Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2015, Article ID 139828, 6 pages
http://dx.doi.org/10.1155/2015/139828

Research Article

A New Comparison Principle for Impulsive Functional Differential Equations

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Received 19 March 2015; Accepted 27 May 2015

Academic Editor: Carlo Bianca

We establish a new comparison principle for impulsive differential systems with time delay. Then, using this comparison principle, we obtain some sufficient conditions for several stabilities of impulsive delay differential equations. Finally, we present an example to show the effectiveness of our results.

1. Introduction

The impulsive functional differential systems provide very important mathematical models for many real phenomena and processes in the field of natural sciences and technology [1–3]. In the last few decades, the stability theory of impulsive differential equations had a rapid development; for instance, see [1–16]. In those works, most researchers utilized Lyapunov functions or Lyapunov functionals coupled with a certain Razumikhin technique. It is well known that comparison principles play an important role in the theory for differential systems, which always reduce the studies from a given complicated differential system to some relatively simpler differential system. Up to now, there exist many results on this subject; see [2, 3, 5–16]. For example, Lakshmikantham et al. [3] presented a comparison principle for impulsive differential systems and applied it to the stability. Significant progress has been made in the theory of impulsive functional differential equations in recent years (see [17, 18]). It is well known that the monotone iterative technique offers an approach for obtaining approximate solutions of nonlinear differential equations. Some recent advances in the field of approximate solutions of nonlinear differential equations can be found in [19–21]. Afterwards, some researchers gave several new comparison principles in the qualitative analysis for the solutions of impulsive systems; see [5, 9, 10]. In particular, there has been a significant development in the studies of comparison principles for delay systems; see [6–8]. At the same time, the comparison principles for differential systems with impulses and delays simultaneously have attracted many researchers, and many interesting results on this subject are obtained; see [9–11].

In view of the importance of comparison principles in the qualitative analysis for differential equations, in this paper we establish a new comparison principle for impulsive delay differential systems. As an application, we use it to deal with the stability of impulsive functional differential equations.

The rest of this paper is organized as follows. In Section 2, we introduce some useful notations and definitions. In Section 3, a new comparison principle and its applications to stability are given. Finally, we give an example to illustrate our results in Section 4.

2. Preliminaries

Let R denote the set of real numbers, R_+ the set of nonnegative real numbers, Z_+ the set of positive integers, and R^n the n-dimensional real Euclidean space equipped with the norm $|\cdot|$.

Consider the following impulsive functional differential equations:

$$\begin{aligned} x'(t) &= f(t, x_t), \quad t \neq t_k, \\ x(t_k) &= x(t_k^-) + I_k(t_k, x(t_k^-)), \quad k \in Z_+, \\ x_{t_0} &= \varphi(s), \quad -\tau \le s \le 0, \end{aligned} \tag{1}$$

where $x \in R^n$ and x' denotes the right-hand derivative of x. The impulse times $\{t_k\}$ satisfy $0 \le t_0 < t_1 < \cdots < t_k < \cdots$ and $\lim_{k\to+\infty} t_k = +\infty$. Also, assume $f \in C([t_{k-1}, t_k) \times \Omega, R^n)$; meanwhile $\varphi \in \Omega$, where Ω is an open set in $PC([-\tau, 0], R^n)$, where $PC([-\tau, 0], R^n) = \{\psi : [-\tau, 0] \to R^n \mid \psi$ is continuous except at a finite number of points t, at which $\psi(t^+)$ and $\psi(t^-)$ exist and $\psi(t^+) = \psi(t)\}$. For $\psi \in \Omega$, the norm of ψ is defined by $\|\psi\| = \sup_{-\tau\le\theta\le0} |\psi(\theta)|$. For each $t \ge t_0$, $x_t \in \Omega$ is defined by $x_t(s) = x(t + s)$, $s \in [-\tau, 0]$. For each $k \in Z_+$, $I_k(t, x) \in C([t_0, \infty) \times R^n, R^n)$. For any $\rho > 0$, there exists a $\rho_1 > 0$ ($0 < \rho_1 < \rho$) such that $x \in S(\rho_1)$ implies that $x + I_k(t_k, x) \in S(\rho)$, where $S(\rho) = \{x : |x| < \rho, x \in R^n\}$.

