$R_i(A)=\sum_{j\neq i}\left|a_{ij}\right|. \tag{1}$$

We know that A is called a strictly diagonally dominant matrix if

$$\left|a_{ii}\right|>R_i(A),\quad \forall i\in N. \tag{2}$$

A is called a generalized Ostrowski matrix if

$$\left|a_{ii}\right|\left|a_{jj}\right|\ge R_i(A)R_j(A),\quad \forall i,j\in N,\ i\neq j. \tag{3}$$

A is called Ostrowski matrix if all strict inequalities in (3) hold (see [1]).

SD_n and $OS_n(GOS_n)$ will be used to denote the sets of all $n\times n$ strictly diagonally dominant matrices and the sets of all $n\times n$ (generalized) Ostrowski matrices, respectively.

As shown in [2], for all $i\in N$, we call $|a_{ii}|-R_i(A)$ and $|a_{ii}||a_{jj}|-R_i(A)R_j(A)$ the ith diagonally and doubly diagonally dominant degree of A, respectively.

The infinity norm of A is defined as

$$\|A\|_\infty=\max_{1\le i\le n}\left\{R_i(A)+\left|a_{ii}\right|\right\}. \tag{4}$$

For $\beta\subseteq N$, denote by $|\beta|$ the cardinality of β and $\overline{\beta}=N/\beta$. If $\beta,\gamma\subseteq N$, then $A(\beta,\gamma)$ is the submatrix of A with row indices in β and column indices in γ. In particular, $A(\beta,\beta)$ is abbreviated to $A(\beta)$. Assuming that $\beta=\{i_1,i_2,\ldots,i_k\}\subset N$, $\overline{\beta}=N/\beta=\{j_1,j_2,\ldots,j_l\}$ and the elements of β and $\overline{\beta}$ are both conventionally arranged in an increasing order. For $1\le t\le l$, we denote

$$A_t=A\left(\beta\cup\{j_t\}\right). \tag{5}$$

If $A(\beta)$ is nonsingular,

$$\frac{A}{\beta}=\frac{A}{A(\beta)}=A\left(\overline{\beta}\right)-A\left(\overline{\beta},\beta\right)\left[A(\beta)\right]^{-1}A\left(\beta,\overline{\beta}\right) \tag{6}$$

is called the Schur complement of A with respect to $A(\beta)$.

The comparison matrix of A, $\mu(A)=(\alpha_{ij})$, is defined by

$$\alpha_{ij}=\begin{cases}\left|a_{ij}\right|, & \text{if } i=j,\\ -\left|a_{ij}\right|, & \text{if } i\neq j.\end{cases} \tag{7}$$

A matrix $A=(a_{ij})\in\mathbb{C}^{n\times n}$ is called an M-matrix if there exist a nonnegative matrix B and a number $s>\rho(B)$ such that $A=sI-B$, where $\rho(B)$ is the spectral radius of B. We know that A is an H-matrix if and only if $\mu(A)$ is an M-matrix, and if

A is an M-matrix, then the Schur complement of A is also an M-matrix and $\det A > 0$ (see [3]). H_n and M_n will denote the set of all $n\times n$ H-matrices and the set of all $n\times n$ M-matrices, respectively.

The Schur complement has been proved to be a useful tool in many fields such as control theory, statistics, and computational mathematics. A lot of work has been done on it (see [2, 4–15]). It is well known that the Schur complements of SD_n and OS_n are SD_n and OS_n, respectively. These properties have been used for the derivation of matrix inequalities in matrix analysis and for the convergence of iterations in numerical analysis (see [16–19]). Meanwhile, estimating the upper bound for the infinity norm of the inverse of the Schur complement is of great significance. We know that the upper bound of $\|A^{-1}\|_\infty$ plays an important role in some iterations for large scale nonhomogeneous system of linear equation $Ax = b$ (see [20]).

The paper is organized as follows. In Section 2, we give several new estimates of diagonally and doubly diagonally dominant degree on the Schur complement of matrices. In Section 3, new inclusion regions for eigenvalues of the Schur complement are obtained. A new upper bound of $\|(A/\beta)^{-1}\|_\infty$ is given in Section 4. In Section 5, we present numerical examples to illustrate the theory results.

2. The Diagonally Dominant Degree for the Schur Complement

In this section, we give several new estimates of diagonally and doubly diagonally dominant degree on the Schur complement of OS_n.

Lemma 1 (see [3]). *If $A \in H_n$, then $[\mu(A)]^{-1} \geq |A^{-1}|$.*

Lemma 2 (see [3]). *If $A \in SD_n$ or $A \in OS_n$, then $A \in H_n$; that is, $\mu(A) \in M_n$.*

Lemma 3 (see [6]). *If $A \in SD_n$ or $A \in OS_n$ and $\beta \subseteq N$, then the Schur complement of A is in $SD_{|\overline{\beta}|}$ or $OS_{|\overline{\beta}|}$, where $\overline{\beta} = N-\beta$ is the complement of β in N and $|\overline{\beta}|$ is the cardinality of $\overline{\beta}$.*

Lemma 4 (see [12]). *Let $A \in SD_n$, $\beta = \{i_1, i_2, \ldots, i_k\} \subset N$, $\overline{\beta} = \{j_1, j_2, \ldots, j_l\}$, and $k + l = n$. For any $j_t \in \overline{\beta}$, denote*

$$B_{j_t} \equiv \begin{pmatrix} x & -\left|a_{j_t i_1}\right| & \cdots & -\left|a_{j_t i_k}\right| \\ -\sum_{v=1}^{l}\left|a_{i_1 j_v}\right| & & & \\ \vdots & & \mu\left(A\left(\beta\right)\right) & \\ -\sum_{v=1}^{l}\left|a_{i_k j_v}\right| & & & \end{pmatrix}, \quad x > 0. \tag{8}$$

Then $B_{j_t} \in GOS_{k+1}$ if and only if

$$x \geq \max_{1\leq w\leq k} \frac{R_{i_w}(A)}{\left|a_{i_w i_w}\right|} \sum_{v=1}^{k}\left|a_{j_t i_v}\right|. \tag{9}$$

When the strict inequality in (9) *holds, $B_{j_t} \in M_{k+1}$, and thus* $\det B_{j_t} > 0$. *If the equality in* (9) *occurs, then* $\det B_{j_t} \geq 0$.

