Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 679316, 12 pages
http://dx.doi.org/10.1155/2013/679316

Research Article

Multiple Positive Solutions to Multipoint Boundary Value Problem for a System of Second-Order Nonlinear Semipositone Differential Equations on Time Scales

Gang Wu,[1,2] Longsuo Li,[1] Xinrong Cong,[1] and Xiufeng Miao[1]

[1] *Department of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang 150001, China*
[2] *Basic Science, Harbin University of Commerce, Harbin, Heilongjiang 150076, China*

Correspondence should be addressed to Longsuo Li; lilongsuo6982@126.com

Received 13 November 2012; Revised 31 January 2013; Accepted 31 January 2013

Academic Editor: Naseer Shahzad

We study a system of second-order dynamic equations on time scales $(p_1u_1^{\nabla})^{\Delta}(t) - q_1(t)u_1(t) + \lambda f_1(t, u_1(t), u_2(t)) = 0, t \in (t_1, t_n), (p_2u_2^{\nabla})^{\Delta}(t) - q_2(t)u_2(t) + \lambda f_2(t, u_1(t), u_2(t)) = 0$, satisfying four kinds of different multipoint boundary value conditions, f_i is continuous and semipositone. We derive an interval of λ such that any λ lying in this interval, the semipositone coupled boundary value problem has multiple positive solutions. The arguments are based upon fixed-point theorems in a cone.

1. Introduction

In this paper, we consider the following dynamic equations on time scales:

$$\begin{gathered}\left(p_1u_1^{\nabla}\right)^{\Delta}(t) - q_1(t)u_1(t) + \lambda f_1(t, u_1(t), u_2(t)) = 0, \\ t \in (t_1, t_n), \quad \lambda > 0, \\ \left(p_2u_2^{\nabla}\right)^{\Delta}(t) - q_2(t)u_2(t) + \lambda f_2(t, u_1(t), u_2(t)) = 0,\end{gathered} \tag{1}$$

satisfying one of the boundary value conditions

$$\begin{gathered}\alpha_1u_1(t_1) - \beta_1p_1(t_1)u_1^{\nabla}(t_1) = 0, \\ \gamma_1u_1(t_n) + \delta_1p_1(t_n)u_1^{\nabla}(t_n) = \sum_{i=2}^{n-2}b_iu_1(t_i), \\ \alpha_2u_2(t_1) - \beta_2p_2(t_1)u_2^{\nabla}(t_1) = 0, \\ \gamma_2u_2(t_n) + \delta_2p_1(t_n)u_2^{\nabla}(t_n) = \sum_{i=2}^{n-2}b_iu_2(t_i),\end{gathered} \tag{2}$$

$$\begin{gathered}\alpha_1u_1(t_1) - \beta_1p_1(t_1)u_1^{\nabla}(t_1) = \sum_{i=2}^{n-2}a_iu_1(t_i), \\ \gamma_1u_1(t_n) + \delta_1p_1(t_n)u_1^{\nabla}(t_n) = 0, \\ \alpha_2u_2(t_1) - \beta_2p_2(t_1)u_2^{\nabla}(t_1) = \sum_{i=2}^{n-2}a_iu_2(t_i), \\ \gamma_2u_2(t_n) + \delta_2p_1(t_n)u_2^{\nabla}(t_n) = 0,\end{gathered} \tag{3}$$

$$\begin{gathered}\alpha_1u_1(t_1) - \beta_1p_1(t_1)u_1^{\nabla}(t_1) = 0, \\ \gamma_1u_1(t_n) + \delta_1p_1(t_n)u_1^{\nabla}(t_n) = \sum_{i=2}^{n-2}b_iu_1(t_i), \\ \alpha_2u_2(t_1) - \beta_2p_2(t_1)u_2^{\nabla}(t_1) = \sum_{i=2}^{n-2}a_iu_2(t_i), \\ \gamma_2u_2(t_n) + \delta_2p_1(t_n)u_2^{\nabla}(t_n) = 0,\end{gathered} \tag{4}$$

$$\alpha_1u_1(t_1) - \beta_1p_1(t_1)u_1^{\nabla}(t_1) = \sum_{i=2}^{n-2}a_iu_1(t_i),$$

$$\gamma_1 u_1(t_n) + \delta_1 p_1(t_n) u_1^{\nabla}(t_n) = 0,$$
$$\alpha_2 u_2(t_1) - \beta_2 p_2(t_1) u_2^{\nabla}(t_1) = 0,$$
$$\gamma_2 u_2(t_n) + \delta_2 p_1(t_n) u_2^{\nabla}(t_n) = \sum_{i=2}^{n-2} b_i u_2(t_i), \tag{5}$$

where

$$p_i, q_i : [t_1, t_n] \longrightarrow (0, +\infty)$$
$$\text{with } p_i \in C^{\Delta}[t_1, t_n), \quad q_i \in C[t_1, t_n] \quad \text{for } i = 1, 2;$$
$$\alpha_i, \beta_i, \gamma_i, \delta_i \in [0, +\infty) \tag{6}$$
$$\text{with } \alpha_i\gamma_i + \alpha_i\delta_i + \beta_i\gamma_i > 0 \quad \text{for } i = 1, 2,$$

and f_i is continuously and nonegative functionsquad, $a_i, b_i \in [0, +\infty)$ for $i \in \{1, 2, \ldots, n\}$; the points $t_i \in \mathbb{T}_{\kappa}^{\kappa}$ for $i \in \{1, 2, \ldots, n\}$ with $t_1 < t_2 < \cdots < t_n$.

In the past few years, the boundary value problems of dynamic equations on time scales have been studied by many authors (see [1–19] and references). Recently, multipoint boundary value problems on time scale have been studied, for instance, see [1–12].

In 2006, Anderson and Ma [1] studied the second-order multiple time-scale eigenvalue problem:

$$\left(py^{\nabla}\right)^{\Delta}(t) - q(t) y(t) + \lambda h(t) f(y) = 0,$$
$$t \in (t_1, t_n), \ \lambda > 0,$$
$$\alpha y(t_1) - \beta p(t_1) y^{\nabla}(t_1) = \sum_{i=2}^{n-2} a_i y(t_i), \tag{7}$$
$$\gamma y(t_n) + \delta p(t_n) y^{\nabla}(t_n) = \sum_{i=2}^{n-2} b_i y(t_i),$$

where the functions $f : [0, +\infty) \to [0, +\infty)$ and $h : [t_1, t_n] \to [0, +\infty)$ are continuous. The authors discuss conditions for the existence of at least one positive solution to the second-order Sturm-Liouville-type multiple eigenvalue problem on time scales.

In 2009, Feng et al. [2] studied

$$\left(py^{\nabla}\right)^{\Delta}(t) - q(t) y(t) = f(t, y),$$
$$t \in (t_1, t_n),$$
$$\alpha y(t_1) - \beta p(t_1) y^{\nabla}(t_1) = \sum_{i=2}^{n-2} a_i y(t_i), \tag{8}$$
$$\gamma y(t_n) + \delta p(t_n) y^{\nabla}(t_n) = \sum_{i=2}^{n-2} b_i y(t_i),$$

where the functions $f(t, y) = \sum_{j=1}^{n} c_j(t) y^{\nu_j}$, $c_j \in C([t_1, t_n], [0, \infty))$, $\nu_i \in [0, \infty)$, $j = 1, 2, \ldots, n$. This paper shows the existence of multiple positive solutions for the boundary value problem on time scales.

In 2009, Topal and Yantir [3] studied the second-order nonlinear m-point boundary value problems

$$u^{\Delta\nabla}(t) + a(t) u^{\Delta}(t) + b(t) u(t)$$
$$+ \lambda q(t) f(t, u(t)) = 0, \quad t \in (0, 1)_{\mathbb{T}}, \tag{9}$$
$$u(\rho(0)) = 0, \qquad u(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i),$$

where $\alpha_i \geq 0, 0 < \eta_i < \eta_{i+1} < 1$; for all $i = 1, 2, \ldots, m-2$; $a \in C([0, 1], [0, +\infty))$, $b \in C([0, 1], (-\infty, 0])$, f, q are continuously and nonegative functions. The authors deal with determining the value of λ, and the existences of multiple positive solutions of the equation are obtained. In 2010, Yuan and Liu [4] also study the second-order m-point boundary value problems; Yuan and Liu shows the existence of multiple positive solutions if f is semipositone and superlinear.

Motivated by the above results mentioned, we study the second-order nonlinear m-point boundary value problem (1) with boundary condition (k), and nonlinear term may be singularity and semipositone.

In this paper, the nonlinear term f_i of (1) is suit to and semipositone and the superlinear case, we shall prove our two existence results for the problem (1) with (k) by using a nonlinear alternative of Leray-Schauder type and Krasnosel'skii fixed-point theorem. This paper is organized as follows. In Section 2, we start with some preliminary lemmas. In Section 3, we give the main result which state the sufficient conditions for (1) with m-point boundary value (k) to have existence of positive solutions ($k = 2, \ldots, 5$).

2. Preliminaries

In this section, we state the preliminary information that we need to prove the main results.