Define $PCB(t) = \{x_t \in \Omega : x_t$ is bounded$\}$. For any $t_0 > 0$, let $PCB_\delta(t_0) = \{\psi \in PCB(t_0) : \|\psi\| < \delta\}$.

In this paper, we suppose that there exists a unique solution of system (1) through each (t_0, φ). Furthermore, we assume that $f(t, 0) = 0$, and $I_k(t, 0) = 0$, $k \in Z_+$, so that $x(t) = 0$ is a solution of system (1), which is called the trivial solution.

We now give some useful notations and definitions that will be used in the sequel.

Definition 1. A function $V : [t_0 - \tau, \infty) \times \Omega \to R_+$ belongs to class $\mathscr{V}_0$, if

(i) V is continuous on each set $[t_{k-1}, t_k) \times \Omega$ and $\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$ exists,

(ii) $V(t, x)$ is locally Lipschitzian in x and $V(t, 0) \equiv 0$.

Definition 2. Let $V \in \mathscr{V}_0$, for any $(t, \psi) \in [t_{k-1}, t_k) \times \Omega$; the upper right-hand Dini derivative of $V(t, x)$ along a solution of system (1) is defined by

$$D^+V(t, \psi(0)) = \limsup_{h\to0^+} \frac{1}{h} \cdot \{V(t + h, \psi(0) + hf(t, \psi)) - V(t, \psi(0))\}. \tag{2}$$

Definition 3. Assume $x(t) = x(t, t_0, \varphi)$ is the solution of system (1) through (t_0, φ). Then the trivial solution of (1) is said to be

(H_1) stable, if, for any $t_0 \in R_+$ and $\varepsilon > 0$, there exists some $\delta = \delta(\varepsilon, t_0) > 0$ such that $\varphi \in PCB_\delta(t_0)$ implies $|x(t, t_0, \varphi)| < \varepsilon$, $t \ge t_0$;

(H_2) attractive, if, for any $t_0 \in R_+$ and $\varepsilon > 0$, there exist some $\delta = \delta(\varepsilon, t_0) > 0$, $T = T(t_0, \varepsilon) > 0$ such that $\varphi \in PCB_\delta(t_0)$ implies $|x(t, t_0, \varphi)| < \varepsilon$, $t \ge t_0 + T$;

(H_3) asymptotically stable if (H_1) and (H_2) simultaneously hold;

(H_4) exponentially stable; assume $\lambda > 0$ is a constant, if, for any $t_0 \in R_+$ and $\varepsilon > 0$, there exists some $\delta = \delta(\varepsilon) > 0$ such that $\varphi \in PCB_\delta(t_0)$ implies $|x(t, t_0, \varphi)| < \varepsilon \cdot e^{-\lambda(t-t_0)}$, $t \ge t_0$.

In the proof of our main results we will use the following lemma.

Lemma 4 (see [6]). *Let $g_0, g \in C[R_+ \times R_+, R]$ satisfy*

$$g_0(t, u) \le g(t, u), \quad (t, u) \in R_+ \times R_+. \tag{3}$$

Then, the right maximal solution $\gamma(t, t_0, u_0)$ of

$$u'(t) = g(t, u), \quad u(t_0) = u_0 \ge 0 \tag{4}$$

and the left maximal solution $\eta(t, T, v_0)$ of

$$u'(t) = g_0(t, u), \quad u(T) = v_0 \ge 0 \tag{5}$$

satisfy the relation

$$\gamma(t, t_0, u_0) \le \eta(t, T, v_0), \quad t \in [t_0, T], \tag{6}$$

whenever $\gamma(T, t_0, u_0) \le v_0$.

3. Comparison Results and Applications

In this section, we will establish a general comparison principle for the impulsive delay differential system (1), by comparing it with a scalar impulsive differential system. Then, applying the comparison principle, we obtain some stability criteria. First of all, we present the following comparison principle.