Lemma 5. *Let $A = (a_{ij}) \in OS_n$ and $\beta = \{i_1, i_2, \ldots, i_k\}$ with an index i_d $(1 \leq d \leq k)$ satisfying $|a_{i_d i_d}| \leq R_{i_d}(A)$, $|a_{i_d i_d}| > \sum_{i_u\in\beta/\{i_d\}}|a_{i_d i_u}|$, $\overline{\beta} = \{j_1, j_2, \ldots, j_l\}$, $1 \leq k < n$, and $A/\beta = (a'_{ts})$. Then, for all $1 \leq t \leq l$,*

$$\begin{aligned} \left|a'_{tt}\right| - R_t\left(\frac{A}{\beta}\right) &\geq \left|a_{j_t j_t}\right| - R_{j_t}(A) \\ &\quad + \frac{\left|a_{i_d i_d}\right| - P_{i_d}(A)}{\left|a_{i_d i_d}\right|}\sum_{v=1}^{k}\left|a_{j_t i_v}\right| \\ &\geq \left|a_{j_t j_t}\right| - \frac{P_{i_d}(A)}{\left|a_{i_d i_d}\right|}R_{j_t}(A) \\ &> 0, \end{aligned} \tag{10}$$

where

$$h = \max\left\{\max_{i\in N/\{i_d\}} \frac{\left|a_{i i_d}\right|}{\left|a_{ii}\right| - \sum_{j\in N/\{i,i_d\}}\left|a_{ij}\right|}, \frac{\left|a_{i_d i_d}\right|}{R_{i_d}(A)}\right\}, \qquad P_{i_d}(A) = hR_{i_d}(A). \tag{11}$$

Proof. From Lemmas 2 and 3, we know that $A(\beta) \in H_k$ and $\mu(A(\beta)) \in M_k$. Further, by Lemma 1, we have

$$\left[\mu\left(A\left(\beta\right)\right)\right]^{-1} \geq \left|\left[A\left(\beta\right)\right]^{-1}\right|. \tag{12}$$

Thus, for any $1 \leq t \leq l$,

$$\begin{aligned} &\left|a'_{tt}\right| - R_t\left(\frac{A}{\beta}\right) \\ &= \left|a'_{tt}\right| - \sum_{s=1,\neq t}^{l}\left|a'_{ts}\right| \\ &= \left|a_{j_t j_t} - \left(a_{j_t i_1}, \ldots, a_{j_t i_k}\right)\left[A\left(\beta\right)\right]^{-1}\begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right| \\ &\quad - \sum_{s\neq t}^{l}\left|a_{j_t j_s} - \left(a_{j_t i_1}, \ldots, a_{j_t i_k}\right)\left[A\left(\beta\right)\right]^{-1}\begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s}\end{pmatrix}\right| \\ &\geq \left|a_{j_t j_t}\right| - R_{j_t}(A) \\ &\quad + \frac{\left|a_{i_d i_d}\right| - P_{i_d}(A)}{\left|a_{i_d i_d}\right|}\sum_{v=1}^{k}\left|a_{j_t i_v}\right| + \frac{P_{i_d}(A)}{\left|a_{i_d i_d}\right|}\sum_{v=1}^{k}\left|a_{j_t i_v}\right| \\ &\quad - \sum_{s=1}^{l}\left(\left|a_{j_t i_1}\right|, \ldots, \left|a_{j_t i_k}\right|\right)\left[\mu\left(A\left(\beta\right)\right)\right]^{-1}\begin{pmatrix} \left|a_{i_1 j_s}\right| \\ \vdots \\ \left|a_{i_k j_s}\right|\end{pmatrix}. \end{aligned} \tag{13}$$

Further,

$$\begin{aligned}
&\left|a'_{tt}\right| - R_t\left(\frac{A}{\beta}\right) \\
&\geq \left|a_{j_t j_t}\right| - R_{j_t}(A) + \frac{\left|a_{i_d i_d}\right| - P_{i_d}(A)}{\left|a_{i_d i_d}\right|}\sum_{v=1}^{k}\left|a_{j_t i_v}\right| \\
&\quad + \frac{1}{\det\left[\mu\left(A\left(\beta\right)\right)\right]} \\
&\quad \times \det\begin{pmatrix} \frac{P_{i_d}(A)}{\left|a_{i_d i_d}\right|}\sum_{v=1}^{k}\left|a_{j_t i_v}\right| & -\left|a_{j_t i_1}\right| & \cdots & -\left|a_{j_t i_k}\right| \\ -\sum_{s=1}^{l}\left|a_{i_1 j_s}\right| & & & \\ \vdots & & \mu\left(A\left(\beta\right)\right) & \\ -\sum_{s=1}^{l}\left|a_{i_k j_s}\right| & & & \end{pmatrix} \\
&\overset{\text{def.}}{=} \left|a_{j_t j_t}\right| - R_{j_t}(A) + \frac{\left|a_{i_d i_d}\right| - P_{i_d}(A)}{\left|a_{i_d i_d}\right|}\sum_{v=1}^{k}\left|a_{j_t i_v}\right| \\
&\quad + \frac{1}{\det\left[\mu\left(A\left(\beta\right)\right)\right]}\det B.
\end{aligned} \tag{14}$$

By Lemma 4, we can prove that $\det B \geq 0$. Thus, inequality (10) holds. □

Remark 6. Note that

$$\frac{P_{i_d}(A)}{\left|a_{i_d i_d}\right|} \leq \frac{R_{i_d}(A)}{\left|a_{i_d i_d}\right|}. \tag{15}$$

This shows that Lemma 5 improves Theorem 2 of [12].

Theorem 7. *Let $A = (a_{ij}) \in OS_n$, $\beta = \{i_1, i_2, \ldots, i_k\} \subset N$, $\overline{\beta} = N/\beta = \{j_1, j_2, \ldots, j_l\}$, $1 \leq k < n$, and $A/\beta = (a'_{ts})$.*

(a) *If there exists an $i_d \in \beta$ $(1 \leq d \leq k)$ such that $|a_{i_d i_d}| \leq R_{i_d}(A)$, then, for all $1 \leq s, t \leq l$, $t \neq s$,*

$$\begin{aligned}
&\left|a'_{tt}\right|\left|a'_{ss}\right| - R_t\left(\frac{A}{\beta}\right)R_s\left(\frac{A}{\beta}\right) \\
&\quad \geq \left[\left|a_{j_t j_t}\right| - \max_{u\in N/\{j_t\}}\frac{P_u(A)}{\left|a_{uu}\right|}R_{j_t}(A)\right] \\
&\quad \times \left[\left|a_{j_s j_s}\right| - \max_{i_v\in\beta}\frac{P_{i_v}(A)}{\left|a_{i_v i_v}\right|}R_{j_s}(A)\right],
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\left|a'_{tt}\right|\left|a'_{ss}\right| + R_t\left(\frac{A}{\beta}\right)R_s\left(\frac{A}{\beta}\right) \\
&\quad \leq \left[\left|a_{j_t j_t}\right| + \max_{u\in N/\{j_t\}}\frac{P_u(A)}{\left|a_{uu}\right|}R_{j_t}(A)\right] \\
&\quad \times \left[\left|a_{j_s j_s}\right| + \max_{i_v\in\beta}\frac{P_{i_v}(A)}{\left|a_{i_v i_v}\right|}R_{j_s}(A)\right],
\end{aligned} \tag{17}$$

where

$$P_i(A) = h\sum_{j\in N/\{i,i_d\}}\left|a_{ij}\right| + \left|a_{ii_d}\right| \quad (i \neq i_d), \tag{18}$$

and $P_{i_d}(A)$ and h are such as in Lemma 5.