In this paper, for our constructions, we shall consider the Banach space $E = C[\rho(t_1), t_n]$ equipped with standard norm $\|x\| = \max_{\rho(t_1) \leq t \leq t_n} |x(t)|$, $x \in E$; for each $(x, y) \in E \times E$, we write $||(x, y)||_1 = ||x|| + ||y||$. Clearly, $(E \times E, ||\cdot||_1)$ is a Banach space. Denote by ϕ_{i1} and ϕ_{i2} ($i = 1, 2$), the solutions of the equation

$$\left(p_i u_i^{\nabla}\right)^{\Delta}(t) - q_i(t) u_i(t) = 0, \quad t \in [t_1, t_n), \tag{10}$$

under the initial conditions

$$u_i(t_1) = \beta_i, \qquad p_i(t_1) u_i^{\nabla}(t_1) = \alpha_i,$$
$$u_i(t_n) = \delta_i, \qquad p_i(t_n) u_i^{\nabla}(t_n) = -\gamma_i, \tag{11}$$

respectively. So that ϕ_{i1} and ϕ_{i2} $(i = 1, 2)$ satisfy

$$\begin{gathered}\left(p_i\phi_{i1}^{\nabla}\right)^{\Delta}(t) - q_i(t)\,\phi_{i1}(t) = 0, \\ t \in [t_1, t_n), \quad \phi_{i1}(t_1) = \beta_i, \\ p_i(t_1)\,\phi_{i1}^{\nabla}(t_1) = \alpha_i, \\ \left(p_i\phi_{i2}^{\nabla}\right)^{\Delta}(t) - q_i(t)\,\phi_{i2}(t) = 0, \\ t \in [t_1, t_n), \quad \phi_{i2}(t_n) = \delta_i, \\ p_i(t_n)\,\phi_{i2}^{\nabla}(t_n) = -\gamma_i,\end{gathered} \tag{12}$$

respectively. For $i = 1, 2$, set $d_i = \alpha_i\phi_{i2}(t_1) - \beta_i p_i(t_1)\phi_{i2}^{\nabla}(t_1) = \gamma_i\phi_{i1}(t_n) + \delta_i p_i(t_n)\phi_{i1}^{\nabla}(t_n)$, the Green's function of the corresponding homogeneous boundary value problem is defined by

$$G_i(t,s) = \frac{1}{d_i}\begin{cases}\phi_{i2}(t)\,\phi_{i1}(s), & \rho(t_1) \le s \le t \le t_n, \\ \phi_{i1}(t)\,\phi_{i2}(s), & \rho(t_1) \le t \le s \le t_n,\end{cases} \quad \text{for } i = 1, 2. \tag{13}$$

From Lemmas 3.1 and 3.3 in [1], we have the following lemma.

Lemma 1. *If $d_i \neq 0$, (u_1, u_2) is a solution of* (1) *with boundary value condition (k) if only and if*

$$u_1(t) = \lambda\int_{t_1}^{t_n} H_{1k}(t,s)\,f_1(s, u_1(s), u_2(s))\,\Delta s, \quad t \in [\rho(t_1), t_n], \tag{14}$$

$$u_2(t) = \lambda\int_{t_1}^{t_n} H_{2k}(t,s)\,f_2(s, u_1(s), u_2(s))\,\Delta s, \tag{15}$$

where $k = 2, \ldots, 5$, and

$$\begin{aligned}H_{i2}(t,s) &= G_i(t,s) + \frac{1}{d_i - \sum_{j=2}^{n-1} b_j\phi_{i1}(t_j)}\sum_{j=2}^{n-1} b_j G_i(t_j, s)\,\phi_{i1}(t), \quad (i = 1, 2), \\ H_{i3}(t,s) &= G_i(t,s) + \frac{1}{d_i - \sum_{j=2}^{n-1} a_j\phi_{i2}(t_j)}\sum_{j=2}^{n-1} a_j G_i(t_j, s)\,\phi_{i2}(t), \quad (i = 1, 2), \\ H_{14}(t,s) &= G_1(t,s) + \frac{1}{d_1 - \sum_{j=2}^{n-1} b_i\phi_{11}(t_i)}\sum_{j=2}^{n-1} b_i G_1(t_i, s)\,\phi_{11}(t) = H_{12}(t,s), \\ H_{24}(t,s) &= G_2(t,s) + \frac{1}{d_2 - \sum_{j=2}^{n-1} a_i\phi_{22}(t_i)}\sum_{j=2}^{n-1} a_i G_2(t_i, s)\,\phi_{22}(t) = H_{23}(t,s), \\ H_{15}(t,s) &= G_1(t,s) + \frac{1}{d_1 - \sum_{j=2}^{n-1} a_i\phi_{12}(t_i)}\sum_{j=2}^{n-1} a_i G_1(t_i, s)\,\phi_{12}(t) = H_{13}(t,s), \\ H_{25}(t,s) &= G_2(t,s) + \frac{1}{d_2 - \sum_{j=2}^{n-1} b_i\phi_{21}(t_i)}\sum_{j=2}^{n-1} b_i G_2(t_i, s)\,\phi_{21}(t) = H_{22}(t,s).\end{aligned} \tag{16}$$

For the rest of the paper, we need the following assumption:

$$0 < \sum_{j=2}^{n-1} b_j\phi_{i1}(t_j), \quad \sum_{j=2}^{n-1} a_j\phi_{i2}(t_j) < d_i, \quad \text{for } i = 1, 2. \tag{C}$$

From ϕ_{i1} is nondecreasing on $[\rho(t_1), t_n]$, ϕ_{i2} is nonincreasing on $[\rho(t_1), t_n]$ (see [2, Proposition 2.3]), it is easy to verify the following inequalities:

$$\begin{gathered}d_i G_i(t,s) \le \phi_{i1}(t)\,\phi_{i2}(t), \\ d_i G_i(t,s) \le \phi_{i1}(s)\,\phi_{i2}(s), \\ d_i G_i(t,s) \ge \frac{1}{\|\phi_{i1}\|\,\|\phi_{i2}\|}\phi_{i1}(t)\,\phi_{i2}(t)\,\phi_{i1}(s)\,\phi_{i2}(s).\end{gathered} \tag{17}$$

Lemma 2. *The Green's function $G_i(t,s)$ has properties*

$$\begin{gathered}G_i(t,s) \le G_i(t,t), \\ \frac{d_i}{\|\phi_{i1}\|\,\|\phi_{i2}\|}G_i(t,t)\,G_i(s,s) \le G_i(t,s) \le G_i(s,s).\end{gathered} \tag{18}$$

Lemma 3. *For $H_{ik}(t,s)$, $k=2,\ldots,5$ and $i=1,2$, one has the conclusions $H_{ik}(t,s)\le C^*G_i(s,s)$ and*

$$\begin{aligned}
&c_*\phi_{i1}(t)G_i(s,s)\le H_{i2}(t,s)\le C^*\phi_{i1}(t),\quad (i=1,2),\\
&c_*\phi_{i2}(t)G_i(s,s)\le H_{i3}(t,s)\le C^*\phi_{i2}(t),\quad (i=1,2),\\
&c_*\phi_{ii}(t)G_1(s,s)\le H_{i4}(t,s)\le C^*\phi_{ii}(t),\quad (i=1,2),\\
&c_*\phi_{12}(t)G_1(s,s)\le H_{15}(t,s)\le C^*\phi_{12}(t),\\
&c_*\phi_{21}(t)G_1(s,s)\le H_{25}(t,s)\le C^*\phi_{21}(t),
\end{aligned}\tag{19}$$

where $C^*=C_1+C_2$ and

$$\begin{aligned}
C_1 &= \max_{i=1,2}\left\{1+\frac{\|\phi_{i1}\|}{d_i-\sum_{j=2}^{n-1}b_j\phi_{i1}(t_j)}\sum_{i=1}^{n-1}b_j\right.\\
&\quad\left.+\frac{\|\phi_{i2}\|}{d_i-\sum_{j=2}^{n-1}a_j\phi_{i2}(t_j)}\sum_{j=2}^{n-1}a_j\right\},\\
C_2 &= \max_{i=1,2}\left\{\frac{\|\phi_{i1}\|+\|\phi_{i2}\|}{d_i}+\frac{1}{d_i-\sum_{j=2}^{n-1}b_j\phi_{i1}(t_j)}\right.\\
&\quad\times\sum_{j=2}^{n-1}b_jG_i(t_j,t_j)+\frac{1}{d_i-\sum_{j=2}^{n-1}b_j\phi_{i2}(t_j)}\\
&\quad\left.\times\sum_{j=2}^{n-1}a_jG_2(t_j,t_j)\right\},\\
c_* &= \min\left\{\frac{d_i}{\|\phi_{i1}\|\|\phi_{i2}\|}\frac{1}{d_i-\sum_{j=2}^{n-1}b_j\phi_{i1}(t_j)}\right.\\
&\quad\times\sum_{j=2}^{n-1}b_jG_i(t_j,t_j),\\
&\quad\frac{d_i}{\|\phi_{i1}\|\|\phi_{i2}\|}\frac{1}{d_i-\sum_{j=2}^{n-1}a_j\phi_{i2}(t_j)}\\
&\quad\left.\times\sum_{j=2}^{n-1}a_jG_i(t_j,t_j);\ i=1,2\right\}.
\end{aligned}\tag{20}$$