Lemma 5. *Assume that $g_0, g \in C[R_+ \times R_+, R]$ satisfy $g_0(t, u) \le g(t, u)$; $(t, u) \in R_+ \times R_+$, $\gamma(t, t_0, u_0)$ is the right maximal solution of*

$$u'(t) = g(t, u), \quad t \ne t_k, \qquad u(t_k) = \psi_k(u(t_k^-)), \quad k = 1, 2, \ldots, \qquad u(t_0) = u_0 \ge 0, \tag{7}$$

where $0 \le t_0 \le t_1 \le \cdots$, $\lim_{k\to\infty} t_k = \infty$, $\psi_k(s) \ge s$, $\psi_k : R_+ \to R$ is nondecreasing, and $\eta(t, T, v_0)$ is the left maximal solution of

$$u'(t) = g_0(t, u), \quad u(T) = v_0 \ge 0. \tag{8}$$

Then

$$\gamma(t, t_0, u_0) \le \eta(t, T, v_0), \quad t \in [t_0, T], \tag{9}$$

whenever $\gamma(T, t_0, u_0) \le v_0$.

Proof. Since $\lim_{k\to\infty} t_k = \infty$, there exists some $k \ge 1$ such that $T \in [t_k, t_{k+1})$. Since $\gamma(T, t_0, u_0) \le v_0$, by Lemma 4, it follows that

$$\gamma(t, t_0, u_0) \le \eta(t, T, v_0), \quad t \in [t_k, T] \ \text{(for some } k \ge 1). \tag{10}$$

When $t = t_k$,

$$\gamma(t_k^-, t_0, u_0) \le \psi_k(\gamma(t_k^-, t_0, u_0)) = \gamma(t_k, t_0, u_0) \le \eta(t_k, T, v_0). \quad (11)$$

Therefore, using Lemma 4 again, we obtain

$$\gamma(t, t_0, u_0) \le \eta(t, T, v_0), \quad t \in [t_{k-1}, t_k]. \quad (12)$$

By induction, we can get

$$\gamma(t, t_0, u_0) \le \eta(t, T, v_0), \quad t_0 \le t \le T. \quad (13)$$

The proof is complete. □

Theorem 6. *Assume that the conditions in Lemma 5 are satisfied; $x(t) = x(t, t_0, \varphi)$ is the solution of system (1) with $x_{t_0} = \varphi$. And the following conditions hold:*

(i) $V \in \mathscr{V}_0$, *if* $V(t+s, x(t+s)) \le \eta(t+s, t, V(t, x(t)))$, $-\tau \le s \le 0$; *then*

$$D^+V(t, x(t)) \le g(t, V(t, x(t))), \quad t \ne t_k; \quad (14)$$

(ii) $V(t_k, x(t_k) + I_k(x(t_k))) \le \psi_k(V(t_k^-, x(t_k^-)))$, *where* $\psi_k(s) \ge s$ *and* $\psi_k(s)$ *is nondecreasing.*

Then $\sup_{-\tau\le s\le 0} V(t_0 + s, \varphi(s)) \le u_0$ *implies*

$$V(t, x(t)) \le \gamma(t, t_0, u_0), \quad t \ge t_0. \quad (15)$$

Proof. Let $x(t) = x(t, t_0, \varphi)$ be a solution of system (1) existing for $t \ge t_0$ such that $\sup_{-\tau\le s\le 0} V(t_0 + s, \varphi(s)) \le u_0$.

For simplicity, let $m(t) = V(t, x(t))$; then $\sup_{-\tau\le s\le 0} m(t_0 + s) = m_{t_0} \le u_0$.

First, we will prove

$$m(s) < u_1(s, \varepsilon), \quad s \in (t_0, t_1], \quad (16)$$

where $\varepsilon > 0$ is small enough, $u_1(s, \varepsilon)$ is the solution of

$$u'(t) = g(t, u) + \varepsilon, \quad u(t_0) = u_0 + \varepsilon, \quad (17)$$

and $\lim_{\varepsilon\to 0} u_1(s, \varepsilon) = \gamma(s, t_0, u_0)$. Note that since $g(t, u)$ is continuous, a solution u_1 exists.

