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## Research Article

# Hybrid Algorithm for Common Fixed Points of Uniformly Closed Countable Families of Hemirelatively Nonexpansive Mappings and Applications

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The authors have obtained the following results: (1) the definition of uniformly closed countable family of nonlinear mappings, (2) strong convergence theorem by the monotone hybrid algorithm for two countable families of hemirelatively nonexpansive mappings in a Banach space with new method of proof, (3) two examples of uniformly closed countable families of nonlinear mappings and applications, (4) an example which is hemirelatively nonexpansive mapping but not weak relatively nonexpansive mapping, and (5) an example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping. Therefore, the results of this paper improve and extend the results of Plubtieng and Ungchittrakool (2010) and many others.

## 1. Introduction and Preliminaries

Let  $E$  be a Banach space with the dual  $E^*$ . We denote by  $J$  the normalized duality mapping from  $E$  to  $2^{E^*}$  defined by

$$Jx = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad (1.1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the generalized duality pairing. It is well known that the normalized duality  $J$  has the following properties: (1) if  $E$  is smooth, then  $J$  is single valued; (2) if  $E$  is strictly convex, then  $J$  is one-to-one (i.e.,  $Jx \cap Jy = \emptyset$  for all  $x \neq y$ ); (3) if  $E$  is reflexive, then  $J$  is surjective; (4) if  $E$  has Frchet differentiable norm, then  $J$  is uniformly norm-to-norm

continuous; (5) if  $E$  is uniformly smooth, then  $J$  is uniformly norm-to-norm continuous on each bounded subset of  $E$ ; (7) if  $E$  is a Hilbert space, then  $J$  is the identity operator.

As we all know that if  $C$  is a nonempty closed convex subset of a Hilbert space  $H$  and  $P_C : H \rightarrow C$  is the metric projection of  $H$  onto  $C$ , then  $P_C$  is nonexpansive. This fact actually characterizes Hilbert spaces, and, consequently, it is not available in more general Banach spaces. In this connection, Alber [1] has recently introduced a generalized projection operator  $\Pi_C$  in a Banach space  $E$  which is an analogue of the metric projection in Hilbert spaces.

Let  $E$  be a smooth Banach space. Consider the function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for } x, y \in E. \quad (1.2)$$

Observe that, in a Hilbert space  $H$ , (1.2) reduces to  $\phi(x, y) = \|x - y\|^2$ ,  $x, y \in H$ .

The generalized projection  $\Pi_C : E \rightarrow C$  is a map that assigns to an arbitrary point  $x \in E$  the minimum point of the functional  $\phi(x, y)$ , that is,  $\Pi_C x = \bar{x}$ , where  $\bar{x}$  is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x), \quad (1.3)$$

existence and uniqueness of the operator  $\Pi_C$  follow from the properties of the functional  $\phi(x, y)$  and strict monotonicity of the mapping  $J$  (see, e.g., [1, 2]). In Hilbert space,  $\Pi_C = P_C$ . It is obvious from the definition of function  $\phi$  that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \quad \forall x, y \in E. \quad (1.4)$$

If  $E$  is a reflexive strictly convex and smooth Banach space, then for  $x, y \in E$ ,  $\phi(x, y) = 0$  if and only if  $x = y$ . It is sufficient to show that if  $\phi(x, y) = 0$ , then  $x = y$ . From (1.4), we have  $\|x\| = \|y\|$ . This implies that  $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$ . From the definitions of  $J$ , we have  $Jx = Jy$ . That is,  $x = y$ ; see [3, 4] for more details.

Let  $C$  be a closed convex subset of  $E$ , and let  $T$  be a mapping from  $C$  into itself with nonempty set of fixed points. We denote by  $F(T)$  the set of fixed points of  $T$ .  $T$  is called hemi-relatively nonexpansive if  $\phi(p, Tx) \leq \phi(p, x)$  for all  $x \in C$  and  $p \in F(T)$ .

A point  $p$  in  $C$  is said to be an asymptotic fixed point of  $T$  if  $C$  contains a sequence  $\{x_n\}$  which converges weakly to  $p$  such that the strong  $\lim_{n \rightarrow \infty} (Tx_n - x_n) = 0$ . The set of asymptotic fixed points of  $T$  will be denoted by  $\hat{F}(T)$ . A hemi-relatively nonexpansive mapping  $T$  from  $C$  into itself is called relatively nonexpansive if  $\hat{F}(T) = F(T)$  (see, [5]).

A point  $p$  in  $C$  is said to be a strong asymptotic fixed point of  $T$  if  $C$  contains a sequence  $\{x_n\}$  which converges strongly to  $p$  such that the strong  $\lim_{n \rightarrow \infty} (Tx_n - x_n) = 0$ . The set of strong asymptotic fixed points of  $T$  will be denoted by  $\tilde{F}(T)$ . A hemi-relatively nonexpansive mapping  $T$  from  $C$  into itself is called weak relatively nonexpansive if  $\tilde{F}(T) = F(T)$  (see, [6]).

The following conclusions are obvious: (1) relatively nonexpansive mapping must be weak relatively nonexpansive mapping; (2) weak relatively nonexpansive mapping must be hemi-relatively nonexpansive mapping.

In this paper, we will give two examples to show that the inverses of above two conclusions are not hold.

In an infinite-dimensional Hilbert space, Mann's iterative algorithm has only weak convergence, in general, even for nonexpansive mappings. Hence in order to have strong convergence, in recent years, the hybrid iteration methods for approximating fixed points of nonlinear mappings has been introduced and studied by various authors.

In 2003, Nakajo and Takahashi [7] proposed the following modification of Mann iteration method for a single nonexpansive mapping  $T$  in a Hilbert space  $H$ :

$$\begin{aligned} x_0 &\in C \text{ chosen only arbitrarily,} \\ y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned} \tag{1.5}$$

where  $C$  is a closed convex subset of  $H$ , and  $P_K$  denotes the metric projection from  $H$  onto a closed convex subset  $K$  of  $H$ . They proved that if the sequence  $\{\alpha_n\}$  is bounded above from one, then the sequence  $\{x_n\}$  generated by (1.5) converges strongly to  $P_{F(T)}(x_0)$ , where  $F(T)$  denote the fixed points set of  $T$ .

The ideas to generalize the process (1.5) from Hilbert space to Banach space have recently been made. By using available properties on uniformly convex and uniformly smooth Banach space, Matsushita and Takahashi [8] presented their ideas as the following method for a single relatively nonexpansive mapping  $T$  in a Banach space  $E$ :

$$\begin{aligned} x_0 &\in C \text{ chosen only arbitrarily,} \\ y_n &= J^{-1}(\alpha_n J x_0 + (1 - \alpha_n) J T x_n), \\ C_n &= \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n}(x_0), \end{aligned} \tag{1.6}$$

where  $J$  is the duality mapping on  $E$ , and  $\Pi_K(\cdot)$  is the generalized projection from  $E$  onto a nonempty closed convex subset  $K$ . They proved the following convergence theorem.