Proof. From Lemma 2 and

$$\begin{aligned}
&\frac{1}{d_i-\sum_{j=2}^{n-1}b_j\phi_{i1}(t_j)}\sum_{j=2}^{n-1}b_jG_i(t_j,s)\phi_{i1}(t)\\
&\quad\le\frac{1}{d_i-\sum_{j=2}^{n-1}b_j\phi_{i1}(t_j)}\sum_{j=2}^{n-1}b_jG_i(s,s)\|\phi_{i1}\|,\\
&\frac{1}{d_i-\sum_{j=2}^{n-1}a_j\phi_{i2}(t_j)}\sum_{j=2}^{n-1}a_jG_i(t_j,s)\phi_{i2}(t)\\
&\quad\le\frac{1}{d_i-\sum_{j=2}^{n-1}a_j\phi_{i2}(t_j)}\sum_{j=2}^{n-1}a_jG_i(s,s)\|\phi_{i2}\|,
\end{aligned}\tag{21}$$

we have

$$H_{ik}(t,s)\le C_1G_i(s,s)\le C^*G_i(s,s).\tag{22}$$

For $k=2$ or 3, we have

$$\begin{aligned}
H_{i2}(t,s) &\le \frac{\|\phi_{i2}(t)\|}{d_i}\phi_{i1}(t)+\frac{1}{d_i-\sum_{j=2}^{n-1}b_i\phi_{i1}(t_i)}\\
&\quad\times\sum_{j=2}^{n-1}b_jG_i(t_j,t_j)\phi_{i1}(t)\\
&\le C^*\phi_{i1}(t),\\
H_{i3}(t,s) &\le \frac{\|\phi_{i1}(t)\|}{d_i}\phi_{i2}(t)+\frac{1}{d_i-\sum_{j=2}^{n-1}a_i\phi_{i2}(t_i)}\\
&\quad\times\sum_{j=2}^{n-1}a_jG_i(t_j,t_j)\phi_{i2}(t)\\
&\le C^*\phi_{i2}(t),\\
H_{i2}(t,s) &\ge \frac{1}{d_i-\sum_{j=2}^{n-1}b_j\phi_{i1}(t_j)}\\
&\quad\times\sum_{j=2}^{n-1}b_jG_i(t_j,s)\phi_{i1}(t)\\
&\ge \frac{1}{d_i-\sum_{j=2}^{n-1}b_j\phi_{i1}(t_j)}\\
&\quad\times\sum_{j=2}^{n-1}b_j\frac{d_i}{\|\phi_1\|\|\phi_2\|}G_i(t_j,t_j)G_i(s,s)\phi_{i1}(t)\\
&\ge c_*\phi_{i1}(t)G_i(s,s),\\
H_{i3}(t,s) &\ge \frac{1}{d_i-\sum_{j=2}^{n-1}a_j\phi_{i2}(t_j)}\\
&\quad\times\sum_{j=2}^{n-1}a_jG_i(t_j,s)\phi_{i2}(t)\\
&\ge \frac{1}{d_i-\sum_{j=2}^{n-1}a_j\phi_{i1}(t_j)}
\end{aligned}$$

$$\times \sum_{j=2}^{n-1} a_j \frac{d_i}{\|\phi_1\| \|\phi_2\|} G_i(t_j, t_j) G_i(s,s) \phi_{i2}(t)$$
$$\geq c_* \phi_{i2}(t) G_i(s,s). \tag{23}$$

So, we have

$$c_* \phi_{i1}(t) G_i(s,s) \leq H_{i2}(t,s) \leq C^* \phi_{i1}(t), \quad (i = 1,2),$$
$$c_* \phi_{i2}(t) G_i(s,s) \leq H_{i3}(t,s) \leq C^* \phi_{i2}(t), \quad (i = 1,2). \tag{24}$$

Since $H_{14}(t,s) = H_{12}(t,s), H_{24}(t,s) = H_{23}(t,s), H_{15}(t,s) = H_{13}(t,s), H_{25}(t,s) = H_{22}(t,s)$, then we also have

$$c_* \phi_{11}(t) G_1(s,s) \leq H_{14}(t,s) = H_{12}(t,s) \leq C^* \phi_{11}(t),$$
$$c_* \phi_{22}(t) G_1(s,s) \leq H_{24}(t,s) = H_{23}(t,s) \leq C^* \phi_{22}(t),$$
$$c_* \phi_{12}(t) G_1(s,s) \leq H_{15}(t,s) = H_{13}(t,s) \leq C^* \phi_{12}(t),$$
$$c_* \phi_{21}(t) G_1(s,s) \leq H_{25}(t,s) = H_{22}(t,s) \leq C^* \phi_{21}(t). \tag{25}$$

The proof is complete. □

The following theorems will play a major role in our next analysis.

Theorem 4 (see [20]). *Let X be a Banach space, and $\Omega \subset X$ closed and convex. Assume U is a relatively open subset of Ω with $0 \in U$, and let $S : \overline{U} \to \Omega$ be a compact, continuous map. Then either*

(1) *S has a fixed point in $\overline{U}$, or*

(2) *there exists $u \in \partial U$ and $\nu \in (0,1)$, with $u = \nu S u$.*

Theorem 5 (see [21]). *Let X be a Banach space, and let $P \subset X$ be a cone in X. Let Ω_1, Ω_2 be bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator such that, either*

(1) $\|Sw\| \leq \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \geq \|w\|$, $w \in P \cap \partial\Omega_2$, *or*

(2) $\|Sw\| \geq \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \leq \|w\|$, $w \in P \cap \partial\Omega_2$.

Then S has a fixed point in $P \cap \overline{\Omega}_2 \setminus \Omega_1$.

3. Main Results

We make the following assumptions:

(H_1) $f_i(t,u_1,u_2) \in C([t_1,t_n] \times [0,+\infty)^2, (-\infty,+\infty))$, moreover there exists a function $g(t) \in L^1([t_1,t_n],(0,+\infty))$ such that $f_i(t,u_1,u_2) \geq -g(t)$, for any $t \in (t_1,t_n)$, $u_i \in [0,+\infty)$, $i = 1,2$.

(H_1^*) $f_i(t,u_1,u_2) \in C((t_1,t_n) \times [0,+\infty)^2, (-\infty,+\infty))$ may be singular at $t = t_1, t_n$; moreover, there exists a function $g(t) \in L^1((t_1,t_n),(0,+\infty))$ such that $f_i(t,u_1,u_2) \geq -g(t)$, for any $t \in (t_1,t_n)$, $u_i \in [0,+\infty)$.

(H_2) $f_i(t,0,0) > 0$, for $t \in [t_1,t_n]$ $(i = 1,2)$.

(H_3) There exists $[\theta_1,\theta_2] \subset (t_1,t_n)$ such that $\lim_{u_1+u_2 \uparrow +\infty} \min_{t\in[\theta_1,\theta_2]} (f_i(t,u_1,u_2)/(u_1+u_2)) = +\infty$ $(i = 1,2)$.

(H_4) $\int_{t_1}^{t_n} G_i(s,s) g(s) \nabla s < +\infty$ and $\int_{t_1}^{t_n} G_i(s,s) f_i(s,z_1,z_2) \nabla s < +\infty$ for any $z_i \in [0,m]$, $m > 0$ is any constant $(i = 1,2)$.

In fact, we only consider the system

$$\left(p_1 x_1^{\nabla}\right)^{\Delta}(t) - q_1(t) x_1(t) + \lambda \Big(f_1\left(t, [x_1(t) - \nu_{1k}(t)]^*, [x_2(t) - \nu_{2k}(t)]^*\right) + g(t) \Big) = 0, \quad \lambda > 0,$$
$$\left(p_2 x_2^{\nabla}\right)^{\Delta}(t) - q_2(t) x_2(t) + \lambda \Big(f_2\left(t, [x_1(t) - \nu_{1k}(t)]^*, [x_2(t) - \nu_{2k}(t)]^*\right) + g(t) \Big) = 0, \quad \lambda > 0, \tag{26}$$

with one of the boundary value conditions

$$\alpha_1 x_1(t_1) - \beta_1 p_1(t_1) x_1^{\nabla}(t_1) = 0,$$
$$\gamma_1 x_1(t_n) + \delta_1 p_1(t_n) x_1^{\nabla}(t_n) = \sum_{i=2}^{n-2} b_i x_1(\eta_i),$$
$$\alpha_2 x_2(t_1) - \beta_2 p_2(t_1) x_2^{\nabla}(t_1) = 0,$$
$$\gamma_2 x_2(t_n) + \delta_2 p_1(t_n) x_2^{\nabla}(t_n) = \sum_{i=2}^{n-2} b_i x_2(\eta_i),$$
$$\alpha_1 x_1(t_1) - \beta_1 p_1(t_1) x_1^{\nabla}(t_1) = \sum_{i=2}^{n-2} a_i x_1(\eta_i),$$
$$\gamma_1 x_1(t_n) + \delta_1 p_1(t_n) x_1^{\nabla}(t_n) = 0,$$
$$\alpha_2 x_2(t_1) - \beta_2 p_2(t_1) x_2^{\nabla}(t_1) = \sum_{i=2}^{n-2} a_i x_2(\eta_i),$$
$$\gamma_2 x_2(t_n) + \delta_2 p_1(t_n) x_2^{\nabla}(t_n) = 0,$$
$$\alpha_1 x_1(t_1) - \beta_1 p_1(t_1) x_1^{\nabla}(t_1) = 0,$$