If (16) is not true, then there exists some $\bar{t} \in (t_0, t_1)$ such that

$$m(\bar{t}) = u_1(\bar{t}, \varepsilon), \quad m(t) < u_1(t, \varepsilon), \quad t \in (t_0, \bar{t}). \quad (18)$$

Therefore,

$$\begin{aligned} D^+m(\bar{t}) &= \lim_{h\to 0^+} \frac{1}{h}(m(\bar{t}+h) - m(\bar{t})) \\ &= \lim_{h\to 0^-} \frac{1}{h}(m(\bar{t}) - m(\bar{t}+h)) \\ &\ge \lim_{h\to 0^-} \frac{1}{h}(u_1(\bar{t}, \varepsilon) - u_1(\bar{t}+h, \varepsilon)) \\ &= \lim_{h\to 0^+} \frac{1}{h}(u_1(\bar{t}+h, \varepsilon) - u_1(\bar{t}, \varepsilon)) \\ &= D^+u_1(\bar{t}, \varepsilon); \end{aligned} \quad (19)$$

here, since $\bar{t}$ is an interior point of the interval (t_0, t_1) in which the functions are continuous, it implies that the left limits equal the right limits. Thus, it follows that

$$D^+m(\bar{t}) \ge D^+u_1(\bar{t}, \varepsilon) = g(\bar{t}, u_1(\bar{t}, \varepsilon)) + \varepsilon. \quad (20)$$

Now consider the left maximal solution $\eta(s, \bar{t}, m(\bar{t}))$, $t_0 \le s \le \bar{t}$, of

$$u'(t) = g_0(t, u), \quad u(\bar{t}) = m(\bar{t}). \quad (21)$$

By Lemma 4, we obtain

$$\gamma(s, t_0, u_0) \le \eta(s, \bar{t}, m(\bar{t})), \quad t_0 \le s \le \bar{t}. \quad (22)$$

Since $\gamma(\bar{t}, t_0, u_0) = \lim_{\varepsilon\to 0} u_1(\bar{t}, \varepsilon) = m(\bar{t}) = \eta(\bar{t}, \bar{t}, m(\bar{t}))$, and $m(t) \le u_1(t, \varepsilon)$, $t \in [t_0, \bar{t}]$, it follows that

$$m(s) \le \gamma(s, t_0, u_0) \le \eta(s, \bar{t}, m(\bar{t})), \quad t_0 \le s \le \bar{t}. \quad (23)$$

Since $m_{t_0} \le u_0$, we have $m(\bar{t}+s) \le \eta(\bar{t}+s, \bar{t}, m(\bar{t}))$, $-\tau \le s \le 0$.

Consequently, condition (i) yields

$$D^+m(\bar{t}) \le g(\bar{t}, m(\bar{t})) = g(\bar{t}, u_1(\bar{t}, \varepsilon)), \quad (24)$$

which contradicts with (20). Hence, (16) is proved.

When $t = t_1$, $m(t_1) \le \psi_1(m(t_1^-)) \le \psi_1(\gamma(t_1^-)) = \gamma_1^+$.

Next, we will prove

$$m(t) < u_2(t, \varepsilon), \quad t \in (t_0, t_2], \quad (25)$$

where $\varepsilon > 0$ is small enough, $u_2(t, \varepsilon)$ is the solution of

$$u'(t) = g(t, u) + \varepsilon, \quad u(t_1) = \gamma_1^+ + \varepsilon, \quad u(t_0) = u_0 + \varepsilon, \quad (26)$$

and $\lim_{\varepsilon\to 0} u_2(t, \varepsilon) = \gamma(t, t_0, u_0)$.

By the above proof, it easily follows that

$$m(t) < u_2(t, \varepsilon), \quad t \in (t_0, t_1]. \quad (27)$$

If (25) is false, then there exists a $t' \in (t_1, t_2)$ such that

$$m(t') = u_2(t', \varepsilon), \quad m(t) < u_2(t, \varepsilon), \quad t \in (t_0, t'). \tag{28}$$

This implies that

$$D^+ m(t') \geq D^+ u_2(t', \varepsilon) = g(t', u_2(t', \varepsilon)) + \varepsilon. \tag{29}$$