**Theorem MT.** *Let  $E$  be a uniformly convex and uniformly smooth Banach space, let  $C$  be a nonempty closed convex subset of  $E$ , let  $T$  be a relatively nonexpansive mapping from  $C$  into itself, and let  $\{\alpha_n\}$  be a sequence of real numbers such that  $0 \leq \alpha_n < 1$  and  $\limsup_{n \rightarrow \infty} \alpha_n < 1$ . Suppose that  $\{x_n\}$  is given by (1.6), where  $J$  is the duality mapping on  $E$ . If  $F(T)$  is nonempty, then  $\{x_n\}$  converges strongly to  $\Pi_{F(T)} x_0$ , where  $\Pi_{F(T)}(\cdot)$  is the generalized projection from  $C$  onto  $F(T)$ .*

Recently, Plubtieng and Ungchittrakool [9] here proposed the following hybrid iteration method for a countable family of relatively nonexpansive mappings in a Banach space and proved the convergence theorem.

**Theorem PU.** Let  $E$  be a uniformly smooth and uniformly convex Banach space and let  $\widehat{C}$  and  $C$  be two nonempty closed convex subsets of  $E$  such that  $\widehat{C} \subset C$ . Let  $\{T_n\}$  be a sequence of relatively nonexpansive mappings from  $C$  into  $E$  such that  $\bigcap_{n=1}^{\infty} F(T_n)$  is nonempty, and let  $\{x_n\}$  be a sequence defined as follows:

$$\begin{aligned} x_0 &\in \widehat{C}, \\ C_1 &= C, \\ x_1 &= \Pi_{C_1} x_0, \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_n x_n), \\ C_{n+1} &= \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \quad n \geq 1, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \end{aligned} \tag{1.7}$$

where  $\alpha_n \in [0, 1]$  satisfies either

- (a)  $0 \leq \alpha_n < 1$  for all  $n \geq 1$  and  $\limsup_{n \rightarrow \infty} \alpha_n < 1$  or
- (b)  $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ .

Suppose that for any bounded subset  $B$  of  $C$  there exists an increasing, continuous and convex function  $h_B$  from  $\mathbb{R}^+$  into  $\mathbb{R}^+$  such that  $h_B(0) = 0$ , and

$$\lim_{l, k \rightarrow \infty} \sup \{h_B(\|T_l z - T_k z\|) : z \in B\} = 0. \tag{1.8}$$

Let  $T$  be a mapping from  $C$  into  $E$  defined by  $Tx = \lim_{n \rightarrow \infty} T_n x$  for all  $x \in C$  and suppose that

$$F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} \widehat{F}(T_n) = \widehat{F}(T). \tag{1.9}$$

Then  $\{x_n\}$ ,  $\{T_n x_n\}$ , and  $\{y_n\}$  converge strongly to  $\Pi_{F(T)} x_0$ .

In this paper, the authors have obtained the following results: (1) the definition of uniformly closed countable family of nonlinear mappings, (2) strong convergence theorem by the monotone hybrid algorithm for a countable family of hemi-relatively nonexpansive mappings in a Banach space with new method of proof, (3) two examples of uniformly closed countable families of nonlinear mappings and applications, (4) an example which is hemi-relatively nonexpansive mapping but not weak relatively nonexpansive mapping, and (5) an example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping. Therefore, the results of this paper improve and extend the results of Plubtieng and Ungchittrakool [9] and many others.

We need the following definitions and lemmas.

**Lemma 1.1** (Kamimura and Takahashi [10]). Let  $E$  be a uniformly convex and smooth Banach space, and let  $\{x_n\}$  and  $\{y_n\}$  be two sequences of  $E$ . If  $\phi(x_n, y_n) \rightarrow 0$  and either  $\{x_n\}$  or  $\{y_n\}$  is bounded, then  $x_n - y_n \rightarrow 0$ .

**Lemma 1.2** (Alber [1]). *Let  $C$  be a nonempty closed convex subset of a smooth Banach space  $E$  and  $x \in E$ . Then,  $x_0 = \Pi_C x$  if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \quad \text{for } y \in C. \quad (1.10)$$

**Lemma 1.3** (Alber [1]). *Let  $E$  be a reflexive, strictly convex, and smooth Banach space, let  $C$  be a nonempty closed convex subset of  $E$ , and let  $x \in E$ . Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \quad \forall y \in C. \quad (1.11)$$

The following lemma is not hard to prove.

**Lemma 1.4.** *Let  $E$  be a strictly convex and smooth Banach space, let  $C$  be a closed convex subset of  $E$ , and let  $T$  be a hemi-relatively nonexpansive mapping from  $C$  into itself. Then  $F(T)$  is closed and convex.*

In this paper, we present the definition of uniformly closed for a sequence of mappings as follows.

*Definition 1.5.* Let  $E$  be a Banach space,  $C$  be a closed convex subset of  $E$ , let  $\{T_n\}_{n=1}^{\infty}$  be a sequence of mappings of  $C$  into  $E$  such that  $\bigcap_{n=1}^{\infty} F(T_n)$  is nonempty. We say that  $\{T_n\}_{n=1}^{\infty}$  is uniformly closed if  $p \in \bigcap_{n=1}^{\infty} F(T_n)$  whenever  $\{x_n\} \subset C$  converges strongly to  $p$  and  $\|x_n - T_n x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Lemma 1.6** (see [11, 12]). *Let  $E$  be a  $p$ -uniformly convex Banach space with  $p \geq 2$ . Then, for all  $x, y \in E$ ,  $j(x) \in J_p(x)$ , and  $j(y) \in J_p(y)$ ,*

$$\langle x - y, j(x) - j(y) \rangle \geq \frac{c^p}{c^{p-2}p} \|x - y\|^p, \quad (1.12)$$

where  $J_p$  is the generalized duality mapping from  $E$  into  $E^*$ , and  $1/c$  is the  $p$ -uniformly convexity constant of  $E$ .

## 2. Main Results

*Definition 2.1.* Let  $E$  be a Banach space, let  $C$  be a nonempty closed convex subset of  $E$ , and let  $\{T_n\}_{n=1}^{\infty} : C \rightarrow E$ ,  $\{S_n\}_{n=1}^{\infty} : C \rightarrow E$  be two sequences of mappings. If for any convergence sequence  $\{x_n\} \subset C$ , the following holds:

$$\lim_{n \rightarrow +\infty} \|T_n x_n - S_n x_n\| = 0. \quad (2.1)$$

Then  $\{T_n\}$  and  $\{S_n\}$  are said to satisfy the only asymptotically condition.