(b) *If $|a_{i_d i_d}| > R_{i_d}(A)$ for any $i_d \in \beta$ $(1 \leq d \leq k)$, then, for all $1 \leq s, t \leq l$, $t \neq s$,*

$$\begin{aligned}
&\left|a'_{tt}\right|\left|a'_{ss}\right| - R_t\left(\frac{A}{\beta}\right)R_s\left(\frac{A}{\beta}\right) \\
&\quad \geq \left[\left|a_{j_t j_t}\right| - \max_{u\in N/\{j_t\}}\frac{R_u(A)}{\left|a_{uu}\right|}R_{j_t}(A)\right] \\
&\quad \times \left[\left|a_{j_s j_s}\right| - \max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_v i_v}\right|}R_{j_s}(A)\right],
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\left|a'_{tt}\right|\left|a'_{ss}\right| + R_t\left(\frac{A}{\beta}\right)R_s\left(\frac{A}{\beta}\right) \\
&\quad \leq \left[\left|a_{j_t j_t}\right| + \max_{u\in N/\{j_t\}}\frac{R_u(A)}{\left|a_{uu}\right|}R_{j_t}(A)\right] \\
&\quad \times \left[\left|a_{j_s j_s}\right| + \max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_v i_v}\right|}R_{j_s}(A)\right],
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
\eta = \max&\left\{\max_{1\leq\omega\leq k}\frac{\sum_{v=1}^{l}\left|a_{i_\omega j_v}\right|}{\left|a_{i_\omega i_\omega}\right| - \sum_{t\neq\omega}^{k}\left|a_{i_\omega i_t}\right|},\right. \\
&\left.\max_{\substack{1\leq\omega\leq k\\1\leq v\leq l}}\frac{\left|a_{i_\omega j_v}\right|}{\sum_{v=1}^{l}\left|a_{i_\omega j_v}\right|}\right\},
\end{aligned} \tag{21}$$

$$Q_{i_\omega}(A) = \eta\sum_{t\neq\omega}^{k}\left|a_{i_\omega i_t}\right| + \sum_{v=1}^{l}\left|a_{i_\omega j_v}\right|, \quad 1 \leq \omega \leq k,$$

and if there exists some $1 \leq \omega \leq k$ such that $\sum_{v=1}^{l}|a_{i_\omega j_v}| = 0$, one denotes $\eta = 1$.

Proof. (a) If there exists an $i_d \in \beta$ such that $|a_{i_d i_d}| \le R_{i_d}(A)$, then, for all $j_t \in \overline{\beta}$,

$$\max_{u\in N/\{j_t\}} \frac{P_u(A)}{|a_{uu}|} = \max_{i_v\in\beta} \frac{P_{i_v}(A)}{|a_{i_v i_v}|} = \frac{P_{i_d}(A)}{|a_{i_d i_d}|}. \tag{22}$$

By Lemma 5, for all $1 \le t \le l$,

$$|a'_{tt}| - R_t\left(\frac{A}{\beta}\right) \ge |a_{j_t j_t}| - \frac{P_{i_d}(A)}{|a_{i_d i_d}|} R_{j_t}(A) > 0. \tag{23}$$

Thus, for all $1 \le t,\ s \le l,\ t \ne s$,

$$\left[|a'_{tt}| - R_t\left(\frac{A}{\beta}\right)\right]\left[|a'_{ss}| - R_s\left(\frac{A}{\beta}\right)\right] > 0. \tag{24}$$

From Lemma 3, A/β is in $\mathrm{OS}_{|\overline{\beta}|}$; that is, for all $1 \le t,\ s \le l,\ t \ne s$,

$$|a'_{tt}||a'_{ss}| - R_t\left(\frac{A}{\beta}\right) R_s\left(\frac{A}{\beta}\right) > 0. \tag{25}$$

Further, for all $1 \le t,\ s \le l,\ t \ne s$,

$$\begin{aligned} |a'_{tt}||a'_{ss}| &- R_t\left(\frac{A}{\beta}\right) R_s\left(\frac{A}{\beta}\right) \\ &\ge \left[|a'_{tt}| - R_t\left(\frac{A}{\beta}\right)\right] \\ &\quad\times \left[|a'_{ss}| - R_s\left(\frac{A}{\beta}\right)\right] \\ &\ge \left[|a_{j_t j_t}| - \max_{u\in N/\{j_t\}} \frac{P_u(A)}{|a_{uu}|} R_{j_t}(A)\right] \\ &\quad\times \left[|a_{j_s j_s}| - \max_{i_v\in\beta} \frac{P_{i_v}(A)}{|a_{i_v i_v}|} R_{j_s}(A)\right]. \end{aligned} \tag{26}$$

Therefore, inequality (16) holds. Similarly, we can prove inequality (17).

(b) If $|a_{i_d i_d}| > R_{i_d}(A)$ for any $i_d \in \beta$ $(1 \le d \le k)$, then, from Lemmas 1 and 2, for all $1 \le t,\ s \le l,\ t \ne s$,

$$\begin{aligned} &|a'_{tt}||a'_{ss}| - R_t\left(\frac{A}{\beta}\right) R_s\left(\frac{A}{\beta}\right) \\ &= \left|a_{j_s j_s} - (a_{j_s i_1}, \dots, a_{j_s i_k}) [A(\beta)]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix}\right| \\ &\quad\times \left|a_{j_t j_t} - (a_{j_t i_1}, \dots, a_{j_t i_k}) [A(\beta)]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix}\right| \\ &\quad - \left[\sum_{v\ne s}^{l} \left|a_{j_s j_v} - (a_{j_s i_1}, \dots, a_{j_s i_k}) \times [A(\beta)]^{-1} \begin{pmatrix} a_{i_1 j_v} \\ \vdots \\ a_{i_k j_v} \end{pmatrix}\right|\right] \\ &\quad\times \left[\sum_{u\ne t}^{l} \left|a_{j_t j_u} - (a_{j_t i_1}, \dots, a_{j_t i_k}) \times [A(\beta)]^{-1} \begin{pmatrix} a_{i_1 j_u} \\ \vdots \\ a_{i_k j_u} \end{pmatrix}\right|\right] \\ &\ge \left[|a_{j_s j_s}| - (|a_{j_s i_1}|, \dots, |a_{j_s i_k}|) \times [\mu(A(\beta))]^{-1} \begin{pmatrix} |a_{i_1 j_s}| \\ \vdots \\ |a_{i_k j_s}| \end{pmatrix}\right] \\ &\quad\times \left[|a_{j_t j_t}| - (|a_{j_t i_1}|, \dots, |a_{j_t i_k}|) \times [\mu(A(\beta))]^{-1} \begin{pmatrix} |a_{i_1 j_t}| \\ \vdots \\ |a_{i_k j_t}| \end{pmatrix}\right] \\ &\quad - \left\{\sum_{v\ne s}^{l} \left[|a_{j_s j_v}| + (|a_{j_s i_1}|, \dots, |a_{j_s i_k}|) \times [\mu(A(\beta))]^{-1} \begin{pmatrix} |a_{i_1 j_v}| \\ \vdots \\ |a_{i_k j_v}| \end{pmatrix}\right]\right\} \\ &\quad\times \left\{\sum_{u\ne t}^{l} \left[|a_{j_t j_u}| + (|a_{j_t i_1}|, \dots, |a_{j_t i_k}|) \times [\mu(A(\beta))]^{-1} \begin{pmatrix} |a_{i_1 j_u}| \\ \vdots \\ |a_{i_k j_u}| \end{pmatrix}\right]\right\} \\ &\overset{\text{def.}}{=} \xi = \left[|a_{j_s j_s}| - (|a_{j_s i_1}|, \dots, |a_{j_s i_k}|) \right. \end{aligned}$$