Now consider the left maximal solution $\eta(s, t', m(t'))$, $t_0 \leq s \leq t'$, of

$$u'(t) = g_0(t, u), \quad u(t') = m(t'). \tag{30}$$

By Lemma 5, we obtain

$$\gamma(s, t_0, u_0) \leq \eta(s, t', m(t')), \quad t_0 \leq s \leq t'. \tag{31}$$

Since $\gamma(t', t_0, u_0) = \lim_{\varepsilon \to 0} u_2(t', \varepsilon) = m(t') = \eta(t', t', m(t'))$, it follows that $m(t) \leq u_2(t, \varepsilon)$, $t \in [t_0, t']$, and $m(s) \leq \gamma(s, t_0, u_0) \leq \eta(s, t', m(t'))$, $t_0 \leq s \leq t'$.

This implies that $m(t'+s) \leq \eta(t'+s, t', m(t'))$, $-\tau \leq s \leq 0$. Consequently, condition (i) yields

$$D^+ m(t') \leq g(t', m(t')) = g(t', u_2(t', \varepsilon)), \tag{32}$$

which contradicts with (29). Hence, (25) is proved.

By induction, we can obtain

$$m(t) < u_k(t, \varepsilon), \quad t \in [t_0, t_k], \tag{33}$$

where $\varepsilon > 0$ is small enough, $u_k(t, \varepsilon)$ is the solution of

$$u'(t) = g(t, u) + \varepsilon, \quad u(t_m) = \gamma_m^+ + \varepsilon, \quad m = 1, 2, \ldots, k-1, \quad u(t_0) = u_0 + \varepsilon, \tag{34}$$

and $\gamma_m^+ = \psi_m(u(t_m^-))$, $\lim_{\varepsilon \to 0} u_k(t, \varepsilon) = \gamma(t, t_0, u_0)$.

This means that $m(t) \leq \gamma(t, t_0, u_0)$, $t \geq t_0$, which completes the proof. □

Remark 7. If $g_0(t, u) = 0$, Theorem 6 is similar to Lemma 2 in [14]. However, it should be noted that inequality $V(t_0, \varphi(0)) \leq u_0$ is not enough for the validity of the claim of Lemma 2. In Theorem 6, we complement and correct the known results in [14].

Next, we give some special cases of Theorem 6, which can be concrete and used easily.

Corollary 8. *Assume that $x(t) = x(t, t_0, \varphi)$ is the solution of (1) with $x_{t_0} = \varphi$. Let $g_0(t, u) = 0$, $\psi_k(s) = (1 + \beta_k)s$, $\beta_k \geq 0$, $k = 1, 2, \ldots$ in Lemma 5, and*

(i) *$V \in \mathscr{V}_0$, if $V(t+s, x(t+s)) \leq V(t, x(t))$, $-\tau \leq s \leq 0$; then*

$$D^+ V(t, x(t)) \leq g(t, V(t, x(t))), \quad t \neq t_k; \tag{35}$$

(ii) *$V(t, x + I_k(x)) \leq (1 + \beta_k) V(t, x(t))$, $t = t_k$.*

Then $\sup_{-\tau \leq s \leq 0} V(t_0 + s, \varphi(s)) \leq u_0$ implies

$$V(t, x(t)) \leq \gamma(t, t_0, u_0), \quad t \geq t_0. \tag{36}$$

Remark 9. In particular, let $g(t, u) = 0$ in Corollary 8; the estimate of $V(t, x(t))$ can be obtained; that is, $V(t, x(t)) \leq u_0 \prod_{t_0 < t_k < t} (1 + \beta_k)$. If $g(t, u) = \lambda'(t) u$, $\lambda'(t) \geq 0$ in Corollary 8, then $V(t, x(t)) \leq u_0 \prod_{t_0 < t_k < t} (1 + \beta_k) e^{\lambda(t) - \lambda(t_0)}$.

Remark 10. If $g(t, u) = 0$, $\beta_k = 0$, $k = 1, 2, \ldots$ in Corollary 8, then comparison system (7) becomes an ordinary differential equation and the corresponding style of Corollary 8 reduces to the result of Liu and Xu [8].