**Theorem 2.2.** *Let  $E$  be a uniformly convex and uniformly smooth Banach space, let  $C$  be a nonempty closed convex subset of  $E$ , let  $\{T_n\}_{n=1}^{\infty} : C \rightarrow E$ ,  $\{S_n\}_{n=1}^{\infty} : C \rightarrow E$  be two uniformly closed sequences of hemi-relatively nonexpansive mappings satisfying the asymptotically condition such that*

$F_1 = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ ,  $F_2 = \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ , and  $F = F_1 \cap F_2 \neq \emptyset$ . Assume that  $\{\alpha_n\}_{n=0}^{\infty}$ ,  $\{\beta_n\}_{n=0}^{\infty}$ ,  $\{\gamma_n\}_{n=0}^{\infty}$  and  $\{\delta_n\}_{n=0}^{\infty}$  are four sequences in  $[0, 1]$  such that  $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ ,  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $0 < \gamma \leq \gamma_n \leq 1$ ,  $\delta_n \rightarrow 0$  for some constant  $\gamma \in (0, 1)$ . Define a sequence  $\{x_n\}$  in  $C$  by the following algorithm:

$$\begin{aligned} x_0 &\in C \text{ arbitrarily,} \\ y_n &= J^{-1}(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JT_n x_n + \delta_n JS_n x_n), \quad n \geq 1, \\ C_n &= \{z \in C_{n-1} : \phi(z, y_n) \leq (1 - \alpha_n)\phi(z, x_n) + \alpha_n\phi(z, x_0)\}, \quad n \geq 1, \\ C_0 &= C, \\ x_{n+1} &= \Pi_{C_n} x_0, \quad n \geq 0. \end{aligned} \tag{2.2}$$

Then  $\{x_n\}$  converges strongly to  $q = \Pi_F x_0$ .

*Proof.* We first show that  $C_n$  is closed and convex for all  $n \geq 0$ . From the definitions of  $C_n$ , it is obvious that  $C_n$  is closed for all  $n \geq 0$ . Next, we prove that  $C_n$  is convex for all  $n \geq 0$ . Since

$$\phi(z, y_n) \leq (1 - \alpha_n)\phi(z, x_n) + \alpha_n\phi(z, x_0) \tag{2.3}$$

is equivalent to

$$2\langle z, (1 - \alpha_n)Jx_n + \alpha_n Jx_0 - Jy_n \rangle \leq (1 - \alpha_n)\|x_n\|^2 + \alpha_n\|x_0\|^2 - \|y_n\|^2, \tag{2.4}$$

it is easy to get that  $C_n$  is convex for all  $n \geq 0$ .

Next, we show that  $F \subset C_n$  for all  $n \geq 1$ . Indeed, for each  $p \in F$ , we have

$$\begin{aligned} \phi(p, y_n) &= \phi\left(p, J^{-1}(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JT_n x_n + \delta_n JS_n x_n)\right) \\ &= \|p\|^2 - 2\langle p, (\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JT_n x_n + \delta_n JS_n x_n) \rangle \\ &\quad + \|\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JT_n x_n + \delta_n JS_n x_n\|^2 \\ &\leq \|p\|^2 - 2\alpha_n\langle p, Jx_0 \rangle - 2\beta_n\langle p, Jx_n \rangle - 2\gamma_n\langle p, JT_n x_n \rangle - 2\delta_n\langle p, JS_n x_n \rangle \\ &\quad + \alpha_n\|x_0\|^2 + \beta_n\|x_n\|^2 + \gamma_n\|T_n x_n\|^2 + \delta_n\|S_n x_n\|^2 \\ &\leq \alpha_n\phi(p, x_0) + \beta_n\phi(p, x_n) + \gamma_n\phi(p, T_n x_n) + \delta_n\phi(p, S_n x_n) \\ &\leq \alpha_n\phi(p, x_0) + (1 - \alpha_n)\phi(p, x_n). \end{aligned} \tag{2.5}$$

So,  $p \in C_n$ , which implies that  $F \subset C_n$  for all  $n \geq 1$ .

Since  $x_{n+1} = \Pi_{C_n} x_0$  and  $C_n \subset C_{n-1}$ , then we get the following:

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0. \tag{2.6}$$

Therefore,  $\{\phi(x_n, x_0)\}$  is nondecreasing. On the other hand, by Lemma 1.3, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_{n-1}}x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0), \quad (2.7)$$

for all  $p \in F(T) \subset C_{n-1}$  and for all  $n \geq 1$ . Therefore,  $\phi(x_n, x_0)$  is also bounded. This together with (3.1) implies that the limit of  $\{\phi(x_n, x_0)\}$  exists. Put the following:

$$\lim_{n \rightarrow \infty} \phi(x_n, x_0) = d. \quad (2.8)$$

From Lemma 1.3, we have, for any positive integer  $m$ , that

$$\begin{aligned} \phi(x_{n+m}, x_{n+1}) &= \phi(x_{n+m}, \Pi_{C_n}x_0) \leq \phi(x_{n+m}, x_0) - \phi(\Pi_{C_n}x_0, x_0) \\ &= \phi(x_{n+m}, x_0) - \phi(x_{n+1}, x_0), \end{aligned} \quad (2.9)$$

for all  $n \geq 0$ . This together with (3.6) implies that

$$\lim_{n \rightarrow \infty} \phi(x_{n+m}, x_{n+1}) = 0 \quad (2.10)$$

is, uniformly for all  $m$ , holds. By using Lemma 1.1, we get that

$$\lim_{n \rightarrow \infty} \|x_{n+m} - x_{n+1}\| = 0 \quad (2.11)$$

is, uniformly for all  $m$ , holds. Then  $\{x_n\}$  is a Cauchy sequence; therefore, there exists a point  $p \in C$  such that  $x_n \rightarrow p$ .

Since  $x_{n+1} = \Pi_{C_n}x_0 \in C_n$ , from the definition of  $C_n$ , we have the following:

$$\phi(x_{n+1}, y_n) \leq (1 - \alpha_n)\phi(x_{n+1}, x_n) + \alpha_n\phi(x_{n+1}, x_0). \quad (2.12)$$

This together with (2.10) and  $\lim_{n \rightarrow \infty} \alpha_n = 0$  implies that

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, y_n) = 0. \quad (2.13)$$

Therefore, by using Lemma 1.1, we obtain the following:

$$\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = 0. \quad (2.14)$$

Since  $J$  is uniformly norm-to-norm continuous on bounded sets, then we have the following:

$$\lim_{n \rightarrow \infty} \|Jx_{n+1} - Jy_n\| = \lim_{n \rightarrow \infty} \|Jx_{n+1} - Jx_n\| = 0. \quad (2.15)$$

Noticing that

$$\begin{aligned}
\|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JT_n x_n + \delta_n JS_n x_n)\| \\
&= \|\alpha_n (Jx_{n+1} - Jx_0) + \beta_n (Jx_{n+1} - Jx_n) + \gamma_n (Jx_{n+1} - JT_n x_n) + \delta_n (Jx_{n+1} - JS_n x_n)\| \\
&\geq \gamma_n \|Jx_{n+1} - JT_n x_n\| - \delta_n \|Jx_{n+1} - JS_n x_n\| \\
&\quad - \alpha_n \|Jx_{n+1} - Jx_0\| - \beta_n \|Jx_{n+1} - Jx_n\|,
\end{aligned} \tag{2.16}$$

which leads to

$$\begin{aligned}
\gamma_n \|Jx_{n+1} - JT_n x_n\| &\leq \|Jx_{n+1} - Jy_n\| + \alpha_n \|Jx_0 - Jx_{n+1}\| \\
&\quad + \beta_n \|Jx_{n+1} - Jx_n\| + \delta_n \|Jx_{n+1} - JS_n x_n\|.
\end{aligned} \tag{2.17}$$