$$\times \left[\mu\left(A\left(\beta\right)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1 j_s}\right|\\ \vdots\\ \left|a_{i_k j_s}\right|\end{pmatrix}\Bigg]$$
$$\times\Bigg[\left|a_{j_t j_t}\right| - \max_{u\in N/\{j_t\}}\frac{R_u(A)}{\left|a_{uu}\right|}R_{j_t}(A)$$
$$+ \max_{u\in N/\{j_t\}}\frac{R_u(A)}{\left|a_{uu}\right|}R_{j_t}(A)$$
$$-\left(\left|a_{j_t i_1}\right|,\dots,\left|a_{j_t i_k}\right|\right)\left[\mu\left(A\left(\beta\right)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1 j_t}\right|\\ \vdots\\ \left|a_{i_k j_t}\right|\end{pmatrix}\Bigg]$$
$$-\left\{\sum_{v\neq s}^{l}\left[\left|a_{j_s j_v}\right| - \left(\left|a_{j_s i_1}\right|,\dots,\left|a_{j_s i_k}\right|\right)\right.\right.$$
$$\left.\left.\times\left[\mu\left(A\left(\beta\right)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1 j_v}\right|\\ \vdots\\ \left|a_{i_k j_v}\right|\end{pmatrix}\right]\right\}$$
$$\times\left\{\sum_{u\neq t}^{l}\left[\left|a_{j_t j_u}\right| - \left(\left|a_{j_t i_1}\right|,\dots,\left|a_{j_t i_k}\right|\right)\right.\right.$$
$$\left.\left.\times\left[\mu\left(A\left(\beta\right)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1 j_u}\right|\\ \vdots\\ \left|a_{i_k j_u}\right|\end{pmatrix}\right]\right\}. \tag{27}$$

Therefore,

$$\xi = \Bigg[\left|a_{j_s j_s}\right| - \left(\left|a_{j_s i_1}\right|,\dots,\left|a_{j_s i_k}\right|\right)$$
$$\times\left[\mu\left(A\left(\beta\right)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1 j_s}\right|\\ \vdots\\ \left|a_{i_k j_s}\right|\end{pmatrix}\Bigg]$$
$$\times\left[\left|a_{j_t j_t}\right| - \max_{u\in N/\{j_t\}}\frac{R_u(A)}{\left|a_{uu}\right|}R_{j_t}(A)\right]$$
$$+\Bigg[\left|a_{j_s j_s}\right| - \left(\left|a_{j_s i_1}\right|,\dots,\left|a_{j_s i_k}\right|\right)$$
$$\times\left[\mu\left(A\left(\beta\right)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1 j_s}\right|\\ \vdots\\ \left|a_{i_k j_s}\right|\end{pmatrix}\Bigg]$$
$$\times\Bigg[\max_{u\in N/\{j_t\}}\frac{R_u(A)}{\left|a_{uu}\right|}R_{j_t}(A)$$
$$-\left(\left|a_{j_t i_1}\right|,\dots,\left|a_{j_t i_k}\right|\right)\left[\mu\left(A\left(\beta\right)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1 j_t}\right|\\ \vdots\\ \left|a_{i_k j_t}\right|\end{pmatrix}\Bigg]$$
$$-\left\{\sum_{v\neq s}^{l}\left[\left|a_{j_s j_v}\right| - \left(\left|a_{j_s i_1}\right|,\dots,\left|a_{j_s i_k}\right|\right)\right.\right.$$
$$\left.\left.\times\left[\mu\left(A\left(\beta\right)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1 j_v}\right|\\ \vdots\\ \left|a_{i_k j_v}\right|\end{pmatrix}\right]\right\}$$
$$\times\left\{\sum_{u\neq t}^{l}\left[\left|a_{j_t j_u}\right| - \left(\left|a_{j_t i_1}\right|,\dots,\left|a_{j_t i_k}\right|\right)\right.\right.$$
$$\left.\left.\times\left[\mu\left(A\left(\beta\right)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1 j_u}\right|\\ \vdots\\ \left|a_{i_k j_u}\right|\end{pmatrix}\right]\right\}. \tag{28}$$

Further,

$$\left|a_{j_s j_s}\right| - \left(\left|a_{j_s i_1}\right|,\dots,\left|a_{j_t i_k}\right|\right)\left[\mu\left(A\left(\beta\right)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1 j_s}\right|\\ \vdots\\ \left|a_{i_k j_s}\right|\end{pmatrix}$$
$$= \left|a_{j_s j_s}\right| - \max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_v i_v}\right|}\sum_{v=1}^{k}\left|a_{j_s i_v}\right| + \max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_v i_v}\right|}\sum_{v=1}^{k}\left|a_{j_s i_v}\right|$$
$$-\left(\eta\left|a_{j_s i_1}\right|,\dots,\eta\left|a_{j_s i_k}\right|\right)\left[\eta\mu\left(A\left(\beta\right)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1 j_s}\right|\\ \vdots\\ \left|a_{i_k j_s}\right|\end{pmatrix}$$
$$= \left|a_{j_s j_s}\right| - \max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_v i_v}\right|}\sum_{v=1}^{k}\left|a_{j_s i_v}\right| + \frac{1}{\det\left[\eta\mu\left(A\left(\beta\right)\right)\right]}$$
$$\times\det\begin{pmatrix}\max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_v i_v}\right|}\sum_{v=1}^{k}\left|a_{j_s i_v}\right| & -\eta\left|a_{j_s i_1}\right| & \cdots & -\eta\left|a_{j_s i_k}\right|\\ -\left|a_{i_1 j_s}\right| & & & \\ \vdots & & \eta\mu\left(A\left(\beta\right)\right) & \\ -\left|a_{i_k j_s}\right| & & & \end{pmatrix}$$
$$\overset{\text{def.}}{=} \left|a_{j_s j_s}\right| - \max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_v i_v}\right|}\sum_{v=1}^{k}\left|a_{j_s i_v}\right| + \frac{1}{\det\left[\eta\mu\left(A\left(\beta\right)\right)\right]}\det B_1. \tag{29}$$

In B_1, for all $p = 1, 2, 3, \ldots, k$,

$$\begin{aligned}
&\eta \left|a_{i_p i_p}\right| \max_{i_v \in \beta} \frac{Q_{i_v}(A)}{\left|a_{i_v i_v}\right|} \sum_{v=1}^{k} \left|a_{j_s i_v}\right| \\
&\geq \eta \left|a_{i_p i_p}\right| \frac{Q_{i_p}(A)}{\left|a_{i_p i_p}\right|} \sum_{v=1}^{k} \left|a_{j_s i_v}\right| \\
&= \eta Q_{i_p}(A) \sum_{v=1}^{k} \left|a_{j_s i_v}\right| \qquad (30)\\
&= \eta \left(\eta \sum_{v \neq p}^{k} \left|a_{i_p i_v}\right| + \sum_{v=1}^{l} \left|a_{i_p j_v}\right| \right) \sum_{v=1}^{k} \left|a_{j_s i_v}\right| \\
&\geq \left(\eta \sum_{v \neq p}^{k} \left|a_{i_p i_v}\right| + \left|a_{i_p j_s}\right| \right) \sum_{v=1}^{k} \left|\eta a_{j_s i_v}\right|.
\end{aligned}$$