$$\gamma_1 x_1(t_n) + \delta_1 p_1(t_n) x_1^{\nabla}(t_n) = \sum_{i=2}^{n-2} b_i x_1(\eta_i),$$
$$\alpha_2 x_2(t_1) - \beta_2 p_2(t_1) x_2^{\nabla}(t_1) = \sum_{i=2}^{n-2} a_i x_2(\eta_i),$$
$$\gamma_2 x_2(t_n) + \delta_2 p_1(t_n) x_2^{\nabla}(t_n) = 0,$$
$$\alpha_1 x_1(t_1) - \beta_1 p_1(t_1) x_1^{\nabla}(t_1) = \sum_{i=2}^{n-2} a_i x_1(\eta_i),$$
$$\gamma_1 x_1(t_n) + \delta_1 p_1(t_n) x_1^{\nabla}(t_n) = 0,$$
$$\alpha_2 x_2(t_1) - \beta_2 p_2(t_1) x_2^{\nabla}(t_1) = 0,$$
$$\gamma_2 x_2(t_n) + \delta_2 p_1(t_n) x_2^{\nabla}(t_n) = \sum_{i=2}^{n-2} b_i x_2(\eta_i), \quad (27)$$

where

$$y(t)^* = \begin{cases} y(t), & y(t) \geq 0, \\ 0, & y(t) < 0, \end{cases} \quad (28)$$

and $v_{ik}(t) = \lambda \int_{t_1}^{t_n} H_{ik}(t,s) g(s) \Delta s$. For $k = 2, \ldots, 5$, from Lemma 1, $(v_{1k}(t), v_{2k}(t))$ is the solution of the equation

$$\begin{aligned} &\left(p_1 v_1^{\nabla}\right)^{\Delta}(t) - q_1(t) v_1(t) + \lambda g(t) = 0, \\ &\qquad \lambda > 0, \ t_1 < t < t_n, \\ &\left(p_2 v_2^{\nabla}\right)^{\Delta}(t) - q_2(t) v_2(t) + \lambda g(t) = 0, \\ &\qquad \lambda > 0, \end{aligned} \quad (29)$$

respectively, satisfying the following boundary value conditions:

$$\alpha_1 v_1(t_1) - \beta_1 p_1(t_1) v_1^{\nabla}(t_1) = 0,$$
$$\gamma_1 v_1(t_n) + \delta_1 p_1(t_n) v_1^{\nabla}(t_n) = \sum_{i=2}^{n-2} b_i v_1(\eta_i),$$
$$\alpha_2 v_2(t_1) - \beta_2 p_2(t_1) v_2^{\nabla}(t_1) = 0,$$
$$\gamma_2 v_2(t_n) + \delta_2 p_1(t_n) v_2^{\nabla}(t_n) = \sum_{i=2}^{n-2} b_i v_2(\eta_i),$$
$$\alpha_1 v_1(t_1) - \beta_1 p_1(t_1) v_1^{\nabla}(t_1) = \sum_{i=2}^{n-2} a_i v_1(\eta_i),$$
$$\gamma_1 v_1(t_n) + \delta_1 p_1(t_n) v_1^{\nabla}(t_n) = 0,$$
$$\alpha_2 v_2(t_1) - \beta_2 p_2(t_1) v_2^{\nabla}(t_1) = \sum_{i=2}^{n-2} a_i v_2(\eta_i),$$
$$\gamma_2 v_2(t_n) + \delta_2 p_1(t_n) v_2^{\nabla}(t_n) = 0,$$
$$\alpha_1 v_1(t_1) - \beta_1 p_1(t_1) v_1^{\nabla}(t_1) = 0,$$
$$\gamma_1 v_1(t_n) + \delta_1 p_1(t_n) v_1^{\nabla}(t_n) = \sum_{i=2}^{n-2} b_i v_1(\eta_i),$$
$$\alpha_2 v_2(t_1) - \beta_2 p_2(t_1) v_2^{\nabla}(t_1) = \sum_{i=2}^{n-2} a_i v_2(\eta_i),$$
$$\gamma_2 v_2(t_n) + \delta_2 p_1(t_n) v_2^{\nabla}(t_n) = 0,$$
$$\alpha_1 v_1(t_1) - \beta_1 p_1(t_1) x_1^{\nabla}(t_1) = \sum_{i=2}^{n-2} a_i v_1(\eta_i),$$
$$\gamma_1 v_1(t_n) + \delta_1 p_1(t_n) x_1^{\nabla}(t_n) = 0,$$
$$\alpha_2 v_2(t_1) - \beta_2 p_2(t_1) x_2^{\nabla}(t_1) = 0,$$
$$\gamma_2 v_2(t_n) + \delta_2 p_1(t_n) x_2^{\nabla}(t_n) = \sum_{i=2}^{n-2} b_i v_2(\eta_i). \quad (30)$$

We will show that there exists a solution (x_{1k}, x_{2k}) to the boundary value problem $(\bar{k})$ of the system (26) with $x_{ik}(t) \geq v_{ik}(t), t \in [\rho(t_1), t_n]$. If this is true, then $u_{ik}(t) = x_{ik}(t) - v_{ik}(t)$ is a nonnegative solution (positive on $(\rho(t_1), t_n)$) of the system (1) with the boundary value problem (k), (where $i = 1, 2; k = 2, \ldots, 5, \bar{k} = k + 11$). Since for any $t \in (t_1, t_n)$, from

$$\begin{aligned} &\left(p_i x_{ik}^{\nabla}\right)^{\Delta}(t) - q_i(t) x_{ik}^{\nabla}(t) \\ &\quad = \left(p_i (u_{ik} + v_{ik})^{\nabla}\right)^{\Delta}(t) - q_i(t) (u_{ik} + v_{ik})^{\nabla}(t) \\ &\quad = -\lambda \left(f_i\left(t, [x_{1k}(t) - v_{1k}(t)]^*, [x_{2k}(t) - v_{2k}(t)]^*\right) + g(t)\right) \\ &\quad = -\lambda \left(f_i(t, u_{1k}(t), u_{2k}(t)) + g(t)\right), \end{aligned} \quad (31)$$

we have

$$\left(p_i u_{ik}^{\nabla}\right)^{\Delta}(t) - q_i(t) u_{ik}(t) = -\lambda f_i(t, u_{1k}(t), u_{2k}(t)). \quad (32)$$

As a result, we will concentrate our study on (26) with the boundary value problem $(\bar{k})$.

Employing Lemma 1, we note that $(x_{1k}(t), x_{2k}(t))$ is a solution of the system (26) with boundary value $(\overline{k})$ if and only if

$$x_{1k}(t) = \lambda \int_{t_1}^{t_n} H_{1k}(t,s)\Big(f_1\Big(s, [x_{1k}(s) - v_{1k}(s)]^*, [x_{2k}(s) - v_{2k}(s)]^*\Big) + g(s)\Big)\Delta s, \quad t \in [\rho(t_1), t_n], \tag{33}$$

$$x_{2k}(t) = \lambda \int_{t_1}^{t_n} H_{2k}(t,s)\Big(f_2\Big(s, [x_{1k}(s) - v_{1k}(s)]^*, [x_{2k}(s) - v_{2k}(s)]^*\Big) + g(s)\Big)\Delta s.$$

We define a cone P_{ij} $(i, j = 1, 2)$ by

$$P_{ij} = \left\{x \in E \mid x(t) \geq \frac{c_*}{C^*}\phi_{ij}(t)\|x\|,\ t \in [\rho(t_1), t_n]\right\}. \tag{34}$$

It is clearly that $P_{ij} \times P_{mn}$ is a cone of $E \times E$, $(i, j, m, n = 1, 2)$. Define the integral operator $T_2 : P_{11} \times P_{21} \to E \times E$, $T_3 : P_{12} \times P_{22} \to E \times E$, $T_4 : P_{12} \times P_{21} \to E \times E$, $T_5 : P_{21} \times P_{12} \to E \times E$, by

$$T_k(x_{1k}, x_{2k}) = (T_{1k}(x_{1k}, x_{2k}), T_{2k}(x_{1k}, x_{2k})), \quad (k = 2, \ldots, 5), \tag{35}$$

where operators T_{ik} are defined by

$$T_{ik}(x_{1k}, x_{2k})(t) = \lambda \int_{t_1}^{t_n} H_{ik}(t,s)\Big(f_i\Big(s, [x_{1k}(s) - v_{1k}(s)]^*, [x_{2k}(s) - v_{2k}(s)]^*\Big) + g(s)\Big)\Delta s, \quad t \in [\rho(t_1), t_n], \tag{36}$$

where $i = 1, 2$. Clearly, if (x_{1k}, x_{2k}) is a fixed point of T_k, then (x_{1k}, x_{2k}) is a solution of system (26) with $(\overline{k})$ $(k = 2, \ldots, 5, \overline{k} = k + 11)$.