Corollary 11. *Assume that $x(t) = x(t, t_0, \varphi)$ is the solution of (1) with $x_{t_0} = \varphi$. Let $g_0(t, u) = -\lambda u$, $\lambda > 0$, in Lemma 5, and*

(i) *$V \in \mathscr{V}_0$, if $V(t+s, x(t+s)) \leq e^{-\lambda s} V(t, x(t))$, $-\tau \leq s \leq 0$; then*

$$D^+ V(t, x(t)) \leq g(t, V(t, x(t))), \quad t \neq t_k; \tag{37}$$

(ii) *$V(t_k, x(t_k) + I_k(x(t_k))) \leq \psi_k(V(t_k^-, x(t_k^-)))$, where $\psi_k(s) \geq s$ and $\psi_k(s)$ is nondecreasing.*

Then $\sup_{-\tau \leq s \leq 0} V(t_0 + s, \varphi(s)) \leq u_0$ implies

$$V(t, x(t)) \leq \gamma(t, t_0, u_0), \quad t \geq t_0. \tag{38}$$

Remark 12. From Corollary 11, we can observe that $g(t, u)$ is not always positive. That is, $g < 0$ is allowed. To the best of our knowledge, no similar work has been carried out on comparison method for impulsive functional differential systems. Hence, our result greatly enriches the theory of comparison principle and can be used for a wider class of impulsive systems.

Next, we will apply the comparison result to establish some stability criteria of system (1). In what follows, let K be the class of continuous strictly increasing functions $a(x)$ defined on R with $a(0) = 0$.

Theorem 13. *Assume that the conditions in Theorem 6 are satisfied. Moreover, if there exists function $a \in K$ such that*

$$a(|x|) \leq V(t, x(t)), \quad (t, x) \in R_+ \times S(\rho), \tag{39}$$

then the stability properties of the trivial solution of comparison system (7) imply the corresponding stability properties of the trivial solution of impulsive functional differential system (1).

Proof. We establish asymptotical stability. First, we prove that the trivial solution of system (1) is stable. Since the trivial solution of system (7) is stable, for any given $t_0 \in R_+$, $\varepsilon > 0$, there exists $\delta_1 = \delta_1(t_0, \varepsilon)$ such that

$$u(t, t_0, u_0) < a(\varepsilon) \quad \text{for } 0 < u_0 < \delta_1,\ t \ge t_0. \tag{40}$$

Since $V(t, 0) = 0$, then there exists $\delta_2 = \delta_2(t_0, \delta_1) > 0$ such that

$$V(t_0 + s, \varphi(s)) \le u_0 < \delta_1 \quad \text{for } \|\varphi\| < \delta_2,\ -\tau \le s \le 0. \tag{41}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, and, from Theorem 6, we have

$$a(|x|) \le V(t, t_0, \varphi) \le u(t, t_0, u_0) < a(\varepsilon) \quad \text{for } \|\varphi\| < \delta. \tag{42}$$

Hence, $|x| < \varepsilon$; that is, the trivial solution of (1) is stable.

Now, we prove that the trivial solution of system (1) is attractive.

For any given $\varphi \in \Omega$, $t_0 \in R_+$, and $\varepsilon > 0$, there exists u_0 satisfing $V(t_0 + s, \varphi(s)) \le u_0$, $-\tau \le s \le 0$. Since $u(t, t_0, u_0)$ is asymptotically stable, hence, there exists $T = T(\varepsilon, t_0, u_0)$ such that

$$u(t, t_0, u_0) < a(\varepsilon) \quad \text{for } t \ge t_0 + T. \tag{43}$$

From Theorem 6, we obtain

$$a(|x|) \le V(t, t_0, \varphi_0) \le u(t, t_0, u_0) < a(\varepsilon) \quad \text{for } t \ge t_0 + T. \tag{44}$$

Hence, $|x| < \varepsilon$ for $t \ge t_0 + T$; that is, the trivial solution of (1) is attractive. Therefore, the trivial solution of system (1) is asymptotically stable. □

Theorem 14. *Assume that the conditions in Theorem 6 are satisfied. Moreover, if there exist constants $p > 0$, $c > 0$ such that the following condition holds*

$$c|x|^p \le V(t, x(t)), \quad (t, x) \in R_+ \times S(\rho), \tag{45}$$

then the exponential stability of the trivial solution of comparison system (7) implies the exponential stability of the trivial solution of impulsive functional differential system (1).

Proof. Since the trivial solution of (7) is exponentially stable, hence, assuming $\lambda > 0$ is a constant, for any given $t_0 \in R_+$, $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $u_0 < \delta$ implies

$$u(t, t_0, u_0) < c \cdot \varepsilon^p \cdot e^{-\lambda(t - t_0)}, \quad t \ge t_0. \tag{46}$$

From Theorem 6, we have

$$c|x|^p \le V(t, t_0, \varphi) \le u(t, t_0, u_0) < c \cdot \varepsilon^p \cdot e^{-\lambda(t - t_0)} \quad \text{for } \|\varphi\| < \delta. \tag{47}$$

Hence, $|x| < \varepsilon \cdot e^{-(\lambda/p)(t - t_0)}$, $t \ge t_0$. Therefore, the trivial solution of system (1) is exponentially stable. This completes the proof. □

4. An Example

In this section, we will give an example to illustrate the effectiveness of our results.

Example 1. Consider the following impulsive delay differential equations:

$$\begin{aligned} x'(t) &= -a(t)x(t) + b(t)x(t - \tau(t)), \quad t \ge 0,\ t \ne t_k, \\ x(t_k) &= \left(1 + \frac{1}{k^2}\right)x(t_k^-), \quad k \in \mathbb{Z}_+, \\ x_{t_0} &= \varphi(s), \quad t_0 - \tau \le s \le t_0, \end{aligned} \tag{48}$$

where $a(t) \ge a > 0$, $0 < |b(t)| \le b$, and $0 \le \tau(t) \le \tau$, for all $t \ge t_0$.

Property 1. The trivial solution of system (48) is exponentially stable if $a > be^{a\tau}$.

Proof. Choose $V(t) = |x(t)|$. Then when $V(t + s, x(t + s)) \le e^{-as}V(t, x(t))$, $-\tau \le s \le 0$; that is, $|x(t - \tau(t))| \le e^{-as}|x(t)|$, and we have

$$\begin{aligned} D^+V(t) &= x'(t) \cdot \operatorname{sgn} x(t) \\ &= -a(t) \cdot x(t) \cdot \operatorname{sgn} x(t) + b(t) \cdot \operatorname{sgn} x(t) \\ &\quad \cdot x(t - \tau(t)) \le -a|x(t)| + b|x(t - \tau(t))| \\ &\le -aV(t) + be^{a\tau}V(t) \le (-a + be^{a\tau})V(t). \end{aligned} \tag{49}$$

Furthermore,

$$\begin{aligned} V(t_k) &= |x(t_k)| = \left(1 + \frac{1}{k^2}\right)|x(t_k^-)| \\ &= \left(1 + \frac{1}{k^2}\right)V(t_k^-). \end{aligned} \tag{50}$$

Then we can give the following comparison system:

$$\begin{aligned} u'(t) &= (-a + be^{a\tau})u, \quad t \ge 0,\ t \ne t_k, \\ u(t_k) &= \left(1 + \frac{1}{k^2}\right)u(t_k^-), \quad k \in Z_+, \\ u_{t_0} &= u_0 \ge 0, \quad t_0 - \tau \le s \le t_0. \end{aligned} \tag{51}$$

We can easily observe the solution of (51); that is, $u(t) = u_0\prod_{t_0 \le t_k < t}(1 + 1/k^2)e^{(-a + be^{a\tau})(t - t_0)}$.

If $-a + be^{a\tau} < 0$, then the solution of (51) is exponentially stable. Hence, by Theorem 14, the solution of (48) is also exponentially stable. This completes the proof. □

Remark 15. In [3, p129], the author gave the sufficient condition for the uniform stability of the functional differential equation of (48) without impulses; that is, $|b(t)| \le a(t)$. It is easy to check that $a > be^{a\tau}$ implies $|b(t)| \le a(t)$. Therefore, Property 1 shows that under proper impulse effect, the exponential stability can be derived, which illustrates that the impulses do contribute the equations stability and attractive properties.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported in part by National Science Foundation of China under Grant (no. 61201430) and Scientific and Technological Program of Huangdao District (no. 2014-1-28).

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