From (2.15) and  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\lim_{n \rightarrow \infty} \delta_n = 0$ ,  $0 < \gamma \leq \gamma_n \leq 1$  we obtain that

$$\lim_{n \rightarrow \infty} \|Jx_{n+1} - JT_n x_n\| = 0. \tag{2.18}$$

Since  $J^{-1}$  is also uniformly norm-to-norm continuous on bounded sets, then we obtain that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - T_n x_n\| = 0. \tag{2.19}$$

This together with (2.11) implies that

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0. \tag{2.20}$$

Since  $\{T_n\}$  and  $\{S_n\}$  satisfy the asymptotically condition, we also have

$$\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0. \tag{2.21}$$

Since  $x_n \rightarrow p$  and  $\{T_n\}_{n=1}^{\infty}$ ,  $\{S_n\}_{n=1}^{\infty}$  are uniformly closed, we have

$$p \in F = \left( \bigcap_{n=1}^{\infty} F(T_n) \right) \cap \left( \bigcap_{n=1}^{\infty} F(S_n) \right). \tag{2.22}$$

Finally, we prove that  $p = \Pi_F x_0$ , from Lemma 1.3, we have

$$\phi(p, \Pi_F x_0) + \phi(\Pi_F x_0, x_0) \leq \phi(p, x_0). \tag{2.23}$$

On the other hand,  $x_{n+1} = \Pi_{C_n} x_0$  and  $F \subset C_n$ , for all  $n$ . Also from Lemma 1.3, we have

$$\phi(\Pi_F x_0, x_{n+1}) + \phi(x_{n+1}, x_0) \leq \phi(\Pi_F x_0, x_0). \tag{2.24}$$



By the definition of  $\phi(x, y)$ , we know that

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_0) = \phi(p, x_0). \quad (2.25)$$

Combining (2.24) and (2.25), we know that  $\phi(p, x_0) = \phi(\Pi_F x_0, x_0)$ . Therefore, it follows from the uniqueness of  $\Pi_F x_0$  that  $p = \Pi_F x_0$ . This completes the proof.  $\square$

When  $\alpha_n \equiv 0$ ,  $\delta_n \equiv 0$  in Theorem 2.2, we obtain the following result.

**Theorem 2.3.** *Let  $E$  be a uniformly convex and uniformly smooth Banach space, let  $C$  be a nonempty closed convex subset of  $E$ , and let  $\{T_n\}_{n=1}^{\infty} : C \rightarrow E$  be a uniformly closed sequence of hemi-relatively nonexpansive mappings such that  $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ . Assume that  $\{\alpha_n\}_{n=0}^{\infty}$  is a sequences in  $[0, 1]$  such that  $0 \leq \alpha_n \leq \alpha < 1$  for some constant  $\alpha \in (0, 1)$ . Define a sequence  $\{x_n\}$  in  $C$  by the following algorithm:*

$$\begin{aligned} x_0 &\in C \text{ arbitrarily,} \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_n x_n), \quad n \geq 1, \\ C_n &= \{z \in C_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \quad n \geq 1, \\ C_0 &= C, \\ x_{n+1} &= \Pi_{C_n} x_0, \quad n \geq 0. \end{aligned} \quad (2.26)$$

Then  $\{x_n\}$  converges strongly to  $q = \Pi_F x_0$ .

### 3. Applications for Equilibrium Problem

Let  $E$  be a real Banach space, and let  $E^*$  be the dual space of  $E$ . Let  $C$  be a closed convex subset of  $E$ . Let  $f$  be a bifunction from  $C \times C$  to  $R = (-\infty, +\infty)$ . The equilibrium problem is to find  $x \in C$  such that

$$f(x, y) \geq 0, \quad \forall y \in C. \quad (3.1)$$

The set of solutions of (1.2) is denoted by  $EP(f)$ . Given a mapping  $T : C \rightarrow E^*$  let  $f(x, y) = \langle Tx, y - x \rangle$  for all  $x, y \in C$ . Then,  $p \in EP(f)$  if and only if  $\langle Tp, y - p \rangle \geq 0$  for all  $y \in C$ , that is,  $p$  is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.2). Some methods have been proposed to solve the equilibrium problem in Hilbert spaces, see, for instance, [13–15].

For solving the equilibrium problem, let us assume that a bifunction  $f$  satisfies the following conditions:

- (A1)  $f(x, x) = 0$ , for all  $x \in E$ ,
- (A2)  $f$  is monotone, that is,  $f(x, y) + f(y, x) \leq 0$ , for all  $x, y \in E$ ,
- (A3) for all  $x, y, z \in E$ , only  $\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$ ,
- (A4) for all  $x \in C$ ,  $f(x, \cdot)$  is convex and lower semicontinuous.

**Lemma 3.1** (Blum and Oettli [13]). *Let  $C$  be a closed convex subset of a smooth, strictly convex, and reflexive Banach space  $E$ , let  $f$  be a bifunction from  $C \times C$  to  $R = (-\infty, +\infty)$  satisfying (A1)–(A4), and let  $r > 0$  and  $x \in E$ . Then, there exists  $z \in C$  such that*

$$f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \quad (3.2)$$

**Lemma 3.2** (Takahashi and Zembayashi [15]). *Let  $C$  be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space  $E$ , and let  $f$  be a bifunction from  $C \times C$  to  $R = (-\infty, +\infty)$  satisfying (A1)–(A4). For  $r > 0$ , define a mapping  $T_r : E \rightarrow C$  as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\} \quad (3.3)$$

for all  $x \in E$ . Then, the following hold:

- (1)  $T_r$  is single-valued;
- (2)  $T_r$  is a firmly nonexpansive-type mapping, that is, for all  $x, y \in E$ ,

$$\langle T_r x - T_r y, J T_r x - J T_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle, \quad (3.4)$$

- (3)  $F(T_r) = \text{EP}(f)$ ;
- (4)  $\text{EP}(f)$  is closed and convex;
- (5)  $T_r$  is also a relatively nonexpansive mapping.

**Lemma 3.3** (Takahashi and Zembayashi [15]). *Let  $C$  be a closed convex subset of a smooth, strictly convex, and reflexive Banach space  $E$ , let  $f$  be a bifunction from  $C \times C$  to  $R = (-\infty, +\infty)$  satisfying (A1)–(A4), let  $r > 0$ , and let  $x \in E$ ,  $q \in F(T_r)$ , then the following holds:*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \quad (3.5)$$

**Lemma 3.4.** *Let  $E$  be a  $p$ -uniformly convex with  $p \geq 2$  and uniformly smooth Banach space, and let  $C$  be a nonempty closed convex subset of  $E$ . Let  $f$  be a bifunction from  $C \times C$  to  $R = (-\infty, +\infty)$  satisfying (A1)–(A4). Let  $\{r_n\}$  be a positive real sequence such that  $\lim_{n \rightarrow \infty} r_n = r > 0$ . Then the sequence of mappings  $\{T_{r_n}\}$  is uniformly closed.*

*Proof.* (1) Let  $\{x_n\}$  be a convergent sequence in  $C$ . Let  $z_n = T_{r_n} x_n$  for all  $n$ , then

$$f(z_n, y) + \frac{1}{r_n} \langle y - z_n, Jz_n - Jx_n \rangle \geq 0, \quad \forall y \in C, \quad (3.6)$$

$$f(z_{n+m}, y) + \frac{1}{r_{n+m}} \langle y - z_{n+m}, Jz_{n+m} - Jx_{n+m} \rangle \geq 0, \quad \forall y \in C. \quad (3.7)$$

Putting  $y = z_{n+m}$  in (3.6) and  $y = z_n$  in (3.7), we have

$$\begin{aligned} f(z_n, z_{n+m}) + \frac{1}{r_n} \langle z_{n+m} - z_n, Jz_n - Jx_n \rangle &\geq 0, \quad \forall y \in C, \\ f(z_{n+m}, z_n) + \frac{1}{r_{n+m}} \langle z_n - z_{n+m}, Jz_{n+m} - Jx_{n+m} \rangle &\geq 0, \quad \forall y \in C. \end{aligned} \quad (3.8)$$

So, from (A2), we have

$$\left\langle z_{n+m} - z_n, \frac{Jz_n - Jx_n}{r_n} - \frac{Jz_{n+m} - Jx_{n+m}}{r_{n+m}} \right\rangle \geq 0, \quad (3.9)$$

and hence

$$\left\langle z_{n+m} - z_n, Jz_n - Jx_n - \frac{r_n}{r_{n+m}} (Jz_{n+m} - Jx_{n+m}) \right\rangle \geq 0. \quad (3.10)$$

Thus, we have

$$\left\langle z_{n+m} - z_n, Jz_n - Jz_{n+m} + Jz_{n+m} - Jx_n - \frac{r_n}{r_{n+m}} (Jz_{n+m} - Jx_{n+m}) \right\rangle \geq 0, \quad (3.11)$$

which implies that

$$\langle z_{n+m} - z_n, Jz_{n+m} - Jz_n \rangle \leq \left\langle z_{n+m} - z_n, Jz_{n+m} - Jx_n - \frac{r_n}{r_{n+m}} (Jz_{n+m} - Jx_{n+m}) \right\rangle \geq 0. \quad (3.12)$$

By using Lemma 1.6, we obtain the following:

$$\begin{aligned} \frac{c^p}{c^{p-2p}} \|z_{n+m} - z_n\|^p &\leq \left\langle z_{n+m} - z_n, Jz_{n+m} - Jx_n - \frac{r_n}{r_{n+m}} (Jz_{n+m} - Jx_{n+m}) \right\rangle \geq 0 \\ &= \left\langle z_{n+m} - z_n, \left(1 - \frac{r_n}{r_{n+m}}\right) Jz_{n+m} + \frac{r_n}{r_{n+m}} (Jx_{n+m} - Jx_n) \right\rangle. \end{aligned} \quad (3.13)$$

Therefore, we get the following:

$$\frac{c^p}{c^{p-2p}} \|z_{n+m} - z_n\|^{p-1} \leq \left| 1 - \frac{r_n}{r_{n+m}} \right| \|Jz_{n+m}\| + \left\| \frac{r_n}{r_{n+m}} Jx_{n+m} - Jx_n \right\|. \quad (3.14)$$

On the other hand, for any  $p \in \text{EP}(f)$ , from  $z_n = T_{r_n} x_n$ , we have

$$\|z_n - p\| = \|T_{r_n} x_n - p\| \leq \|x_n - p\|, \quad (3.15)$$

so that  $\{z_n\}$  is bounded. Since  $\lim_{n \rightarrow \infty} r_n = r > 0$ , this together with (3.14) implies that  $\{z_n\}$  is a Cauchy sequence. Hence  $T_{r_n}x_n = z_n$  is convergent.

(2) By using Lemma 3.2, we know that

$$\bigcap_{n=1}^{\infty} F(T_{r_n}) = \text{EP}(f) \neq \emptyset. \quad (3.16)$$

(3) From (1) we know that,  $\lim_{n \rightarrow \infty} T_{r_n}x$  exists for all  $x \in C$ . So, we can define a mapping  $T$  from  $C$  into itself by

$$Tx = \lim_{n \rightarrow \infty} T_{r_n}x, \quad \forall x \in C. \quad (3.17)$$

It is obvious that  $T$  is nonexpansive. It is easy to see that

$$\text{EP}(f) = \bigcap_{n=1}^{\infty} F(T_{r_n}) \subset F(T). \quad (3.18)$$

On the other hand, let  $w \in F(T)$  and  $w_n = T_{r_n}w$ , we have

$$f(w_n, y) + \frac{1}{r_n} \langle y - w_n, Jw_n - Jw \rangle \geq 0, \quad \forall y \in C. \quad (3.19)$$

By (A2) we know that

$$\frac{1}{r_n} \langle y - w_n, Jw_n - Jw \rangle \geq f(y, w_n), \quad \forall y \in C. \quad (3.20)$$

Since  $w_n \rightarrow Tw = w$  and from (A4), we have  $f(y, w) \leq 0$ , for all  $y \in C$ . Then, for  $t \in (0, 1]$  and  $y \in C$ ,

$$\begin{aligned} 0 &= f(ty + (1-t)w, ty + (1-t)w) \\ &\leq tf(ty + (1-t)w, y) + (1-t)f(ty + (1-t)w, w) \\ &\leq tf(ty + (1-t)w, y). \end{aligned} \quad (3.21)$$

Therefore, we have

$$f(ty + (1-t)w, y) \geq 0. \quad (3.22)$$

Letting  $t \downarrow 0$  and using (A3), we get the following:

$$f(w, y) \geq 0, \quad \forall y \in C, \quad (3.23)$$

and hence  $w \in \text{EP}(f)$ . From above two respects, we know that  $F(T) = \bigcap_{n=0}^{\infty} F(T_{r_n})$ .