For $k = 2, \ldots, 5$, from (35) and Lemma 3, we have $T_k(x_{1k}, x_{2k})(t) \geq 0$ on $[0, 1]$, for $(x_{1k}, x_{2k}) \in P_{ij} \times P_{mn}$, we have

$$\begin{aligned} T_{ik}(x_{1k}, x_{2k})(t) &= \lambda \int_{t_1}^{t_n} H_{ik}(t,s)\Big(f_i\Big(s, [x_{1k}(s) - v_{1k}(s)]^*, [x_{2k}(s) - v_{2k}(s)]^*\Big) + g(s)\Big)\Delta s \\ &\leq C^*\lambda \times \int_{\rho(t_1)}^{\sigma(t_n)} G_i(s,s)\big(f_i(s, [x(s) - v(s)]^*) + g(s)\big)\Delta s, \end{aligned} \tag{37}$$

then $\|T_{ik}(x_{1k}, x_{2k})\| \leq C^*\lambda \int_{\rho(t_1)}^{\sigma(t_n)} G_i(s,s)(f(t, [x(t) - v(t)]^*) + g(t))\Delta s$.

On the other hand, when $k = 2$, we have

$$\begin{aligned} T_{i2}(x_{1k}, x_{2k})(t) &= \lambda \int_{t_1}^{t_n} H_{i2}(t,s)\Big(f_i\Big(s, [x_{1k}(s) - v_{1k}(s)]^*, [x_{2k}(s) - v_{2k}(s)]^*\Big) + g(s)\Big)\Delta s \\ &\geq \lambda \int_{t_1}^{t_n} c_*\phi_{i1}(t) G_i(s,s)\Big(f_i\Big(s, [x_{1k}(s) - v_{1k}(s)]^*, [x_{2k}(s) - v_{2k}(s)]^*\Big) + g(s)\Big)\Delta s \\ &\geq \frac{c_*}{C^*}\phi_{i1}(t)\lambda \times \int_{\rho(t_1)}^{\sigma(t_n)} C^* G_i(s,s)\big(f_i(s, [x(s) - v(s)]^*) + g(s)\big)\Delta s \\ &\geq \frac{c_*}{C^*}\phi_{i1}(t)\|T_{i2}(x_{1k}, x_{2k})\|. \end{aligned} \tag{38}$$

Thus, $T_{i2}(P_{11} \times P_{21}) \subset P_{i1}$. Hence $T_2(P_{11} \times P_{21}) \subset P_{11} \times P_{21}$.

When $k = 3$, we have

$$\begin{aligned} T_{i3}(x_{1k}, x_{2k})(t) &= \lambda \int_{t_1}^{t_n} H_{i3}(t,s)\Big(f_i\Big(s, [x_{1k}(s) - v_{1k}(s)]^*, [x_{2k}(s) - v_{2k}(s)]^*\Big) + g(s)\Big)\Delta s \end{aligned}$$

$$\geq \lambda \int_{t_1}^{t_n} c_* \phi_{i2}(t) G_i(s,s) \Big(f_i \big(s, [x_{1k}(s) - v_{1k}(s)]^* , [x_{2k}(s) - v_{2k}(s)]^* \big) + g(s) \Big) \Delta s$$
$$\geq \frac{c_*}{C^*} \phi_{i2}(t) \lambda \times \int_{\rho(t_1)}^{\sigma(t_n)} C^* G_i(s,s) \big(f_i(s, [x(s) - v(s)]^*) + g(s) \big) \Delta s$$
$$\geq \frac{c_*}{C^*} \phi_{i2}(t) \| T_{i3}(x_{1k}, x_{2k}) \| . \tag{39}$$

Thus, $T_{i3}(P_{12} \times P_{22}) \subset P_{i2}$. Hence $T_3(P_{12} \times P_{22}) \subset P_{12} \times P_{22}$.

Similarly discussion, we also have $T_4(P_{12} \times P_{21}) \subset P_{12} \times P_{21}$, $T_5(P_{21} \times P_{12}) \subset P_{21} \times P_{12}$. In addition, standard arguments show that T_k is a completely continuous operator.

For simplicity, we adopt the notation: $P_{14} = P_{25} := P_{12}$ and $P_{24} = P_{15} := P_{21}$, then, we can write $T_k(P_{1(k-1)} \times P_{2(k-1)}) \subset P_{1(k-1)} \times P_{2(k-1)}$, that is, $T_{ik}(P_{1(k-1)} \times P_{2(k-1)}) \subset P_{i(k-1)}$, ($i = 1, 2, k = 2, \ldots, 5$).

Theorem 6. *Suppose that (H_1)-(H_2) hold. Then there exists a constant $\overline{\lambda} > 0$ such that, for any $0 < \lambda \leq \overline{\lambda}$,* (1) *with boundary value condition (k) has at least one positive solution ($k = 2, \ldots, 5$).*

Proof. Fix $\delta \in (0, 1)$ and k ($k = 2, \ldots, 5$). From (H_2) let $0 < \varepsilon < 1$ be such that

$$f_i(t, z_1, z_2) \geq \delta f_i(t, 0, 0), \quad \text{for } t_1 \leq t \leq t_n,\ 0 \leq z_i \leq \varepsilon,\ i = 1, 2. \tag{40}$$

Let $\overline{f}(\varepsilon) = \max_{t_1 \leq t \leq t_n, 0 \leq z_1, z_2 \leq \varepsilon} \{\max_{i=1,2}\{f_i(t, z_1, z_2)\} + g(t)\}$, and $c = \int_{t_1}^{t_n} C^* G_i(s,s) \Delta s$. We have

$$\lim_{z \downarrow 0} \frac{\overline{f}(z)}{z} = +\infty. \tag{41}$$

Set $\overline{\lambda} = \varepsilon / 4c\overline{f}(\varepsilon)$, since for any $0 < \lambda \leq \overline{\lambda}$, fix the $\lambda \in (0, \overline{\lambda}]$, we always have

$$\lim_{z \downarrow 0} \frac{\overline{f}(z)}{z} = +\infty, \qquad \frac{\overline{f}(\varepsilon)}{\varepsilon} < \frac{1}{4c\lambda}, \tag{42}$$

Then there exists a $R_0 \in (0, \varepsilon]$ such that

$$\frac{\overline{f}(R_0)}{R_0} = \frac{1}{4c\lambda}. \tag{43}$$

Let $U_k = \{(x_{1k}, x_{2k}) \in P_{1(k-1)} \times P_{2(k-1)} : \|(x_{1k}, x_{2k})\|_1 < R_0\}$, $(x_{1k}, x_{2k}) \in \partial U_k$ and $\nu \in (0, 1)$ be such that $(x_{1k}, x_{2k}) = \nu T_k(x_{1k}, x_{2k})$, that is, $x_{ik} = \nu T_{ik}(x_{1k}, x_{2k})$ ($i = 1, 2$). We claim that $\|(x_{1k}, x_{2k})\|_1 \neq R_0$. In fact for $(x_{1k}, x_{2k}) \in \partial U_k$ and $\|(x_{1k}, x_{2k})\|_1 = R_0$, we have

$$x_{ik} = \nu T_{ik}(x_{1k}, x_{2k})$$
$$\leq \lambda \int_{t_1}^{t_n} H_{ik}(t,s) \Big(f_i \big(s, [x_{1k}(s) - v_{1k}(s)]^*, [x_{2k}(s) - v_{2k}(s)]^* \big) + g(s) \Big) \Delta s$$
$$\leq \lambda \int_{t_1}^{t_n} C^* G_i(s,s) \Big(f_i \big(s, [x_{1k}(s) - v_{1k}(s)]^* , [x_{2k}(s) - v_{2k}(s)]^* \big) + g(s) \Big) \Delta s$$
$$\leq \lambda \int_{t_1}^{t_n} C^* G_i(s,s) \overline{f}(R_0) \Delta s$$
$$\leq \lambda c \overline{f}(R_0). \tag{44}$$

It follows that

$$R_0 = \|(x_{1k}, x_{2k})\|_1 \leq 2\lambda c \overline{f}(R_0), \tag{45}$$

that is,

$$\frac{\overline{f}(R_0)}{R_0} \geq \frac{1}{2c\lambda} > \frac{1}{4c\lambda} = \frac{\overline{f}(R_0)}{R_0}, \tag{46}$$

which implies that $\|(x_{1k}, x_{2k})\|_1 \neq R_0$. By the nonlinear alternative of Leray-Schauder type, T_k has a fixed point $(x_{i1}, x_{i2}) \in \overline{U_k}$. Moreover combining (40) and the fact that $R_0 < \varepsilon$, we obtain

$$x_{ik} = \lambda \int_{t_1}^{t_n} H_{ik}(t,s) \Big(f_i \big(t, [x_{1k}(t) - v_{1k}(t)]^* , [x_{2k}(t) - v_{2k}(t)]^* \big) + g(t) \Big) \Delta s$$
$$\geq \lambda \int_{t_1}^{t_n} H_{ik}(t,s) (\delta f(s, 0, 0) + g(t)) \Delta s$$
$$\geq \lambda \int_{t_1}^{t_n} H_{ik}(t,s) g(t) \Delta s$$
$$= v_{ik}(t) \quad \text{for } t \in (\rho(t_1), t_n). \tag{47}$$

Then T_k has a positive fixed point (x_{i1}, x_{i2}) and $\|(x_{i1}, x_{i2})\|_1 \leq R_0 < 1$; that is, (x_{i1}, x_{i2}) is a positive solution of the boundary value problem (26) with $x_{ik} > v_{ik}(t)$ for $t \in (t_1, t_n)$.