And for all $p, q = 1, 2, 3, \ldots, k$, $p \neq q$,

$$\begin{aligned}
&\eta \left|a_{i_p i_p}\right| \eta \left|a_{i_q i_q}\right| > \eta R_{i_p}(A) \eta R_{i_q}(A) \\
&= \left(\eta \sum_{v \neq p}^{k} \left|a_{i_p i_v}\right| + \eta \sum_{v=1}^{l} \left|a_{i_p j_v}\right| \right) \left(\eta \sum_{v \neq q}^{k} \left|a_{i_q i_v}\right| + \eta \sum_{v=1}^{l} \left|a_{i_q j_v}\right| \right) \\
&\geq \left(\eta \sum_{v \neq p}^{k} \left|a_{i_p i_v}\right| + \left|a_{i_p j_s}\right| \right) \left(\eta \sum_{v \neq q}^{k} \left|a_{i_q i_v}\right| + \left|a_{i_q j_s}\right| \right).
\end{aligned} \qquad (31)$$

Hence, by (30) and (31), we have $B_1 \in \mathrm{GOS}_{k+1}$ and so $\det B_1 \geq 0$. Further, by (29), we obtain

$$\begin{aligned}
&\left|a_{j_s j_s}\right| - \left(\left|a_{j_s i_1}\right|, \ldots, \left|a_{j_t i_k}\right|\right) [\mu(A(\beta))]^{-1} \begin{pmatrix} \left|a_{i_1 j_s}\right| \\ \vdots \\ \left|a_{i_k j_s}\right| \end{pmatrix} \\
&\geq \left|a_{j_s j_s}\right| - \max_{i_v \in \beta} \frac{Q_{i_v}(A)}{\left|a_{i_v i_v}\right|} \sum_{v=1}^{k} \left|a_{j_s i_v}\right| \\
&\geq \left|a_{j_s j_s}\right| - \max_{i_v \in \beta} \frac{Q_{i_v}(A)}{\left|a_{i_v i_v}\right|} R_{j_s}(A).
\end{aligned} \qquad (32)$$

By (28) and a similar method as the proof of Theorem 2.1 in [2], we can prove $\xi > 0$. Therefore, by (29) and (32), we obtain inequality (19). Similarly, we can prove inequality (20). □

Remark 8. Note that

$$0 \leq h, \qquad \eta \leq 1. \qquad (33)$$

This shows that Theorem 7 improves Theorem 2.1 of [2].

3. Eigenvalue Inclusion Regions of the Schur Complement

In this section, we present new inclusion regions for eigenvalues of the Schur complement of OS_n.

Lemma 9 (*Brauer Ovals theorem*). *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. Then the eigenvalues of A are in the union of the following sets:*

$$U_{ij} = \left\{ z \in C \,\|z - a_{ii}\|\, z - a_{jj} \,|\leq R_i(A) R_j(A) \right\}, \qquad \forall i, j = N, \quad i \neq j. \qquad (34)$$

Theorem 10. *Let $A = (a_{ij}) \in \mathrm{OS}_n$, $\beta = \{i_1, i_2, \ldots, i_k\} \subset N$, $\overline{\beta} = N/\beta = \{j_1, j_2, \ldots, j_l\}$, $1 \leq k < n$, and $A/\beta = (a'_{ts})$, and let λ be eigenvalue of A/β.*

(a) *If there exists an $i_d \in \beta$ $(1 \leq d \leq k)$ such that $|a_{i_d i_d}| \leq R_{i_d}(A)$, then there exist $1 \leq t,\ s \leq l$, $t \neq s$, such that*

$$\begin{aligned}
&\left|\lambda - \frac{\det(A_t)}{\det A(\beta)}\right| \left|\lambda - \frac{\det(A_s)}{\det A(\beta)}\right| \\
&\leq 2 \left[\left|a_{j_s j_s}\right| \max_{u \in N/\{j_t\}} \frac{P_u(A)}{\left|a_{uu}\right|} R_{j_t}(A) + \left|a_{j_t j_t}\right| \max_{i_v \in \beta} \frac{P_{i_v}(A)}{\left|a_{i_v i_v}\right|} R_{j_s}(A) \right],
\end{aligned} \qquad (35)$$

$$\begin{aligned}
&\left|\lambda - \frac{\det(A_t)}{\det A(\beta)}\right| \left|\lambda - \frac{\det(A_s)}{\det A(\beta)}\right| \\
&\leq \left[\left|a_{j_t j_t}\right| + \max_{u \in N/\{j_t\}} \frac{P_u(A)}{\left|a_{uu}\right|} R_{j_t}(A) \right] \times \left[\left|a_{j_s j_s}\right| + \max_{i_v \in \beta} \frac{P_{i_v}(A)}{\left|a_{i_v i_v}\right|} R_{j_s}(A) \right],
\end{aligned} \qquad (36)$$

where $P_{i_d}(A)$ is such as in Lemma 5 and $P_{i_v}(A)$ $(v \neq d)$ is such as in Theorem 7.

(b) *If $|a_{i_d i_d}| > R_{i_d}(A)$ for any $i_d \in \beta$ $(1 \leq d \leq k)$, then there exist $1 \leq t,\ s \leq l$, $t \neq s$, such that*

$$\begin{aligned}
&\left|\lambda - \frac{\det(A_t)}{\det A(\beta)}\right| \left|\lambda - \frac{\det(A_s)}{\det A(\beta)}\right| \\
&\leq 2 \left[\left|a_{j_s j_s}\right| \max_{u \in N/\{j_t\}} \frac{R_u(A)}{\left|a_{uu}\right|} R_{j_t}(A) + \left|a_{j_t j_t}\right| \max_{i_v \in \beta} \frac{Q_{i_v}(A)}{\left|a_{i_v i_v}\right|} R_{j_s}(A) \right],
\end{aligned} \qquad (37)$$

$$\begin{aligned}
&\left|\lambda - \frac{\det(A_t)}{\det A(\beta)}\right| \left|\lambda - \frac{\det(A_s)}{\det A(\beta)}\right| \\
&\leq \left[\left|a_{j_t j_t}\right| + \max_{u \in N/\{j_t\}} \frac{R_u(A)}{\left|a_{uu}\right|} R_{j_t}(A) \right] \times \left[\left|a_{j_s j_s}\right| + \max_{i_v \in \beta} \frac{Q_{i_v}(A)}{\left|a_{i_v i_v}\right|} R_{j_s}(A) \right],
\end{aligned} \qquad (38)$$

where $Q_{i_v}(A)$ is such as in Theorem 7.

Proof. By Lemma 9, we know that there exist $1 \le t, s \le l$, $t \neq s$, such that

$$\left|\lambda - a'_{tt}\right|\left|\lambda - a'_{ss}\right| \le R_t\left(\frac{A}{\beta}\right) R_s\left(\frac{A}{\beta}\right). \quad (39)$$

(a) If there exists $i_d \in \beta$ satisfying $|a_{i_d i_d}| \le R_{i_d}(A)$, by (16), we have

$$\begin{aligned}
&R_t\left(\frac{A}{\beta}\right) R_s\left(\frac{A}{\beta}\right) \\
&\le \left|a'_{tt}\right|\left|a'_{ss}\right| - \left[\left|a_{j_t j_t}\right| - \max_{u \in N/\{j_t\}} \frac{P_u(A)}{|a_{uu}|} R_{j_t}(A)\right] \\
&\quad \times \left[\left|a_{j_s j_s}\right| - \max_{i_v \in \beta} \frac{P_{i_v}(A)}{|a_{i_v i_v}|} R_{j_s}(A)\right] \\
&= \left|a_{j_s j_s} - \left(a_{j_s i_1}, \ldots, a_{j_s i_k}\right)[A(\beta)]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix}\right| \\
&\quad \times \left|a_{j_t j_t} - \left(a_{j_t i_1}, \ldots, a_{j_t i_k}\right)[A(\beta)]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix}\right| \\
&\quad - \left[\left|a_{j_t j_t}\right| - \max_{u \in N/\{j_t\}} \frac{P_u(A)}{|a_{uu}|} R_{j_t}(A)\right] \\
&\quad \times \left[\left|a_{j_s j_s}\right| - \max_{i_v \in \beta} \frac{P_{i_v}(A)}{|a_{i_v i_v}|} R_{j_s}(A)\right] \\
&\le \left[\left|a_{j_t j_t}\right| + \max_{u \in N/\{j_t\}} \frac{P_u(A)}{|a_{uu}|} R_{j_t}(A)\right] \\
&\quad \times \left[\left|a_{j_s j_s}\right| + \max_{i_v \in \beta} \frac{P_{i_v}(A)}{|a_{i_v i_v}|} R_{j_s}(A)\right] \\
&\quad - \left[\left|a_{j_t j_t}\right| - \max_{u \in N/\{j_t\}} \frac{P_u(A)}{|a_{uu}|} R_{j_t}(A)\right] \\
&\quad \times \left[\left|a_{j_s j_s}\right| - \max_{i_v \in \beta} \frac{P_{i_v}(A)}{|a_{i_v i_v}|} R_{j_s}(A)\right] \\
&= 2\left[\left|a_{j_s j_s}\right| \max_{u \in N/\{j_t\}} \frac{P_u(A)}{|a_{uu}|} R_{j_t}(A) \right. \\
&\quad \left. + \left|a_{j_t j_t}\right| \max_{i_v \in \beta} \frac{P_{i_v}(A)}{|a_{i_v i_v}|} R_{j_s}(A)\right].
\end{aligned} \quad (40)$$

On the other hand, for all $1 \le t \le l$,

$$\begin{aligned}
&\left|\lambda - a'_{tt}\right| \\
&= \left|\lambda - a_{j_t j_t} + \left(a_{j_t i_1}, \ldots, a_{j_t i_k}\right)[A(\beta)]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix}\right| \\
&= \left|\lambda - \det\left(\frac{A_t}{A(\beta)}\right)\right| = \left|\lambda - \frac{\det(A_t)}{\det A(\beta)}\right|.
\end{aligned} \quad (41)$$

Therefore, by (39), (40), and (41), we obtain inequality (35). With a Similar method, we can prove inequality (36).

(b) If $|a_{i_d i_d}| > R_{i_d}(A)$ for any $i_d \in \beta$ $(1 \le d \le k)$, then by (19), (32), and a similar method as the part (a), we obtain inequality (37). Similarly, we can prove inequality (38). □

4. Upper Bound for the Infinity Norm on the Inverse of the Schur Complement

In this section, we present a new upper bound of $\|(A/\beta)^{-1}\|_\infty$.

Lemma 11 (see [2]). *Let $A = (a_{ij}) \in OS_n$ and $M = (m_{ij}) \in \mathbb{C}^{n\times n}$. Then,*

$$\left\|A^{-1}M\right\|_\infty \le \max_{\substack{1\le i,j\le n \\ i\neq j}} \frac{\left|a_{jj}\right| \sum_{v=1}^{n} \left|m_{iv}\right| + R_i(A) \sum_{v=1}^{n} \left|m_{jv}\right|}{\left|a_{ii}\right|\left|a_{jj}\right| - R_i(A) R_j(A)}. \quad (42)$$

Theorem 12. *Let $A = (a_{ij}) \in SD_n$, $M = (m_{ij}) \in \mathbb{C}^{l\times l}$, $\beta = \{i_1, i_2, \ldots, i_k\} \subset N$, $\overline{\beta} = N/\beta = \{j_1, j_2, \ldots, j_l\}$, $1 \le k < n$, and $A/\beta = (a'_{ts})$. Then,*

$$\begin{aligned}
&\left\|\left(\frac{A}{\beta}\right)^{-1} M\right\|_\infty \\
&\le \max_{\substack{1\le t,s\le l \\ t\neq s}} \left(\Delta_{j_t j_s} \right. \\
&\qquad \times \left(\left(\left|a_{j_t j_t}\right| - \max_{u \in N/\{j_t\}} \frac{R_u(A)}{|a_{uu}|} R_{j_t}(A)\right)\right. \\
&\qquad \left.\left. \times \left(\left|a_{j_s j_s}\right| - \max_{i_v \in \beta} \frac{Q_{i_v}(A)}{|a_{i_v i_v}|} R_{j_s}(A)\right)\right)^{-1}\right),
\end{aligned} \quad (43)$$

$$\left\|\left(\frac{A}{\beta}\right)^{-1}\right\|_{\infty} \le \max_{\substack{1\le t,s\le l\\ t\ne s}}\left(\left(\left(\left|a_{j_tj_t}\right| + R_{j_s}(A) + \max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_vi_v}\right|}\left(R_{j_t}(A)+R_{j_s}(A)\right)\right)\right.\right.$$
$$\times\left(\left(\left|a_{j_tj_t}\right| - \max_{u\in N/\{j_t\}}\frac{R_u(A)}{\left|a_{uu}\right|}R_{j_t}(A)\right)\right.$$
$$\left.\left.\times\left(\left|a_{j_sj_s}\right| - \max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_vi_v}\right|}R_{j_s}(A)\right)\right)^{-1}\right), \quad (44)$$

where

$$\Delta_{j_tj_s} = \left(\left|a_{j_tj_t}\right| + \max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_vi_v}\right|}R_{j_t}(A)\right)\sum_{v=1}^{l}\left|m_{sv}\right| + \left(R_{j_s}(A) + \max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_vi_v}\right|}R_{j_s}(A)\right)\sum_{v=1}^{l}\left|m_{tv}\right|, \quad (45)$$

and $Q_{i_v}(A)$ *is such as in Theorem 7.*

Proof. By Lemma 11, we have

$$\left\|\left(\frac{A}{\beta}\right)^{-1}M\right\|_{\infty} \le \max_{\substack{1\le t,s\le l\\ t\ne s}}\frac{\left|a'_{tt}\right|\sum_{v=1}^{l}\left|m_{sv}\right| + R_s(A/\beta)\sum_{v=1}^{l}\left|m_{tv}\right|}{\left|a'_{tt}\right|\left|a'_{ss}\right| - R_t(A/\beta)R_s(A/\beta)}. \quad (46)$$

Similar to (29), we obtain

$$\left|a'_{tt}\right| = \left|a_{j_tj_t} - \left(a_{j_ti_1},\dots,a_{j_ti_k}\right)\left[A(\beta)\right]^{-1}\begin{pmatrix}a_{i_1j_t}\\ \vdots\\ a_{i_kj_t}\end{pmatrix}\right|$$
$$\le \left|a_{j_tj_t}\right| + \left(\left|a_{j_ti_1}\right|,\dots,\left|a_{j_ti_k}\right|\right)\left[\mu\left(A(\beta)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1j_t}\right|\\ \vdots\\ \left|a_{i_kj_t}\right|\end{pmatrix}$$
$$\le \left|a_{j_tj_t}\right| + \max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_vi_v}\right|}R_{j_t}(A). \quad (47)$$

Thus, by Theorem 1 of [12], we have

$$R_s\left(\frac{A}{\beta}\right) \le \left|a'_{ss}\right| - \left|a_{j_sj_s}\right| + R_{j_s}(A) \le \max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_vi_v}\right|}R_{j_s}(A) + R_{j_s}(A). \quad (48)$$

Since $A\in \mathrm{SD}_n$, then $A\in \mathrm{OS}_n$. Thus, by Theorem 7, we have

$$\left|a'_{tt}\right|\left|a'_{ss}\right| - R_t\left(\frac{A}{\beta}\right)R_s\left(\frac{A}{\beta}\right) \ge \left[\left|a_{j_tj_t}\right| - \max_{u\in N/\{j_t\}}\frac{R_u(A)}{\left|a_{uu}\right|}R_{j_t}(A)\right] \times\left[\left|a_{j_sj_s}\right| - \max_{i_v\in\beta}\frac{Q_{i_v}(A)}{\left|a_{i_vi_v}\right|}R_{j_s}(A)\right]. \quad (49)$$

Further, by (46), (47), (48), and (49), we obtain inequality (43).

Let $M = I = \mathrm{diag}(1,1,\dots,1)$; we can prove inequality (44). □

5. Numerical Examples

In this section, we present several numerical examples to illustrate the theory results.

Example 1 (see Example 2 in [2]). Let

$$A = \begin{pmatrix} 1.3 & 0.2 & 0.3 & 0.4 & 0.5\\ 0.2 & 2 & 0.4 & 0.5 & 0.1\\ 0.3 & 0.4 & 2 & 0.1 & 0.2\\ 0.4 & 0.5 & 0.1 & 3 & 0.3\\ 0.5 & 0.1 & 0.2 & 0.3 & 3\end{pmatrix}, \quad \beta = \{1,2\}. \quad (50)$$

By Theorem 10, the eigenvalues of A/β are in the set

$$\Gamma_1 = \{\lambda \mid |\lambda - 1.87||\lambda - 2.78| \le 11.20\} \cup \{\lambda \mid |\lambda - 1.87||\lambda - 2.81| \le 10.40\} \cup \{\lambda \mid |\lambda - 2.78||\lambda - 2.81| \le 14.40\}. \quad (51)$$

From Theorem 3.1 of [2], the eigenvalues of A/β are in the set

$$\Gamma'_1 = \{\lambda \mid |\lambda - 1.87||\lambda - 2.78| \le 12.06\} \cup \{\lambda \mid |\lambda - 1.87||\lambda - 2.81| \le 11.20\} \cup \{\lambda \mid |\lambda - 2.78||\lambda - 2.81| \le 15.51\}. \quad (52)$$

Evidently, $\Gamma_1 \subset \Gamma'_1$, and we use Figure 1 to show this fact. And the eigenvalues of A/β are denoted by "+" in Figure 1.

Example 2. Let

$$A = \begin{pmatrix} 1.6 & 0.1 & 0.5 & 0.2 & 0.2\\ 0.3 & 1.5 & 0.2 & 0.2 & 0.1\\ 0.2 & 0.2 & 1.8 & 0.3 & 0.4\\ 0.5 & 0.3 & 0.5 & 1.0 & 0.2\\ 0.5 & 0.2 & 0.2 & 0.3 & 1.9\end{pmatrix}, \quad \beta = \{2,4\}. \quad (53)$$

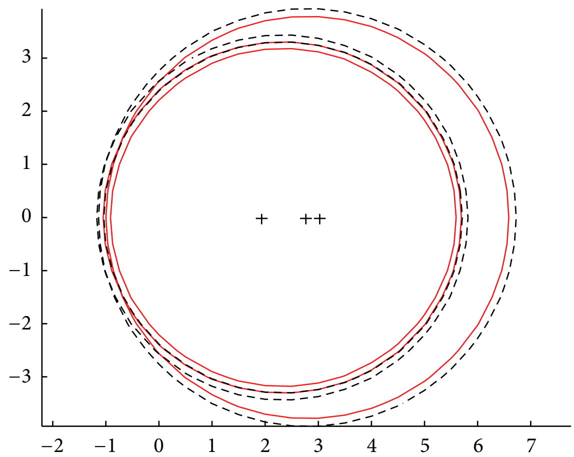

FIGURE 1: The red dotted line and black dashed line denote the corresponding discs Γ_1 and Γ_1', respectively.

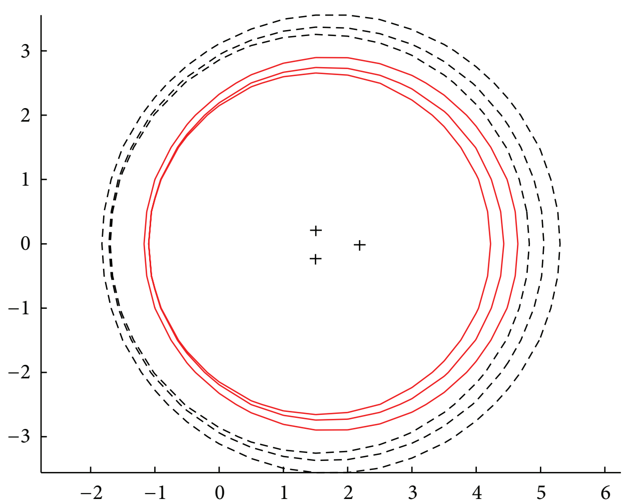

FIGURE 2: The red dotted line and black dashed line denote the corresponding discs Γ_2 and Γ_2', respectively.

By Theorem 10, the eigenvalues of A/β are in the set

$$\begin{aligned}\Gamma_2 &= \{\lambda \,\|\lambda - 1.49\|\, \lambda - 1.64 \,| \le 7.12\} \\ &\quad \cup \{\lambda \,\|\lambda - 1.49\|\, \lambda - 1.84 \,| \le 7.64\} \\ &\quad \cup \{\lambda \,\|\lambda - 1.64\|\, \lambda - 1.84 \,| \le 8.50\}.\end{aligned} \tag{54}$$

From Theorem 3.1 of [2], the eigenvalues of A/β are in the set

$$\begin{aligned}\Gamma_2' &= \{\lambda \,\|\lambda - 1.49\|\, \lambda - 1.64 \,| \le 10.68\} \\ &\quad \cup \{\lambda \,\|\lambda - 1.49\|\, \lambda - 1.84 \,| \le 11.46\} \\ &\quad \cup \{\lambda \,\|\lambda - 1.64\|\, \lambda - 1.84 \,| \le 12.75\}.\end{aligned} \tag{55}$$

Evidently, $\Gamma_2 \subset \Gamma_2'$, and we use Figure 2 to show this fact. And the eigenvalues of A/β are denoted by "+" in Figure 2.