Next we show  $\{T_{r_n}\}$  is uniformly closed. Assume  $x_n \rightarrow x$  and  $\|x_n - T_{r_n}x_n\| \rightarrow 0$ , from above results, we know that  $Tx = \lim_{n \rightarrow \infty} T_{r_n}x$ . On the other hand, from  $\|x_n - T_{r_n}x_n\| \rightarrow 0$ , we also get  $\lim_{n \rightarrow \infty} T_{r_n}x = x$ , so that  $x \in F(T) = \bigcap_{n=1}^{\infty} F(T_{r_n})$ . That is, the sequence of mappings  $\{T_{r_n}\}$  is uniformly closed. This completes the proof.  $\square$

**Theorem 3.5.** *Let  $E$  be a  $p$ -uniformly convex with  $p \geq 2$  and uniformly smooth Banach space, and let  $C$  be a nonempty closed convex subset of  $E$ . Let  $f$  and  $g$  be two bifunctions from  $C \times C$  to  $R = (-\infty, +\infty)$  satisfying (A1)–(A4). Assume that  $\{\alpha_n\}_{n=0}^{\infty}$ ,  $\{\beta_n\}_{n=0}^{\infty}$ ,  $\{\gamma_n\}_{n=0}^{\infty}$ , and  $\{\delta_n\}_{n=1}^{\infty}$  are four sequences in  $[0, 1]$  such that  $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ ,  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\lim_{n \rightarrow \infty} \delta_n = 0$ , and  $0 < \gamma \leq \gamma_n \leq 1$  for some constant  $\gamma \in (0, 1)$ . Let  $\{x_n\}$  be a sequence generated by*

$$\begin{aligned} x_0 &\in C \text{ arbitrarily,} \\ y_n &= J^{-1}(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JT_{r_n}x_n + \delta_n JS_{r_n}x_n), \quad n \geq 1, \\ C_n &= \{z \in C_{n-1} : \phi(z, y_n) \leq (1 - \alpha_n)\phi(z, x_n) + \alpha_n\phi(z, x_0)\}, \quad n \geq 1, \\ C_0 &= C, \\ x_{n+1} &= \Pi_{C_n}x_0, \quad n \geq 0, \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} T_r(x) &= \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C \right\}, \quad \forall x \in E, \\ S_r(x) &= \left\{ z \in C : g(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C \right\}, \quad \forall x \in E, \end{aligned} \tag{3.25}$$

and  $\lim_{n \rightarrow \infty} r_n = r > 0$ . Assume that the mappings  $T_n$  and  $S_n$  satisfy the asymptotically condition. Then  $\{x_n\}$  converges strongly to  $q = \Pi_{EP(f) \cap EP(g)}x_0$ .

*Proof.* By Lemma 3.4,  $\{T_{r_n}\}_{n=1}^{\infty}$ ,  $\{S_{r_n}\}_{n=1}^{\infty}$  are uniformly closed; therefore, by using Theorem 2.2 and Lemma 3.2, we can obtain the conclusion of Theorem 3.5. This completes the proof.  $\square$

When  $\alpha_n \equiv 0$ ,  $\delta_n \equiv 0$  in the Theorem 3.5, we obtain the following result.

**Theorem 3.6.** *Let  $E$  be a  $p$ -uniformly convex with  $p \geq 2$  and uniformly smooth Banach space, and let  $C$  be a nonempty closed convex subset of  $E$ . Let  $f$  be a bifunction from  $C \times C$  to  $R = (-\infty, +\infty)$  satisfying (A1)–(A4). Assume that  $\{\alpha_n\}_{n=0}^{\infty}$  is a sequences in  $[0, 1]$  such that  $0 \leq \alpha_n \leq \alpha < 1$  for some constant  $\alpha \in (0, 1)$ . Define a sequence  $\{x_n\}$  in  $C$  by the following algorithm:*

$$\begin{aligned} x_0 &\in C \text{ arbitrarily,} \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_{r_n}x_n), \quad n \geq 1, \\ C_n &= \{z \in C_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \quad n \geq 1, \\ C_0 &= C, \\ x_{n+1} &= \Pi_{C_n}x_0, \quad n \geq 0, \end{aligned} \tag{3.26}$$

where  $\lim_{n \rightarrow \infty} r_n = r > 0$ . Then  $\{x_n\}$  converges strongly to  $q = \Pi_{EP(f)}x_0$ .

#### 4. Applications for Maximal Monotone Operators

In this section, we apply our above results to prove some strong convergence theorem concerning maximal monotone operators in a Banach space  $E$ .

Let  $A$  be a multivalued operator from  $E$  to  $E^*$  with domain  $D(A) = \{z \in E : Az \neq \emptyset\}$  and range  $R(A) = \{z \in E : z \in D(A)\}$ . An operator  $A$  is said to be monotone if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0 \quad (4.1)$$

for each  $x_1, x_2 \in D(A)$  and  $y_1 \in Ax_1, y_2 \in Ax_2$ . A monotone operator  $A$  is said to be maximal if its graph  $G(A) = \{(x, y) : y \in Ax\}$  is not properly contained in the graph of any other monotone operator. We know that if  $A$  is a maximal monotone operator, then  $A^{-1}0$  is closed and convex. The following result is also wellknown.

**Theorem 4.1.** *Let  $E$  be a reflexive, strictly convex, and smooth Banach space, and let  $A$  be a monotone operator from  $E$  to  $E^*$ . Then  $A$  is maximal if and only if  $R(J + rA) = E^*$  for all  $r > 0$ .*

Let  $E$  be a reflexive, strictly convex, and smooth Banach space, and let  $A$  be a maximal monotone operator from  $E$  to  $E^*$ . Using Theorem 4.1 and strict convexity of  $E$ , we obtain that for every  $r > 0$  and  $x \in E$ , there exists a unique  $x_r$  such that

$$Jx \in Jx_r + rAx_r. \quad (4.2)$$

Then we can define a single-valued mapping  $J_r : E \rightarrow D(A)$  by  $J_r = (J + rA)^{-1}J$  and such a  $J_r$  is called the resolvent of  $A$ , we know that  $A^{-1}0 = F(J_r)$  for all  $r > 0$ .

**Theorem 4.2.** *Let  $E$  be a uniformly convex and uniformly smooth Banach space, let  $A$  be a maximal monotone operator from  $E$  to  $E^*$ , and let  $J_r$  be a resolvent of  $A$  for  $r > 0$ . Then for any sequence  $\{r_n\}_{n=1}^{\infty}$  such that  $\liminf_{n \rightarrow \infty} r_n > 0$ ,  $\{J_{r_n}\}_{n=1}^{\infty}$  is a uniformly closed sequence of hemi-relatively nonexpansive mappings.*

*Proof.* Firstly, we show that  $\{J_{r_n}\}_{n=1}^{\infty}$  is uniformly closed. Let  $\{z_n\} \subset E$  be a sequence such that  $z_n \rightarrow p$  and  $\lim_{n \rightarrow \infty} \|z_n - J_{r_n}z_n\| = 0$ . Since  $J$  is uniformly norm-to-norm continuous on bounded sets, we obtain that

$$\frac{1}{r_n}(Jz_n - JJ_{r_n}z_n) \rightarrow 0. \quad (4.3)$$

It follows from

$$\frac{1}{r_n}(Jz_n - JJ_{r_n}z_n) \in AJ_{r_n}z_n \quad (4.4)$$

and the monotonicity of  $A$  that

$$\left\langle w - J_{r_n}z_n, w^* - \frac{1}{r_n}(Jz_n - JJ_{r_n}z_n) \right\rangle \geq 0 \quad (4.5)$$

for all  $w \in D(A)$  and  $w^* \in Aw$ . Letting  $n \rightarrow \infty$ , we have  $\langle w - p, w^* \rangle \geq 0$  for all  $w \in D(A)$  and  $w^* \in Aw$ . Therefore, from the maximality of  $A$ , we obtain that  $p \in A^{-1}0 = F(J_{r_n})$  for all  $n \geq 1$ , that is,  $p \in \bigcap_{n=1}^{\infty} F(J_{r_n})$ .

Next we show that  $J_{r_n}$  is a hemi-relatively nonexpansive mapping for all  $n \geq 1$ . For any  $w \in E$  and  $p \in F(J_{r_n}) = A^{-1}0$ , from the monotonicity of  $A$ , we have

$$\begin{aligned}
 \phi(p, J_{r_n}w) &= \|p\|^2 - 2\langle p, JJ_{r_n}w \rangle + \|J_{r_n}w\|^2 \\
 &= \|p\|^2 + 2\langle p, Jw - JJ_{r_n}w - Jw \rangle + \|J_{r_n}w\|^2 \\
 &= \|p\|^2 + 2\langle p, Jw - JJ_{r_n}w \rangle - 2\langle p, Jw \rangle + \|J_{r_n}w\|^2 \\
 &= \|p\|^2 - 2\langle J_{r_n}w - p - J_{r_n}w, Jw - JJ_{r_n}w \rangle - 2\langle p, Jw \rangle + \|J_{r_n}w\|^2 \\
 &= \|p\|^2 - 2\langle J_{r_n}w - p, Jw - JJ_{r_n}w \rangle \\
 &\quad + 2\langle J_{r_n}w, Jw - JJ_{r_n}w \rangle - 2\langle p, Jw \rangle + \|J_{r_n}w\|^2 \\
 &\leq \|p\|^2 + 2\langle J_{r_n}w, Jw - JJ_{r_n}w \rangle - 2\langle p, Jw \rangle + \|J_{r_n}w\|^2 \\
 &= \|p\|^2 - 2\langle p, Jw \rangle + \|w\|^2 - \|J_{r_n}w\|^2 + 2\langle J_{r_n}w, Jw \rangle - \|w\|^2 \\
 &= \phi(p, w) - \phi(J_{r_n}w, w) \\
 &\leq \phi(p, w).
 \end{aligned} \tag{4.6}$$

This implies that  $J_{r_n}$  is a hemi-relatively nonexpansive mapping for all  $n \geq 1$ . This completes the proof.  $\square$

**Theorem 4.3.** *Let  $E$  be a uniformly convex and uniformly smooth Banach space, let  $A$  and  $B$  be two maximal monotone operators from  $E$  to  $E^*$  with nonempty common zero point set  $A^{-1}(0) \cap B^{-1}(0)$ , let  $J_r^A$  be a resolvent of  $A$  and  $J_r^B$  a resolvent of  $B$  for  $r > 0$ . Assume that  $\{\alpha_n\}_{n=0}^{\infty}$ ,  $\{\beta_n\}_{n=0}^{\infty}$ ,  $\{\gamma_n\}_{n=0}^{\infty}$ , and  $\{\delta_n\}_{n=1}^{\infty}$  are four sequences in  $[0, 1]$  such that  $\alpha_n + \beta_n + \gamma_n = 1$ ,  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\lim_{n \rightarrow \infty} \delta_n = 0$ , and  $0 < \gamma \leq \gamma_n \leq 1$  for some constant  $\gamma \in (0, 1)$ . Let  $\{x_n\}$  be a sequence generated by*

$$\begin{aligned}
 &x_0 \in C \text{ arbitrarily,} \\
 &y_n = J^{-1}\left(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JJ_{r_n}^A x_n + \delta_n JJ_{r_n}^B x_n\right), \quad n \geq 1, \\
 &C_n = \{z \in C_{n-1} : \phi(z, y_n) \leq (1 - \alpha_n)\phi(z, x_n) + \alpha_n\phi(z, x_0)\}, \quad n \geq 1, \\
 &C_0 = C, \\
 &x_{n+1} = \Pi_{C_n} x_0, \quad n \geq 0,
 \end{aligned} \tag{4.7}$$

where  $\liminf_{n \rightarrow \infty} r_n > 0$ . Assume that the mappings  $J_{r_n}^A$  and  $J_{r_n}^B$  satisfy the asymptotically condition. Then  $\{x_n\}$  converges strongly to  $q = \Pi_{A^{-1}(0) \cap B^{-1}(0)} x_0$ .

*Proof.* From Theorem 4.2,  $\{J_{r_n}\}_{n=1}^{\infty}$  is uniformly closed countable family of hemi-relatively nonexpansive mappings, on the other hand,  $A^{-1}(0) = \bigcap_{n=1}^{\infty} F(J_{r_n})$ , by using Theorem 2.2, we can obtain the conclusion of Theorem 4.3.  $\square$

When  $\alpha_n \equiv 0, \delta_n \equiv 0$  in Theorem 4.3, we obtain the following result.

**Theorem 4.4.** *Let  $E$  be a uniformly convex and uniformly smooth Banach space, let  $A$  be a maximal monotone operator from  $E$  to  $E^*$  with nonempty zero point set  $A^{-1}(0)$ , and let  $J_r$  be a resolvent of  $A$  for  $r > 0$ . Assume that  $\{\alpha_n\}_{n=0}^{\infty}$  is a sequence in  $[0, 1]$  such that  $0 \leq \alpha_n \leq \alpha < 1$  for some constant  $\alpha \in (0, 1)$ . Define a sequence  $\{x_n\}$  in  $C$  by the following algorithm:*

$$\begin{aligned} x_0 &\in C \text{ arbitrarily,} \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JJ_{r_n}x_n), \quad n \geq 1, \\ C_n &= \{z \in C_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \quad n \geq 1, \\ C_0 &= C, \\ x_{n+1} &= \Pi_{C_n}x_0, \quad n \geq 0, \end{aligned} \tag{4.8}$$

where  $\liminf_{n \rightarrow \infty} r_n > 0$ . Then  $\{x_n\}$  converges strongly to  $q = \Pi_{A^{-1}(0)}x_0$ .

## 5. Examples

Firstly, we give an example which is hemi-relatively nonexpansive mapping but not weak relatively nonexpansive mapping.

*Example 5.1.* Let  $E = \mathbb{R}^n$  and  $x_0 \neq 0$  be a any element of  $E$ . We define a mapping  $T : E \rightarrow E$  as follows:

$$T(x) = \begin{cases} \left(\frac{1}{2} + \frac{1}{2^{n+1}}\right)x_0 & \text{if } x = \left(\frac{1}{2} + \frac{1}{2^n}\right)x_0, \\ -x & \text{if } x \neq \left(\frac{1}{2} + \frac{1}{2^n}\right)x_0, \end{cases} \tag{5.1}$$

for  $n = 1, 2, 3, \dots$  Next we show that  $T$  is a hemi-relatively nonexpansive mapping but no weak relatively nonexpansive mapping. First, it is obvious that  $F(T) = \{0\}$ . In addition, it is easy to see that

$$\|Tx\| \leq \|x\|, \quad \forall x \in E. \tag{5.2}$$



This implies that

$$\|Tx\|^2 - \|x\|^2 \leq 2\langle 0, JTx - Jx \rangle = 2\langle p, JTx - Jx \rangle \quad (5.3)$$

for all  $p \in F(T)$ . It follows from above inequality that

$$\|p\|^2 - 2\langle p, JTx \rangle + \|Tx\|^2 \leq \|p\|^2 - 2\langle p, Jx \rangle + \|x\|^2, \quad (5.4)$$

for all  $p \in F(T)$  and  $x \in E$ . That is

$$\phi(p, Tx) \leq \phi(p, x), \quad (5.5)$$

for all  $p \in F(T)$  and  $x \in E$ ; hence  $T$  is a hemi-relatively nonexpansive mapping. Finally, we show that  $T$  is not weak relatively nonexpansive mapping. In fact that, letting

$$x_n = \left(\frac{1}{2} + \frac{1}{2^n}\right)x_0, \quad n = 1, 2, 3, \dots \quad (5.6)$$

from the definition of  $T$ , we have

$$Tx_n = \left(\frac{1}{2} + \frac{1}{2^{n+1}}\right)x_0, \quad n = 1, 2, 3, \dots \quad (5.7)$$

which implies that  $\|x_n - Tx_n\| \rightarrow 0$  and  $x_n \rightarrow x_0$  ( $x_n \rightharpoonup x_0$ ) as  $n \rightarrow \infty$ . That is  $x_0 \in \tilde{F}(T)$  but  $x_0 \notin F(T)$ .

Next, we give an example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping.

*Example 5.2.* Let  $E = l^2$ , where

$$l^2 = \left\{ \xi = (\xi_1, \xi_2, \xi_3, \dots, \xi_n, \dots) : \sum_{n=1}^{\infty} |\xi_n|^2 < \infty \right\},$$

$$\|\xi\| = \left( \sum_{n=1}^{\infty} |\xi_n|^2 \right)^{1/2}, \quad \forall \xi \in l^2, \quad (5.8)$$

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \quad \forall \xi = (\xi_1, \xi_2, \xi_3, \dots, \xi_n, \dots), \eta = (\eta_1, \eta_2, \eta_3, \dots, \eta_n, \dots) \in l^2.$$

It is well known that  $l^2$  is a Hilbert space, so that  $(l^2)^* = l^2$ . Let  $\{x_n\} \subset E$  be a sequence defined by

$$\begin{aligned} x_0 &= (1, 0, 0, 0, \dots), \\ x_1 &= (1, 1, 0, 0, \dots), \\ x_2 &= (1, 0, 1, 0, 0, \dots), \\ x_3 &= (1, 0, 0, 1, 0, 0, \dots), \\ &\vdots \\ x_n &= (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \dots, \xi_{n,k}, \dots), \end{aligned} \tag{5.9}$$

where

$$\xi_{n,k} = \begin{cases} 1 & \text{if } k = 1, n + 1, \\ 0 & \text{if } k \neq 1, k \neq n + 1, \end{cases} \tag{5.10}$$

for all  $n \geq 1$ . Define a mapping  $T : E \rightarrow E$  as follows:

$$T(x) = \begin{cases} \frac{n}{n+1}x_n & \text{if } x = x_n \ (\exists n \geq 1), \\ -x & \text{if } x \neq x_n \ (\forall n \geq 1). \end{cases} \tag{5.11}$$

*Conclusion 1.*  $\{x_n\}$  converges weakly to  $x_0$ .

*Proof.* For any  $f = (\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_k, \dots) \in l^2 = (l^2)^*$ , we have

$$f(x_n - x_0) = \langle f, x_n - x_0 \rangle = \sum_{k=2}^{\infty} \zeta_k \xi_{n,k} = \zeta_{n+1} \longrightarrow 0, \tag{5.12}$$

as  $n \rightarrow \infty$ . That is,  $\{x_n\}$  converges weakly to  $x_0$ . □

*Conclusion 2.*  $\{x_n\}$  is not a Cauchy sequence, so that, it does not converge strongly to any element of  $l^2$ .

*Proof.* In fact, we have  $\|x_n - x_m\| = \sqrt{2}$  for any  $n \neq m$ . Then  $\{x_n\}$  is not a Cauchy sequence. □

*Conclusion 3.*  $T$  has a unique fixed point  $0$ , that is,  $F(T) = \{0\}$ .

*Proof.* The conclusion is obvious. □

*Conclusion 4.*  $x_0$  is an asymptotic fixed point of  $T$ .

*Proof.* Since  $\{x_n\}$  converges weakly to  $x_0$  and

$$\|Tx_n - x_n\| = \left\| \frac{n}{n+1}x_n - x_n \right\| = \frac{1}{n+1}\|x_n\| \rightarrow 0 \quad (5.13)$$

as  $n \rightarrow \infty$ , so that,  $x_0$  is an asymptotic fixed point of  $T$ .  $\square$

*Conclusion 5.*  $T$  has a unique strong asymptotic fixed point  $0$ , so that  $F(T) = \tilde{F}(T)$ .

*Proof.* In fact that, for any strong convergent sequence  $\{z_n\} \subset E$  such that  $z_n \rightarrow z_0$  and  $\|z_n - Tz_n\| \rightarrow 0$  as  $n \rightarrow \infty$ , from Conclusion 2, there exists sufficiently large nature number  $N$  such that  $z_n \neq x_m$ , for any  $n, m > N$ . Then  $Tz_n = -z_n$  for  $n > N$ , and it follows from  $\|z_n - Tz_n\| \rightarrow 0$  that  $2z_n \rightarrow 0$  and hence  $z_n \rightarrow z_0 = 0$ .  $\square$

*Conclusion 6.*  $T$  is a weak relatively nonexpansive mapping.

*Proof.* Since  $E = l^2$  is a Hilbert space, we have

$$\phi(0, Tx) = \|0 - Tx\|^2 = \|Tx\|^2 \leq \|x\|^2 = \|x - 0\|^2 = \phi(0, x), \quad \forall x \in E. \quad (5.14)$$

From Conclusion 2, we have  $F(T) = \tilde{F}(T)$ , then  $T$  is a weak relatively nonexpansive mapping.  $\square$

*Conclusion 7.*  $T$  is not a relatively nonexpansive mapping.

*Proof.* From Conclusions 3 and 4, we have  $F(T) \neq \hat{F}(T)$ , so that  $T$  is not a relatively nonexpansive mapping.  $\square$

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