Let $u_{ik}(t) = x_{ik}(t) - v_{ik}(t) \geq 0$ ($i = 1, 2$), then (u_{1k}, u_{2k}) is a nonnegative solution (positive on $(\rho(t_1), t_n)$) of the boundary value problem (1). □

Theorem 7. *Suppose that (H_1^*) and (H_3)-(H_4) hold. Then there exists a constant $\lambda^* > 0$ such that, for any $0 < \lambda \leq \lambda^*$,* (1) *with boundary value condition (k) has at least one positive solution ($k = 2, \ldots, 5$).*

Proof. We fix k $(k = 2, \dots, 5)$. Let $\Omega_1 = \{(x_{1k}, x_{2k}) \in E \times E : \|x_{ik}\| < R_1, i = 1, 2\}$, where $R_1 = \max\{1, r\}$ and $r = (C^{*2}/c_*) \int_{t_1}^{t_n} g(s)\Delta s$. Choose

$$\lambda^* = \min\left\{1, \frac{R_1}{2}(R+1)^{-1}, \frac{R_1}{2r}\right\}, \tag{48}$$

where $R = \int_{t_1}^{t_n} C^* G_i(s,s)(\max_{0\le z_1,z_2\le R_1} f_i(s,z_1,z_2) + g(s))\Delta s$ and $R \ge 0$.

Then for any $(x_{1k}, x_{2k}) \in (P_{1(k-1)} \times P_{2(k-1)}) \cap \partial\Omega_1$, $x_{ik}(s) - v_{ik}(s) \le x_{ik}(s) \le \|x_{ik}\| \le R_1$ $(i = 1, 2)$ and for $0 < \lambda \le \lambda^*$, we have

$$\begin{aligned} &\|T_{ik}(x_{1k}, x_{2k})(t)\| \\ &\le \lambda \int_{t_1}^{t_n} C^* G_i(s,s)\Big(f_i\big(s, [x_{1k}(s) - v_{1k}(s)]^*, \\ &\qquad [x_{2k}(s) - v_{2k}(s)]^*\big) + g(s)\Big)\Delta s \\ &\le \lambda \int_{t_1}^{t_n} C^* G_i(s,s)\left(\max_{0\le z_1,z_2\le R_1} f_i(s,z_1,z_2) + g(s)\right)\Delta s \\ &\le \lambda R \\ &\le \frac{R_1}{2}. \end{aligned} \tag{49}$$

This implies

$$\begin{aligned} &\|T_k(x_{1k}, x_{2k})\|_1 \le R_1 \le \|(x_{1k}, x_{2k})\|_1, \\ &(x_{1k}, x_{2k}) \in (P_{1(k-1)} \times P_{2(k-1)}) \cap \partial\Omega_1. \end{aligned} \tag{50}$$

Choose a constant $N > 1$ such that

$$\lambda N\gamma \frac{c_*}{2(\|\phi_{i1}\| + \|\phi_{i2}\|)} \int_{\theta_1}^{\theta_2} G_i(s,s)\phi_{i1}(s)\phi_{i2}(s)\Delta s \ge 1, \tag{51}$$

where $\gamma = \min_k \min_{\theta_1\le t\le\theta_2}\{\phi_{1k}(t), \phi_{2k}(t)\}$.

By assumption (H_3) and (H_4), there exists a constant $B > R_1$ such that

$$\begin{aligned} &\frac{f_i(t,z_1,z_2)}{z_1+z_2} > N, \quad \text{that is, } f_i(t,z_1,z_2) > N(z_1+z_2), \\ &\qquad \text{for } t \in [\theta_1, \theta_2],\ z_1 + z_2 > B\ (i = 1, 2). \end{aligned} \tag{52}$$

Choose $R_2 = \max\{R_1 + 1, 2\lambda r, 2C^*(B+1)/c_*\gamma\}$ and let $\Omega_2 = \{(x_{1k}, x_{2k}) \in E \times E : \|x_{ik}\| < R_2, i = 1, 2\}$. We note that $x(t) \ge (c_*/C^*)\phi_{ij}(t)\|x\|$ for all $x \in P_{ij}$, by Lemma 3, we have $H_{ik}(t,s) \le (C^{*2}/c_*)(x(t)/\|x\|)$. Then for any $(x_{1k}, x_{2k}) \in (P_{1(k-1)} \times P_{2(k-1)}) \cap \partial\Omega_2$, we have $\|x_{1k}\| = R_2$ or $\|x_{2k}\| = R_2$. Without loss of generality let $\|x_{1k}\| = R_2$, so we have

$$\begin{aligned} x_{1k}(t) - v_{1k}(t) &= x_{1k}(t) - \lambda \int_{t_1}^{t_n} H_{1k}(t,s) g(s)\Delta s \\ &\ge x_{1k}(t) - \lambda \int_{t_1}^{t_n} \frac{C^{*2}}{c_*}\frac{x_{1k}(t)}{\|x_{1k}\|} g(s)\Delta s \\ &= x_{1k}(t) - \lambda \frac{x_{1k}(t)}{\|x_{1k}\|}\int_{t_1}^{t_n} \frac{C^{*2}}{c_*} g(s)\Delta s \\ &\ge x_{1k}(t) - \frac{x_{1k}(t)}{\|x_{1k}\|}\lambda r \\ &\ge x_{1k}(t) - \frac{x_{1k}(t)}{R_2}\lambda r \\ &\ge \left(1 - \frac{\lambda r}{R_2}\right) x_{1k}(t) \\ &\ge \frac{1}{2} x_{1k}(t) \ge 0, \quad t \in [\rho(t_1), t_n]. \end{aligned} \tag{53}$$

Thus

$$\begin{aligned} &\min_{\theta_1\le t\le\theta_2}\left\{[x_{1k}(t) - v_{1k}(t)]^* + [x_{2k}(t) - v_{2k}(t)]^*\right\} \\ &\ge \min_{\theta_1\le t\le\theta_2}\{x_{1k}(t) - v_{1k}(t)\} \ge \min_{\theta_1\le t\le\theta_2}\left\{\frac{1}{2}x_{1k}(t)\right\} \\ &\ge \min_{\theta_1\le t\le\theta_2}\left\{\frac{c_*}{2C^*}\phi_{1k}(t)\|x_{1k}\|, \frac{c_*}{2C^*}\phi_{2k}(t)\|x_{1k}\|\right\} \\ &= \frac{c_*}{2C^*} R_2 \min_{\theta_1\le t\le\theta_2}\{\phi_{1k}(t), \phi_{2k}(t)\} \ge B + 1 > B. \end{aligned} \tag{54}$$

Now since $B > R_1$, it follows that

$$\begin{aligned} &T_{ik}(x_{1k}, x_{2k})(t) \\ &= \lambda \int_{t_1}^{t_n} H_{ik}(t,s)\Big(f_i\big(s, [x_{1k}(s) - v_{1k}(s)]^*, \\ &\qquad [x_{2k}(s) - v_{2k}(s)]^*\big) + g(s)\Big)\Delta s \\ &\ge \lambda \int_{\theta_1}^{\theta_2} H_{ik}(t,s)\Big(f_i\big(s, [x_{1k}(s) - v_{1k}(s)]^*, \\ &\qquad [x_{2k}(s) - v_{2k}(s)]^*\big) + g(s)\Big)\Delta s \\ &\ge \lambda \int_{\theta_1}^{\theta_2} H_{ik}(t,s) f_i\big(s, [x_{1k}(s) - v_{1k}(s)]^*, \\ &\qquad [x_{2k}(s) - v_{2k}(s)]^*\big)\Delta s \end{aligned}$$

$$\begin{aligned}
&\geq \lambda \int_{\theta_1}^{\theta_2} H_{ik}(t,s) N\left([x_{1k}(s)-v_{1k}(s)]^* \right.\\
&\qquad\qquad \left. + [x_{2k}(s)-v_{2k}(s)]^*\right)\Delta s\\
&\geq \lambda \int_{\theta_1}^{\theta_2} H_{ik}(t,s) N\left(x_{1k}(s)-v_{1k}(s)\right)\Delta s\\
&\geq \lambda \int_{\theta_1}^{\theta_2} c_* \min_{\theta_1\leq t\leq\theta_2}\{\phi_{1k}(t),\phi_{2k}(t)\} G_i(s,s) N\left(x_{1k}(s)-v_{1k}(s)\right)\Delta s\\
&\geq \lambda \int_{\theta_1}^{\theta_2} c_* \min_{\theta_1\leq t\leq\theta_2}\{\phi_{1k}(t),\phi_{2k}(t)\} G_i(s,s) \frac{N}{2} x_{1k}(s)\, ds\\
&\geq \lambda \min_{\theta_1\leq t\leq\theta_2}\{\phi_{1k}(t),\phi_{2k}(t)\}\\
&\quad\times \int_{\theta_1}^{\theta_2} c_* G_i(s,s) \frac{c_* N}{2(\|\phi_{i1}\|+\|\phi_{i2}\|)}\phi_{i1}(s)\phi_{i2}(s)\|x_{1k}\|\Delta s\\
&\geq \lambda N\gamma \frac{c_*}{2(\|\phi_{i1}\|+\|\phi_{i2}\|)}\int_{\theta_1}^{\theta_2} G_i(s,s)\phi_{i1}(s)\phi_{i2}(s)\Delta s R_2\\
&\geq R_2, \quad t\in[\theta_1,\theta_2].
\end{aligned} \tag{55}$$

This implies

$$\begin{gathered}
\|T_k(x_{1k},x_{2k})\|_1 \geq \|(x_{1k},x_{2k})\|_1,\\
(x_{1k},x_{2k}) \in (P_{1(k-1)}\times P_{2(k-1)})\cap\partial\Omega_2.
\end{gathered} \tag{56}$$

For the Krasnosel'skii's fixed point theorem, one deduces that T_k has a fixed point (x_{1k},x_{2k}) with $R_1 < \|(x_{1k},x_{2k})\| < R_2 \Leftrightarrow R_1 < \|x_{1k}\| + \|x_{2k}\| < R_2$.