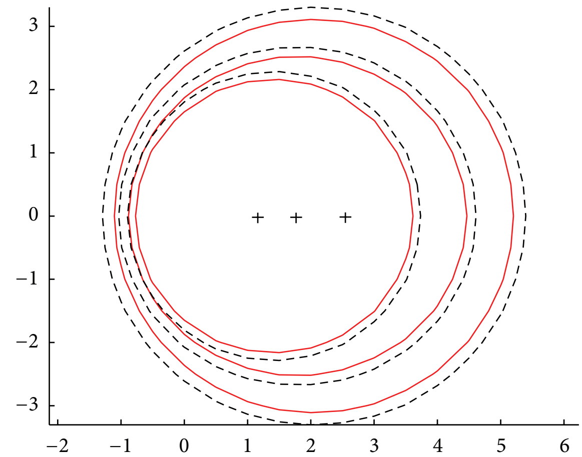

FIGURE 3: The red dotted line and black dashed line denote the corresponding discs Γ_3 and Γ_3', respectively.

Example 3. Let

$$A = \begin{pmatrix} 1.5 & 0.2 & 0.3 & 0.2 & 0.1 & 0.2 \\ 0.2 & 1.2 & 0.1 & 0.3 & 0.1 & 0.4 \\ 0.6 & 0.2 & 1.6 & 0.1 & 0.2 & 0.1 \\ 0.5 & 0.2 & 0.1 & 1.8 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 & 0.3 & 1.3 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.3 & 1.2 & 2.5 \end{pmatrix}, \qquad \beta = \{1, 3, 5\}. \tag{56}$$

By Theorem 10, the eigenvalues of A/β are in the set

$$\begin{aligned}\Gamma_3 &= \{\lambda \,\|\lambda - 1.16\|\, \lambda - 1.68 \,| \le 4.79\} \\ &\quad \cup \{\lambda \,\|\lambda - 1.16\|\, \lambda - 2.42 \,| \le 6.78\} \\ &\quad \cup \{\lambda \,\|\lambda - 1.68\|\, \lambda - 2.42 \,| \le 9.85\}.\end{aligned} \tag{57}$$

From Theorem 3.1 of [2], the eigenvalues of A/β are in the set

$$\begin{aligned}\Gamma_3' &= \{\lambda \,\|\lambda - 1.16\|\, \lambda - 1.68 \,| \le 5.35\} \\ &\quad \cup \{\lambda \,\|\lambda - 1.16\|\, \lambda - 2.42 \,| \le 7.60\} \\ &\quad \cup \{\lambda \,\|\lambda - 1.68\|\, \lambda - 2.42 \,| \le 11.09\}.\end{aligned} \tag{58}$$

Evidently, $\Gamma_3 \subset \Gamma_3'$, and we use Figure 3 to show this fact. And the eigenvalues of A/β are denoted by "+" in Figure 3.

Meanwhile, by Theorem 12,

$$\left\| \left(\frac{A}{\beta} \right)^{-1} \right\|_\infty \le 12.22. \tag{59}$$

From Theorem 4.2 of [2],

$$\left\| \left(\frac{A}{\beta} \right)^{-1} \right\|_\infty \le 20.60. \tag{60}$$

Remark 13. Numerical examples show that the new eigenvalue inclusion set is tighter than that in Theorem 3.1 of [2] and the new upper bound of $\|(A/\beta)^{-1}\|_\infty$ is sharper than that in Theorem 4.2 of [2].

Acknowledgments

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References

[1] L. Cvetković, V. Kostić, and S. Rauški, "A new subclass of H-matrices," *Applied Mathematics and Computation*, vol. 208, no. 1, pp. 206–210, 2009.

[2] J. Liu, J. Zhang, and Y. Liu, "The Schur complement of strictly doubly diagonally dominant matrices and its application," *Linear Algebra and its Applications*, vol. 437, no. 1, pp. 168–183, 2012.

[3] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, New York, NY, USA, 1991.

[4] D. Carlson and T. L. Markham, "Schur complements of diagonally dominant matrices," *Czechoslovak Mathematical Journal*, vol. 29, no. 2, pp. 246–251, 1979.

[5] K. D. Ikramov, "Invariance of the Brauer diagonal dominance in gaussian elimination," *Moscow University Computational Mathematics and Cybernetics*, vol. 2, pp. 91–94, 1989.

[6] B. Li and M. J. Tsatsomeros, "Doubly diagonally dominant matrices," *Linear Algebra and its Applications*, vol. 261, pp. 221–235, 1997.

[7] J. Liu, J. Li, Z. Huang, and X. Kong, "Some properties of Schur complements and diagonal-Schur complements of diagonally dominant matrices," *Linear Algebra and its Applications*, vol. 428, no. 4, pp. 1009–1030, 2008.

[8] J. Liu, Y. Huang, and F. Zhang, "The Schur complements of generalized doubly diagonally dominant matrices," *Linear Algebra and its Applications*, vol. 378, pp. 231–244, 2004.

[9] J. Liu and Y. Huang, "Some properties on Schur complements of H-matrices and diagonally dominant matrices," *Linear Algebra and its Applications*, vol. 389, pp. 365–380, 2004.

[10] J. Liu and Z. Huang, "The Schur complements of γ-diagonally and product γ-diagonally dominant matrix and their disc separation," *Linear Algebra and its Applications*, vol. 432, no. 4, pp. 1090–1104, 2010.

[11] J. Liu, Z. Huang, and J. Zhang, "The dominant degree and disc theorem for the Schur complement of matrix," *Applied Mathematics and Computation*, vol. 215, no. 12, pp. 4055–4066, 2010.

[12] J. Liu and F. Zhang, "Disc separation of the Schur complement of diagonally dominant matrices and determinantal bounds," *SIAM Journal on Matrix Analysis and Applications*, vol. 27, no. 3, pp. 665–674, 2005.

[13] Y.-T. Li, S.-P. Ouyang, S.-J. Cao, and R.-W. Wang, "On diagonal-Schur complements of block diagonally dominant matrices," *Applied Mathematics and Computation*, vol. 216, no. 5, pp. 1383–1392, 2010.

[14] R. L. Smith, "Some interlacing properties of the Schur complement of a Hermitian matrix," *Linear Algebra and its Applications*, vol. 177, pp. 137–144, 1992.

[15] F. Z. Zhang, *The Schur Complement and Its Applications*, Springer, New York, NY, USA, 2005.

[16] J. W. Demmel, *Applied Numerical Linear Algebra*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.

[17] G. H. Golub and C. F. van Loan, *Matrix Computationss*, vol. 18, Johns Hopkins University Press, Baltimore, Md, USA, 3rd edition, 1996.

[18] R. Kress, *Numerical analysis*, vol. 181 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1998.

[19] S. Xiang and S. Zhang, "A convergence analysis of block accelerated over-relaxation iterative methods for weak block H-matrices to partition π," *Linear Algebra and its Applications*, vol. 418, no. 1, pp. 20–32, 2006.

[20] N. Morača, "Upper bounds for the infinity norm of the inverse of SDD and S-SDD matrices," *Journal of Computational and Applied Mathematics*, vol. 206, no. 2, pp. 666–678, 2007.