Since $r \leq R_1 < \|x_{ik}\| < R_2$ $(i=1,2)$, then

$$\begin{aligned}
x_{ik}(t)-v_{ik}(t) &= x_{ik}(t) - \lambda\int_{t_1}^{t_n} H_{ik}(t,s) g(s)\Delta s\\
&\geq x_{ik}(t) - \lambda\int_{t_1}^{t_n} \frac{C^{*2}}{c_*}\frac{x_{ik}(t)}{\|x_{ik}\|} g(s)\Delta s\\
&= x_{ik}(t) - \lambda\frac{x_{ik}(t)}{\|x_{ik}\|} r\\
&\geq x_{ik}(t) - \lambda x_{ik}(t)\\
&= (1-\lambda)x_{ik}(t)\\
&\geq (1-\lambda)\frac{c_*}{C^*}\frac{\phi_{i1}(t)\phi_{i2}(t)}{\|\phi_{i1}\|+\|\phi_{i2}\|}\|x_{ik}\|\\
&> 0, \quad t\in(\rho(t_1),t_n).
\end{aligned} \tag{57}$$

Thus (x_{1k},x_{2k}) is a positive solution of the boundary value problem (26) with $x_{ik}(t) > v_{ik}(t)$ $(i=1,2)$ for $t\in(\rho(t_1),t_n)$.

Let $u_{ik}(t) = x_{ik}(t) - v_{ik}(t) \geq 0$ $(i=1,2)$, then (u_{1k},u_{2k}) is a nonnegative solution (positive on $(\rho(t_1),t_n)$) of the boundary value problem (1). □

Since condition (H_1) implies conditions (H_1^*) and (H_4) then from the proof of Theorems 6 and 7, we immediately have the following theorem.

Theorem 8. *Suppose that (H_1)–(H_3) hold. Then* (1) *with boundary value condition (k) has at least two positive solutions for $\lambda > 0$ sufficiently small $(k = 2,\ldots,5)$.*

In fact with $0 < \lambda < \min\{\overline{\lambda},\lambda^*\}$ then (1) with boundary value condition (k) has at least two positive solutions.

Remark 9. In Theorems 6–8, we use the assumption condition 16. If we have not the condition 16, that is, $a_i = b_i = 0$, then the system (1) and boundary condition (k) are

$$\begin{aligned}
&\left(p_1 u_1^{\nabla}\right)^{\Delta}(t) - q_1(t)u_1(t) + \lambda f_1(t,u_1(t),u_2(t)) = 0,\\
&\qquad\qquad t\in(t_1,t_n),\ \lambda>0,\\
&\left(p_2 u_2^{\nabla}\right)^{\Delta}(t) - q_2(t)u_2(t) + \lambda f_2(t,u_1(t),u_2(t)) = 0,\\
&\alpha_1 u_1(t_1) - \beta_1 p_1(t_1) u_1^{\nabla}(t_1) = 0,\\
&\gamma_1 u_1(t_n) + \delta_1 p_1(t_n) u_1^{\nabla}(t_n) = 0,\\
&\alpha_2 u_2(t_1) - \beta_2 p_2(t_1) u_2^{\nabla}(t_1) = 0,\\
&\gamma_2 u_2(t_n) + \delta_2 p_1(t_n) u_2^{\nabla}(t_n) = 0.
\end{aligned} \tag{58}$$

From Lemma 2, an argument similar to those in Theorems 6–8 yields the following theorems.

Theorem 10. *Suppose that (H_1) and (H_2) hold. Then there exists a constant $\overline{\lambda} > 0$ such that, for any $0 < \lambda \leq \overline{\lambda}$, the boundary value problem* (58) *has at least one positive solution.*

Theorem 11. *Suppose that (H_1^*) and (H_3)-(H_4) hold. Then there exists a constant $\lambda^* > 0$ such that, for any $0 < \lambda \leq \lambda^*$, the boundary value problem* (58) *has at least one positive solution.*

Theorem 12. *Suppose that (H_1)–(H_4) hold. Then the boundary value problem* (58) *has at least two positive solutions for $\lambda > 0$ sufficiently small.*

4. Example

To illustrate the usefulness of the results, we give some examples.

Example 13. Consider the boundary value problem

$$\begin{aligned}
&u'' - u = -\lambda\left((u+v)^a + \frac{1}{(t-t^2)^{1/2}}\cos(2\pi(u+v))\right),\\
&\qquad\qquad -1<t<1,\ \lambda>0,\\
&v'' - v = -\lambda\left((u-1)^2 + v^2 + \frac{1}{(t-t^2)^{1/2}}\sin(2\pi u)\right),\\
&u(-1) = v(1) = 0, \qquad u(1) = au(0), \qquad v(-1) = bv(0),
\end{aligned} \tag{59}$$

where $a > 1$. Then if $\lambda > 0$ is sufficiently small, (59) has a positive solution u with $u(t) > 0$ for $t\in(0,1)$.

To see this, we will apply Theorem 7 with

$$f_1(t,u,v) = (u+v)^a + \frac{1}{(t^2-t^4)^{1/4}}\cos(2\pi(u+v)),$$
$$f_2(t,u,v) = (u-1)^2 + v^2 + \frac{1}{(t^2-t^4)^{1/4}}\sin(2\pi u), \quad (60)$$
$$g_1(t) = g_2(t) = g(t) = \frac{1}{(t^2-t^4)^{1/4}}.$$

Clearly for $t \in (0,1)$,

$$f_i(t,u,v) + g(t) > 0, \quad \text{for } t \in (0,1) \;\; i = 1,2,$$
$$\lim_{u+v\uparrow+\infty}\inf \frac{f_i(t,u,v)}{u+v} = +\infty \quad \text{for all } t \in [\theta_1,\theta_2] \subset (0,1). \quad (61)$$

Now (H_1^*), (H_3), and (H_4) hold. We note that the boundary condition of (59) is in accord with (4), and from [1], we have

$$\phi_{11} = \phi_{21} = \frac{e^{t+1}-e^{-t-1}}{2}, \qquad \phi_{12} = \phi_{22} = \frac{e^{-t+1}-e^{t-1}}{2},$$
$$d_1 = d_2 = \sinh(2). \quad (62)$$

Then

$$G_1(t,s) = G_2(t,s) = \frac{1}{d_1}\begin{cases}\phi_{12}(t)\phi_{11}(s), & \rho(t_1) \le s \le t \le t_n,\\ \phi_{11}(t)\phi_{12}(s), & \rho(t_1) \le t \le s \le t_n,\end{cases}$$
$$H_{14}(t,s) = G_1(t,s) + \frac{1}{d_1 - a\phi_{11}(0)} aG_1(0,s)\phi_{11}(t), \quad (63)$$
$$H_{24}(t,s) = G_2(t,s) + \frac{1}{d_2 - b\phi_{22}(0)} bG_2(0,s)\phi_{22}(t).$$

Note $r = \int_{t_1}^{t_n}(C^{*2}/c_*)g(s)\Delta s$. Let $R_1 = r+1$ and we have

$$\begin{aligned}R &= \int_{t_1}^{t_n} C^* G_i(s,s)\left(\max_{0\le z_1,z_2\le R_1} f_i(s,z_1,z_2) + g(s)\right)\Delta s\\
&\le \int_{t_1}^{t_n} C^* G_i(s,s)\left[2^{a+2}R_1^{a+2} + \frac{2}{(s^2-s^4)^{1/4}}\right]\Delta s\\
&\le \int_{-1}^{1} \frac{C^* e^4}{4}\left[2^{a+2}R_1^{a+2} + \frac{2}{(s^2-s^4)^{1/4}}\right]\Delta s\\
&= \int_{0}^{1} \frac{C^* e^4}{2}\left[2^{a+2}R_1^{a+2} + \frac{2}{(s^2-s^4)^{1/4}}\right]\Delta s\\
&\le \int_{0}^{1} \frac{C^* e^4}{2}\left[2^{a+2}R_1^{a+2} + \frac{2}{(s-s^2)^{1/2}}\right]\Delta s\\
&\le 2^{a+1}C^* e^4\left(R_1^{a+2} + \pi\right).\end{aligned} \quad (64)$$

Also let

$$\lambda^* = \min\left\{1, \frac{R_1}{2^{a+2}C^* e^4}\left(R_1^{a+2}+\pi\right)^{-1}, \frac{R_1}{2r}\right\}. \quad (65)$$

Now, if $\lambda < \lambda^*$, Theorem 7 guarantees that (59) has a positive solutions (u,v) with $\|u\| \ge 1$ and $\|v\| \ge 1$.

Example 14. Consider the boundary value problem:

$$\begin{aligned}&\left(p_1 u_1^{\nabla}\right)^{\Delta}(t) - q_1(t)u_1(t)\\
&\quad = -\lambda\left(e^{u_1} + u_2^2 + 7\cos(2\pi t u_1)\right),\\
&\qquad t_1 < t < t_n, \quad \lambda > 0,\\
&\left(p_2 u_2^{\nabla}\right)^{\Delta}(t) - q_2(t)u_2(t)\\
&\quad = -\lambda\left((u_1-1)^2 + u_2^2 + 5\sin(2\pi t u_2)\right)\end{aligned} \quad (66)$$

satisfying one of the boundary value conditions (k), ($k = 2,\ldots,5$).

Then if $\lambda > 0$ is sufficiently small, (66) has two solutions (u_{11},u_{12}), (u_{21},u_{22}) with $u_{ij}(t) > 0$ for $t \in (0,1)$, $i,j = 1,2$.

To see this, we will apply Theorem 8 with

$$f_1(t,u_1,u_2) = e^{u_1} + u_2^2 + 7\cos(2\pi t u_1),$$
$$f_2(t,u_1,u_2) = (u_1-1)^2 + u_2^2 + 5\sin(2\pi t u_2), \quad (67)$$
$$g_1(t) = g_2(t) = g(t) = 8.$$

Clearly, for $t \in (0,1)$,

$$\begin{aligned}&f_i(t,u_1,u_2) + g(t) \ge 1 > 0,\\
&f_1(t,0,0) = 8 > 0,\\
&f_2(t,0,0) = 3 > 0,\\
&\lim_{u+v\uparrow+\infty}\inf\frac{f_i(t,u_1,u_2)}{u_1+u_2} = +\infty, \quad i = 1,2.\end{aligned} \quad (68)$$

Now (H_1)–(H_4) hold. Let $\delta = 1/100$, $\varepsilon = 1/8$, and we have

$$f_i(t,u_1,u_2) \ge \delta f_i(t,0,0), \quad \text{for } 0 \le t \le 1, \; 0 \le u_i \le \varepsilon, \; i = 1,2. \quad (69)$$

Furthermore let $\overline{f}(\varepsilon) = \max_{0\le t\le 1, 0\le u_1,u_2\le\varepsilon}\{\max_{i=1,2} f_i(t,u_1,u_2) + g(t)\}$, and $c = \int_{t_1}^{t_n} C^* G_i(s,s)\Delta s$. Note

$$\frac{\varepsilon}{4c\overline{f}(\varepsilon)} \ge \frac{1}{32c(e+8)} > \frac{1}{352c}. \quad (70)$$

Let $\overline{\lambda} = 1/352c$. Now, if $0 < \lambda < \overline{\lambda}$ then $0 < \lambda < \varepsilon/4c\overline{f}(\varepsilon)$ and Theorem 6 guarantees that (66) has positive solutions (u_{11},u_{12}) with $\|u_{1j}\| \le (1/8)$ $(j = 1,2)$.

Next note $r = 8{C^*}^2(t_n - t_1)/c_*$ and let $R_1 = r + 2$ so we have

$$\begin{aligned} R &= \int_0^1 C^* G_i(s,s)\left(\max_{0\le z_1,z_2\le R_1} f_i(s,z_1,z_2) + g(s)\right)\Delta s \\ &\le \int_0^1 C^* G_i(s,s)\left(e^{R_1} + 2R_1^2 + 7 + 8\right)\Delta s \\ &\le \int_0^1 C^* G_i(s,s)\,\Delta s\left(e^{R_1} + 2R_1^2 + 15\right) \\ &\le \left(e^{R_1} + 2R_1^2 + 15\right)c. \end{aligned} \tag{71}$$

Also let

$$\lambda^* = \min\left\{1, \frac{R_1}{2\left(e^{R_1} + 2R_1^2 + 15\right)c}, \frac{R_1}{2r}\right\}. \tag{72}$$

Now, if $\lambda < \lambda^*$, Theorem 7 guarantees that (59) has a positive solutions (u_{21}, u_{22}) with $\|u_{2j}\| \ge 2$, $j = 1, 2$.

Thus, if $\lambda < \min\{\overline{\lambda}, \lambda^*\}$, Theorem 8 guarantees that (66) has two solutions (u_{11}, u_{12}) and (u_{21}, u_{22}) with $u_{ij} > 0$ for $t \in (0, 1)$, $i, j = 1, 2$.

References

[1] D. R. Anderson and R. Ma, "Second-order n-point eigenvalue problems on time scales," *Advances in Difference Equations*, vol. 2006, Article ID 59572, 17 pages, 2006.

[2] M. Feng, X. Zhang, and W. Ge, "Multiple positive solutions for a class of m-point boundary value problems on time scales," *Advances in Difference Equations*, vol. 2009, Article ID 219251, 14 pages, 2009.

[3] S. G. Topal and A. Yantir, "Positive solutions of a second order m-point BVP on time scales," *Nonlinear Dynamics and Systems Theory*, vol. 9, no. 2, pp. 185–197, 2009.

[4] C. Yuan and Y. Liu, "Multiple positive solutions of a second order nonlinear semipositone m-point boundary value problem on time scales," *Abstract and Applied Analysis*, vol. 2010, Article ID 261741, 19 pages, 2010.

[5] X. Lin and Z. Du, "Positive solutions of m-point boundary value problem for second-order dynamic equations on time scales," *Journal of Difference Equations and Applications*, vol. 14, no. 8, pp. 851–864, 2008.

[6] H.-R. Sun and W.-T. Li, "Positive solutions for nonlinear three-point boundary value problems on time scales," *Journal of Mathematical Analysis and Applications*, vol. 299, no. 2, pp. 508–524, 2004.

[7] Y. Pang and Z. Bai, "Upper and lower solution method for a fourth-order four-point boundary value problem on time scales," *Applied Mathematics and Computation*, vol. 215, no. 6, pp. 2243–2247, 2009.

[8] F. M. Atici and S. G. Topal, "The generalized quasilinearization method and three point boundary value problems on time scales," *Applied Mathematics Letters*, vol. 18, no. 5, pp. 577–585, 2005.

[9] S. Liang and J. Zhang, "The existence of countably many positive solutions for nonlinear singular m-point boundary value problems on time scales," *Journal of Computational and Applied Mathematics*, vol. 223, no. 1, pp. 291–303, 2009.

[10] S. Liang, J. Zhang, and Z. Wang, "The existence of three positive solutions of m-point boundary value problems for some dynamic equations on time scales," *Mathematical and Computer Modelling*, vol. 49, no. 7-8, pp. 1386–1393, 2009.

[11] L. Hu, "Positive solutions to singular third-order three-point boundary value problems on time scales," *Mathematical and Computer Modelling*, vol. 51, no. 5-6, pp. 606–615, 2010.

[12] İ. Yaslan, "Multiple positive solutions for nonlinear three-point boundary value problems on time scales," *Computers & Mathematics with Applications*, vol. 55, no. 8, pp. 1861–1869, 2008.

[13] J. Li and J. Shen, "Existence results for second-order impulsive boundary value problems on time scales," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 70, no. 4, pp. 1648–1655, 2009.

[14] J.-P. Sun, "Twin positive solutions of nonlinear first-order boundary value problems on time scales," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 68, no. 6, pp. 1754–1758, 2008.

[15] R. P. Agarwal, M. Bohner, and D. O'Regan, "Time scale boundary value problems on infinite intervals," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 27–34, 2002, Dynamic equations on time scales.

[16] H. Chen, H. Wang, Q. Zhang, and T. Zhou, "Double positive solutions of boundary value problems for p-Laplacian impulsive functional dynamic equations on time scales," *Computers & Mathematics with Applications*, vol. 53, no. 10, pp. 1473–1480, 2007.

[17] R. Ma and H. Luo, "Existence of solutions for a two-point boundary value problem on time scales," *Applied Mathematics and Computation*, vol. 150, no. 1, pp. 139–147, 2004.

[18] D. R. Anderson, G. S. Guseinov, and J. Hoffacker, "Higher-order self-adjoint boundary-value problems on time scales," *Journal of Computational and Applied Mathematics*, vol. 194, no. 2, pp. 309–342, 2006.

[19] *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.

[20] R. P. Agarwal, M. Meehan, and D. O'Regan, *Fixed Point Theory and Applications*, vol. 141, Cambridge University Press, Cambridge, UK, 2001.

[21] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5, Academic Press, San Diego, Calif, USA, 1988.

Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Differential Equations
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization