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Research Article

An Efficient Method for Systems of Variable Coefficient Coupled Burgers' Equation with Time-Fractional Derivative

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A new homotopy perturbation method (NHPM) is applied to system of variable coefficient coupled Burgers' equation with time-fractional derivative. The fractional derivatives are described in the Caputo fractional derivative sense. The concept of new algorithm is introduced briefly, and NHPM is examined for two systems of nonlinear Burgers' equation. In this approach, the solution is considered as a power series expansion that converges rapidly to the nonlinear problem. The new approximate analytical procedure depends on two iteratives. The modified algorithm provides approximate solutions in the form of convergent series with easily computable components. Results indicate that the introduced method is promising for solving other types of systems of nonlinear fractional-order partial differential equations.

1. Introduction

In recent years, the differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, medical sciences, biological research, as well as various chemical, biochemical, and physical fields, viscoelasticity, biology, physics, and engineering. Consequently, considerable attention has been given to the solutions of fractional differential equations and integral equations of physical interest [1–4]. Various powerful methods have been presented so far such as homotopy perturbation method [5, 6], variational iteration method [7], differential transform method [8], homotopy analysis method [9], and Adomian decomposition method [10, 11] for solving different kinds of fractional partial differential equations. In this paper, we construct the solution of a system of variable coefficient coupled Burgers' equation with time-fractional derivative by extending the idea of [12, 13]. A new version of homotopy perturbation method is proposed, which we called it NHPM and then applied it to the nonlinear systems of variable coefficient coupled Burgers' equation with

time-fractional derivative that can be written as the following basic form:

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= r_1(t) \frac{\partial^2 u}{\partial x^2} + s_1(t) u \frac{\partial u}{\partial x} + p_1(t) \frac{\partial(uv)}{\partial x}, \\ \frac{\partial^\alpha v}{\partial t^\alpha} &= r_2(t) \frac{\partial^2 v}{\partial x^2} + s_2(t) v \frac{\partial v}{\partial x} + p_2(t) \frac{\partial(uv)}{\partial x},\end{aligned}\quad (1)$$

subject to the initial condition

$$u(x, 0) = f(x), \quad v(x, 0) = g(x), \quad (2)$$

where the subscripts $r_1(t)$, $r_2(t)$, $s_1(t)$, $s_2(t)$, $p_1(t)$, and $p_2(t)$ are arbitrary smooth functions of t .

The paper is organized as follows. In Section 2, we begin with an introduction to some necessary definitions of fractional calculus theory. In Section 3, we illustrated a basic idea of the new method. In Section 4, the uses of the new method for solving nonlinear variable coefficient coupled Burgers' equation are presented. Two examples are solved by the proposed method in this section. Conclusion will appear in Section 5.

2. Fractional Calculus

We give some basic definitions and properties of the fractional calculus theory used in this work. Some of these are Riemann-Liouville, Grunwald-Letnikov, Caputo, and generalized functions approach. The most commonly used definitions are the Riemann-Liouville and Caputo derivatives.

Definition 1. The Riemann-Liouville fractional integral operator J^μ of order μ on the usual Lebesgue space $L_1[a, b]$ is given by

$$J^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t) dt, \quad \mu > 0, \quad (3)$$

$$J^0 f(x) = f(x).$$

It has the following properties:

- (i) J^μ exists for any $x \in [a, b]$,
- (ii) $J^\mu J^\beta = J^{\mu+\beta}$,
- (iii) $J^\mu J^\beta = J^\beta J^\mu$,
- (iv) $J^\mu J^\beta f(x) = J^\beta J^\mu f(x)$,
- (v) $J^\mu (x-a)^\gamma = (\Gamma(\gamma+1)/\Gamma(\mu+\gamma+1))(x-a)^{\mu+\gamma}$,

where $f \in L_1[a, b]$, $\mu, \beta \geq 0$, and $\gamma > -1$.

The Riemann-Liouville fractional derivative is mostly used by mathematicians, but this approach is not suitable for physical problems of the real world since it requires the definition of fractional order initial conditions which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer-order initial conditions for fractional order differential equations.

Definition 2. The Caputo definition of fractal derivative operator is given by

$$D^\mu f(x) = J^{m-\mu} D^m f(x)$$

$$= \frac{1}{\Gamma(m-\mu)} \int_0^t (x-\tau)^{m-\mu-1} f^m(\tau) d\tau, \quad (4)$$

where $m-1 < \mu \leq m$, $m \in \mathbb{N}$, $x > 0$.

Lemma 3. If $m-1 < \mu \leq m$, $m \in \mathbb{N}$, and $f \in L_1[a, b]$, then

$$D^\mu J^\mu f(x) = f(x),$$

$$J^\mu D^\mu f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0^+) \frac{(x-a)^k}{k!}, \quad (5)$$

$$x > 0.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we have considered some systems of linear and nonlinear FPDEs, where fractional derivatives are taken in Caputo sense as follows.

Definition 4. For n to be the smallest integer that exceeds α , the Caputo time-fractional derivative operator of $\alpha > 0$ is defined as

$$D_t^\alpha u(x, t)$$

$$= \frac{\partial^\alpha u(x, t)}{\partial t^\alpha}$$

$$= \begin{cases} \frac{1}{\Gamma(n-\alpha)} \times \int_0^t (x-\tau)^{n-\alpha-1} \frac{\partial^n u(x, \tau)}{\partial \tau^n} d\tau, & \text{for } n-1 < \alpha < n \\ \frac{\partial^n u(x, \tau)}{\partial \tau^n}, & \text{for } \alpha = n \in \mathbb{N}. \end{cases} \quad (6)$$

3. Analysis of New Homotopy Perturbation Method

Let us consider the system of nonlinear fractional differential equations

$$D_t^\alpha u_i(x, t) = A_i(u_i) + f_i(t, x), \quad (7)$$

$$x, t \in \Omega, \quad i = 1, 2, \dots, n,$$

with the following initial conditions:

$$u_i(x, 0) = \alpha_i, \quad i = 1, 2, \dots, n, \quad (8)$$

where A_i are the operators, f_i are known functions, and u_i are sought functions. Assume that operators A_i can be written as

$$A_i(u_i) = L_i(u_i) + N_i(u_i), \quad (9)$$

where L_i are the linear operators and N_i are the nonlinear operators. Hence, (7) can be rewritten as follows:

$$D_t^\alpha u_i(x, t) = L_i(u_i) + N_i(u_i) + f_i(x, t). \quad (10)$$

For solving system (7) by NHPM, we construct the following homotopy:

$$H(U_i; p) = (1-p)(D_t^\alpha U_i(x, t) - u_{i,0})$$

$$+ p(D_t^\alpha u_i(x, t) - L_i(U_i) - N_i(U_i) - f_i(t, x))$$

$$= 0, \quad (11)$$

where $p \in [0, 1]$ is an embedding or homotopy parameter, $H(t, x; p) : \Omega \times [0, 1] \rightarrow R$, and $u_{i,0}$ are the initial approximation of solution of the problem in (10).

Clearly, the homotopy equations $H(U_i : 0) = 0$ and $H_i(U_i : 1) = 0$ are equivalent to the equations $D_t^\alpha U_i(x, t) - u_{i,0} = 0$ and $D_t^\alpha U_i(x, t) - L_i(U_i) + N_i(U_i) + f_i(t, x) = 0$, respectively. Thus, a monotonous change of parameter p from zero to one corresponds to a continuous change of the trivial problem $D_t^\alpha U_i(x, t) - u_{i,0} = 0$ to the original problem. Next, we

assume that the solution of equation $H(U_i, p)$ can be written as a power series in embedding parameter p as follows:

$$U_i = U_{i,0} + pU_{i,1}, \quad i = 1, 2, \dots, n. \quad (12)$$

Now, let us write (12) in the following form:

$$D_t^\alpha U_i(x, t) = u_{i,0} + p(L_i(U_i) + N_i(U_i) + f_i(t, x)). \quad (13)$$

Applying the inverse operator, J_t^α , which is the Riemann-Liouville fractional integral of order $\alpha \geq 0$, on both sides of (13), we have

$$\begin{aligned} U_i(x, t) &= U_i(x, 0) + J_t^\alpha u_{i,0} + pJ_t^\alpha (L_i(U_i) + N_i(U_i) + f_i(t, x)). \end{aligned} \quad (14)$$

Suppose that the initial approximation of (10) has the form

$$u_{i,0}(x, t) = \sum_{n=0}^{\infty} a_{i,n}(x) p_n(t), \quad i = 1, 2, \dots, n, \quad (15)$$

where $a_{i,n}(x)$, $n = 0, 1, 2, \dots$ are unknown coefficients and $p_n(t)$, $n = 0, 1, 2, \dots$ are specific functions on the problem. By substituting (12) and (15) into (14), we get

$$\begin{aligned} &U_{i,0} + pU_{i,1} \\ &= U_i(x, 0) + J_t^\alpha \left(\sum_{n=0}^{\infty} a_{i,n}(x) p_n(t) \right) \\ &\quad + pJ_t^\alpha (L_i(U_{i,0} + pU_{i,1}) + N_i(U_{i,0} + pU_{i,1}) + f_i(t, x)). \end{aligned} \quad (16)$$

Equating the coefficients of like powers of p , we get the following set of equations:

$$\begin{aligned} \text{coefficient of } p^0 : U_{i,0}(x, t) &= U_i(x, 0) + J_t^\alpha \left(\sum_{n=0}^{\infty} a_{i,n}(x) p_n(t) \right), \\ \text{coefficient of } p^1 : U_{i,1}(x, t) &= J_t^\alpha (L_i(U_{i,0}) + N_i(U_{i,0}) + f_i(t, x)). \end{aligned} \quad (17)$$

Now, we solve these equations in such a way that $U_{i,1}(x, t) = 0$. Therefore, the approximate solution may be obtained as

$$\begin{aligned} u_i(x, t) &= U_{i,0}(x, t) \\ &= U_i(x, 0) + J_t^\alpha \left(\sum_{n=0}^{\infty} a_{i,n}(x) p_n(t) \right). \end{aligned} \quad (18)$$

4. Examples

In this section, to illustrate the method and to show the ability of the method, two examples are presented.

Example 1. Consider the following variable coefficient coupled Burgers' equation:

$$\begin{aligned} D_t^\alpha u(x, t) &= \frac{t}{1-t} \frac{\partial^2 u(x, t)}{\partial x^2} - u(x, t) \frac{\partial u(x, t)}{\partial x} \\ &\quad + \frac{1+t}{1-t} \frac{\partial (u(x, t) v(x, t))}{\partial x}, \\ D_t^\alpha v(x, t) &= \frac{t}{1+t} \frac{\partial^2 v(x, t)}{\partial x^2} + v(x, t) \frac{\partial v(x, t)}{\partial x} \\ &\quad - \frac{1-t}{1+t} \frac{\partial (u(x, t) v(x, t))}{\partial x}, \\ &t \neq -1, 1, \end{aligned} \quad (19)$$

subject to the initial condition

$$u(x, 0) = v(x, 0) = x. \quad (20)$$

The exact solutions of (19) for the special case $\alpha = 1$ are $u(x, t) = x/(1-t)$ and $v(x, t) = x/(1+t)$.

To obtain the solution of (19) by NHPM, we construct the following homotopy:

$$\begin{aligned} &(1-p)(D_t^\alpha U(x, t) - u_0(x, t)) \\ &\quad + p \left(D_t^\alpha U(x, t) - \frac{t}{1-t} \frac{\partial^2 U(x, t)}{\partial x^2} \right. \\ &\quad \left. + U(x, t) \frac{\partial U(x, t)}{\partial x} - \frac{1+t}{1-t} \frac{\partial (U(x, t) V(x, t))}{\partial x} \right) \\ &= 0, \\ &(1-p)(D_t^\alpha V(x, t) - v_0(x, t)) \\ &\quad + p \left(D_t^\alpha V(x, t) - \frac{t}{1+t} \frac{\partial^2 V(x, t)}{\partial x^2} \right. \\ &\quad \left. - V(x, t) \frac{\partial V(x, t)}{\partial x} + \frac{1-t}{1+t} \frac{\partial (U(x, t) V(x, t))}{\partial x} \right) \\ &= 0. \end{aligned} \quad (21)$$

Applying the inverse operator J_t^α of D_t^α on both sides of the above equation, we obtain

$$\begin{aligned} U(x, t) &= U(x, 0) + J_t^\alpha u_0(x, t) \\ &\quad - pJ_t^\alpha \left(u_0(x, t) - \frac{t}{1-t} \frac{\partial^2 U(x, t)}{\partial x^2} \right. \\ &\quad \left. + U(x, t) \frac{\partial U(x, t)}{\partial x} \right. \\ &\quad \left. - \frac{1+t}{1-t} \frac{\partial (U(x, t) V(x, t))}{\partial x} \right), \end{aligned}$$

$$\begin{aligned}
 V(x, t) = & V(x, 0) + J_t^\alpha v_0(x, t) \\
 & - p J_t^\alpha \left(v_0(x, t) - \frac{t}{1+t} \frac{\partial^2 V(x, t)}{\partial x^2} \right. \\
 & \quad \left. - V(x, t) \frac{\partial V(x, t)}{\partial x} \right. \\
 & \quad \left. + \frac{1-t}{1+t} \frac{\partial (U(x, t) V(x, t))}{\partial x} \right). \quad (22)
 \end{aligned}$$

For solving system (22), by new homotopy perturbation method, we use the Taylor series of

$$\begin{aligned}
 \frac{1}{1-t} &= \sum_{n=0}^{\infty} t^n, \\
 \frac{1}{1+t} &= \sum_{n=0}^{\infty} (-1)^n t^n. \quad (23)
 \end{aligned}$$

The solution of (19) has the following form:

$$\begin{aligned}
 U(x, t) &= U_0(x, t) + p U_1(x, t), \\
 V(x, t) &= V_0(x, t) + p V_1(x, t). \quad (24)
 \end{aligned}$$

Substituting (23) and (24) in (22) and equating the coefficients of like powers of p , we get the following set of equations:

$$\begin{aligned}
 U_0(x, t) &= U(x, 0) + J_t^\alpha u_0(x, t), \\
 V_0(x, t) &= V(x, 0) + J_t^\alpha v_0(x, t), \\
 U_1(x, t) &= J_t^\alpha \left(-u_0(x, t) + t \sum_{n=0}^{\infty} t^n \frac{\partial^2 U_0(x, t)}{\partial x^2} \right. \\
 &\quad \left. - U_0(x, t) \frac{\partial U_0(x, t)}{\partial x} \right. \\
 &\quad \left. + (1+t) \sum_{n=0}^{\infty} t^n \frac{\partial (U_0(x, t) V_0(x, t))}{\partial x} \right), \\
 V_1(x, t) &= J_t^\alpha \left(-v_0(x, t) + t \sum_{n=0}^{\infty} (-1)^n t^n \frac{\partial^2 V_0(x, t)}{\partial x^2} \right. \\
 &\quad \left. + V_0(x, t) \frac{\partial V_0(x, t)}{\partial x} - (1-t) \right. \\
 &\quad \left. \times \sum_{n=0}^{\infty} (-1)^n t^n \frac{\partial (U_0(x, t) V_0(x, t))}{\partial x} \right). \quad (25)
 \end{aligned}$$

Assuming $u_0(x, t) = \sum_{n=0}^{\infty} a_n(x) p_n(t)$, $v_0(x, t) = \sum_{n=0}^{\infty} b_n(x) p_n(t)$, $p_n(t) = t^{n\alpha}$, $U(x, 0) = u(x, 0)$, and $V(x, 0) = v(x, 0)$ and solving the above equation for $U_1(x, t)$ and $V_1(x, t)$ lead to the result

$$\begin{aligned}
 U_1(x, t) &= (x - a_0(x)) \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
 &\quad + \left(-a_1(x) + b_0(x) + x \frac{db_0(x)}{dx} + 4x \right) \\
 &\quad \times \frac{\Gamma(\alpha + 1) t^{2\alpha}}{\Gamma(2\alpha + 1)} \\
 &\quad + \left(-a_2(x) - a_0(x) \frac{da_0(x)}{dx} + \frac{d^2 a_0(x)}{dx^2} \right. \\
 &\quad \left. + \frac{1}{2} b_1(x) + \frac{1}{2} x \frac{db_1(x)}{dx} + \frac{da_0(x)}{dx} b_0(x) \right. \\
 &\quad \left. + a_0(x) \frac{db_0(x)}{dx} + 2a_0(x) + 2x \frac{da_0(x)}{dx} \right. \\
 &\quad \left. + 2b_0(x) + 2x \left(\frac{db_0(x)}{dx} \right) + 4x \right) \\
 &\quad \times \frac{\Gamma(2\alpha + 1) t^{3\alpha}}{\Gamma(3\alpha + 1)} \\
 &\quad + \left(4x + 2a_0(x) + \dots + 2x \frac{db_0(x)}{dx} \right) \\
 &\quad \times \frac{\Gamma(3\alpha + 1) t^{4\alpha}}{\Gamma(4\alpha + 1)} + \dots, \\
 V_1(x, t) &= (-x - b_0(x)) \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
 &\quad + \left(-b_1(x) - a_0(x) - x \frac{da_0(x)}{dx} + 4x \right) \\
 &\quad \times \frac{\Gamma(\alpha + 1) t^{2\alpha}}{\Gamma(2\alpha + 1)} \\
 &\quad + \left(-b_2(x) + b_0(x) \frac{db_0(x)}{dx} - \frac{d^2 a_0(x)}{dx^2} \right. \\
 &\quad \left. - \frac{1}{2} a_1(x) - \frac{1}{2} x \frac{da_1(x)}{dx} - \frac{da_0(x)}{dx} b_0(x) \right. \\
 &\quad \left. - a_0(x) \frac{db_0(x)}{dx} + 2a_0(x) + 2x \left(\frac{da_0(x)}{dx} \right) \right. \\
 &\quad \left. + 2b_0(x) + 2x \left(\frac{db_0(x)}{dx} \right) - 4x \right) \\
 &\quad \times \frac{\Gamma(2\alpha + 1) t^{3\alpha}}{\Gamma(3\alpha + 1)} \\
 &\quad + \left(4x - 2a_0(x) + \dots - 2x \left(\frac{db_0(x)}{dx} \right) \right) \\
 &\quad \times \frac{\Gamma(3\alpha + 1) t^{4\alpha}}{\Gamma(4\alpha + 1)} + \dots. \quad (26)
 \end{aligned}$$

Vanishing $U_1(x, t)$ and $V_1(x, t)$ lets the coefficients $a_i, b_i, i = 0, 1, 2, \dots$ have the following values:

$$\begin{aligned}
a_0(x) &= x, & a_1(x) &= 2x, & a_2(x) &= 3x, \\
a_3(x) &= 4x, & a_4(x) &= 5x, & a_5(x) &= 6x, \dots, \\
b_0(x) &= -x, & b_1(x) &= 2x, & b_2(x) &= -3x, \\
b_3(x) &= 4x, & b_4(x) &= -5x, & b_5(x) &= 6x.
\end{aligned}
\tag{27}$$

Therefore, we obtain the solutions of (19) as

$$\begin{aligned}
u(x, t) &= x + x \frac{t^\alpha}{\Gamma(\alpha + 1)} + 2x \frac{\Gamma(\alpha + 1) t^{2\alpha}}{\Gamma(2\alpha + 1)} \\
&\quad + 3x \frac{\Gamma(2\alpha + 1) t^{3\alpha}}{\Gamma(3\alpha + 1)} + 4x \frac{\Gamma(3\alpha + 1) t^{4\alpha}}{\Gamma(4\alpha + 1)} + \dots \\
&= x \left(1 + \sum_{n=1}^{\infty} \frac{n \Gamma((n-1)\alpha + 1) t^{n\alpha}}{\Gamma(n\alpha + 1)} \right), \\
v(x, t) &= x - x \frac{t^\alpha}{\Gamma(\alpha + 1)} + 2x \frac{\Gamma(\alpha + 1) t^{2\alpha}}{\Gamma(2\alpha + 1)} \\
&\quad - 3x \frac{\Gamma(2\alpha + 1) t^{3\alpha}}{\Gamma(3\alpha + 1)} + 4x \frac{\Gamma(3\alpha + 1) t^{4\alpha}}{\Gamma(4\alpha + 1)} - \dots \\
&= x \left(1 + \sum_{n=1}^{\infty} \frac{n(-1)^n \Gamma((n-1)\alpha + 1) t^{n\alpha}}{\Gamma(n\alpha + 1)} \right).
\end{aligned}
\tag{28}$$

If we put $\alpha \rightarrow 1$ in (28) or solve (19) with $\alpha = 1$, we obtain the exact solution

$$\begin{aligned}
u(x, t) &= x (1 + t + t^2 + t^3 + \dots) \\
&= \frac{x}{1-t}, \\
v(x, t) &= x (1 - t + t^2 - t^3 + \dots) \\
&= \frac{x}{1+t}.
\end{aligned}
\tag{29}$$

Example 2. Consider the following variable coefficient coupled Burgers' equation:

$$\begin{aligned}
D_t^\alpha u(x, t) &= -\frac{\partial^2 u(x, t)}{\partial x^2} + 2e^{2t} u(x, t) \frac{\partial u(x, t)}{\partial x} \\
&\quad - \sin(2t) \frac{\partial(u(x, t) v(x, t))}{\partial x}, \\
D_t^\alpha v(x, t) &= \frac{\partial^2 v(x, t)}{\partial x^2} - 2e^{-2t} \cos(2t) v(x, t) \frac{\partial v(x, t)}{\partial x} \\
&\quad + \cos(2t) \frac{\partial(u(x, t) v(x, t))}{\partial x},
\end{aligned}
\tag{30}$$

subject to the initial condition

$$u(x, 0) = v(x, 0) = e^x. \tag{31}$$

The exact solution for $\alpha = 1$ is $u(x, t) = e^{x-t}$ and $v(x, t) = e^{x+t}$.

To obtain the solution of (30) by NHPM, we construct the following homotopy:

$$\begin{aligned}
(1-p) (D_t^\alpha U(x, t) - u_0(x, t)) \\
+ p \left(D_t^\alpha U(x, t) + \frac{\partial^2 U(x, t)}{\partial x^2} \right. \\
\left. - 2e^{2t} \sin(2t) U(x, t) \frac{\partial U(x, t)}{\partial x} \right. \\
\left. + \sin(2t) \frac{\partial(U(x, t) V(x, t))}{\partial x} \right) = 0, \\
(1-p) (D_t^\alpha V(x, t) - v_0(x, t)) \\
+ p \left(D_t^\alpha V(x, t) - \frac{\partial^2 V(x, t)}{\partial x^2} \right. \\
\left. + 2e^{-2t} \cos(2t) V(x, t) \frac{\partial V(x, t)}{\partial x} \right. \\
\left. - \cos(2t) \frac{\partial(U(x, t) V(x, t))}{\partial x} \right) = 0.
\end{aligned}
\tag{32}$$

Applying the inverse operator J_t^α of D_t^α on both sides of the above equation, we obtain

$$\begin{aligned}
U(x, t) &= U(x, 0) + J_t^\alpha u_0(x, t) \\
&\quad - p J_t^\alpha \left(u_0(x, t) + \frac{\partial^2 U(x, t)}{\partial x^2} \right. \\
&\quad \left. - 2e^{2t} \sin(2t) U(x, t) \frac{\partial U(x, t)}{\partial x} \right. \\
&\quad \left. + \sin(2t) \frac{\partial(U(x, t) V(x, t))}{\partial x} \right), \\
V(x, t) &= V(x, 0) + J_t^\alpha v_0(x, t) \\
&\quad - p J_t^\alpha \left(v_0(x, t) - \frac{\partial^2 V(x, t)}{\partial x^2} \right. \\
&\quad \left. + 2e^{-2t} \cos(2t) V(x, t) \frac{\partial V(x, t)}{\partial x} \right. \\
&\quad \left. - \cos(2t) \frac{\partial(U(x, t) V(x, t))}{\partial x} \right).
\end{aligned}
\tag{33}$$

For solving system (33), by new homotopy perturbation method, we use the Taylor series of

$$\begin{aligned}
\sin(2t) &= \sum_{n=0}^{\infty} (-1)^n \frac{(2t)^{2n+1}}{(2n+1)!}, \\
\cos(2t) &= \sum_{n=0}^{\infty} (-1)^n \frac{(2t)^{2n}}{(2n)!},
\end{aligned}$$

$$\begin{aligned}\exp(2t) &= \sum_{n=0}^{\infty} \frac{(2t)^n}{n!}, \\ \exp(-2t) &= \sum_{n=0}^{\infty} (-1)^n \frac{(2t)^n}{n!}.\end{aligned}\quad (34)$$

The solution of (30) has the following form:

$$\begin{aligned}U(x, t) &= U_0(x, t) + pU_1(x, t), \\ V(x, t) &= V_0(x, t) + pV_1(x, t).\end{aligned}\quad (35)$$

Substituting (34) and (35) in (33) and equating the coefficients of like powers of p , we get the following set of equations:

$$\begin{aligned}U_0(x, t) &= U(x, 0) + J_t^\alpha u_0(x, t), \\ V_0(x, t) &= V(x, 0) + J_t^\alpha v_0(x, t),\end{aligned}$$

$$U_1(x, t)$$

$$\begin{aligned}&= J_t^\alpha \left(-u_0(x, t) - \frac{\partial^2 U_0(x, t)}{\partial x^2} \right. \\ &\quad + 2 \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \sum_{n=0}^{\infty} (-1)^n \frac{(2t)^{2n+1}}{(2n+1)!} U_0(x, t) \frac{\partial U_0(x, t)}{\partial x} \\ &\quad \left. - \sum_{n=0}^{\infty} (-1)^n \frac{(2t)^{2n+1}}{(2n+1)!} \frac{\partial U_0(x, t) V_0(x, t)}{\partial x} \right),\end{aligned}$$

$$V_1(x, t)$$

$$\begin{aligned}&= J_t^\alpha \left(-v_0(x, t) + \frac{\partial^2 V_0(x, t)}{\partial x^2} \right. \\ &\quad - 2 \sum_{n=0}^{\infty} (-1)^n \frac{(2t)^n}{n!} \sum_{n=0}^{\infty} (-1)^n \frac{(2t)^{2n}}{(2n)!} \\ &\quad \times V_0(x, t) \frac{\partial V_0(x, t)}{\partial x} \\ &\quad \left. + \sum_{n=0}^{\infty} (-1)^n \frac{(2t)^{2n}}{(2n)!} \frac{\partial U_0(x, t) V_0(x, t)}{\partial x} \right).\end{aligned}\quad (36)$$

Assuming $u_0(x, t) = \sum_{n=0}^{\infty} a_n(x) p_n(t)$, $v_0(x, t) = \sum_{n=0}^{\infty} b_n(x) p_n(t)$, $p_n(t) = t^{n\alpha}$, $U(x, 0) = u(x, 0)$, and $V(x, 0) = v(x, 0)$ and solving the above equation for $U_1(x, t)$ and $V_1(x, t)$ lead to the result

$$\begin{aligned}U_1(x, t) &= (-a_0(x) - e^x) \frac{t^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \left(-a_1(x) - \frac{d^2 a_0(x)}{dx^2} \right) \frac{\Gamma(\alpha+1) t^{2\alpha}}{\Gamma(2\alpha+1)}\end{aligned}$$

$$\begin{aligned}&+ \left(-a_2(x) - \frac{1}{2} \frac{d^2 a_1(x)}{dx^2} - 2b_0(x) e^x \right. \\ &\quad + 2a_0(x) e^x + 2e^x \frac{da_0(x)}{dx} \\ &\quad \left. - 2e^x \left(\frac{db_0(x)}{dx} \right) + 8e^{2x} \right) \frac{\Gamma(2\alpha+1) t^{3\alpha}}{\Gamma(3\alpha+1)} \\ &+ \left(-a_3(x) - 2a_0(x) \left(\frac{db_0(x)}{dx} \right) \right. \\ &\quad \left. + \dots + 8a_0(x) e^x \right) \\ &\times \frac{\Gamma(3\alpha+1) t^{4\alpha}}{\Gamma(4\alpha+1)} + \dots, \\ V_1(x, t) &= (-b_0(x) + e^x) \frac{t^\alpha}{\Gamma(\alpha+1)} \\ &+ \left(-b_1(x) - b_0(x) e^x + a_0(x) e^x \right. \\ &\quad + e^x \frac{da_0(x)}{dx} - e^x \frac{db_0(x)}{dx} \\ &\quad + \frac{d^2 b_0(x)}{dx^2} + 4e^{2x} \left. \right) \frac{\Gamma(\alpha+1) t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ \left(-b_2(x) - \frac{1}{2} e^x \frac{db_1(x)}{dx} \right. \\ &\quad - \frac{1}{2} b_1(x) e^x + \frac{1}{2} a_1(x) e^x \\ &\quad + a_0(x) \frac{db_0(x)}{dx} + 4b_0(x) e^x \left. \right) \\ &\times \frac{\Gamma(2\alpha+1) t^{3\alpha}}{\Gamma(3\alpha+1)} \\ &+ \left(-b_1(x) \frac{db_0(x)}{dx} - \frac{1}{3} b_2(x) e^x \right. \\ &\quad \left. + \dots + \frac{1}{3} \frac{d^2 b_2(x)}{dx^2} \right) \frac{\Gamma(3\alpha+1) t^{4\alpha}}{\Gamma(4\alpha+1)} + \dots.\end{aligned}\quad (37)$$

Vanishing $U_1(x, t)$ and $V_1(x, t)$ lets the coefficients $a_i, b_i, i = 0, 1, 2, \dots$ have the following values:

$$\begin{aligned}a_0(x) &= -e^x, & a_1(x) &= e^x, & a_2(x) &= -\frac{1}{2!} e^x, \\ a_3(x) &= \frac{1}{3!} e^x, & a_4(x) &= -\frac{1}{4!} e^x, \dots, \\ b_0(x) &= e^x, & b_1(x) &= e^x, & b_2(x) &= \frac{1}{2!} e^x, \\ b_3(x) &= \frac{1}{3!} e^x, & b_4(x) &= \frac{1}{4!} e^x, \dots.\end{aligned}\quad (38)$$

Therefore, we obtain the solutions of (30) as

$$\begin{aligned}
 u(x, t) &= e^x - e^x \frac{t^\alpha}{\Gamma(\alpha+1)} + e^x \frac{\Gamma(\alpha+1) t^{2\alpha}}{\Gamma(2\alpha+1)} \\
 &\quad - \frac{1}{2!} e^x \frac{\Gamma(2\alpha+1) t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{1}{3!} e^x \frac{\Gamma(3\alpha+1) t^{4\alpha}}{\Gamma(4\alpha+1)} - \dots \\
 &= e^x \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma((n-1)\alpha+1) t^{n\alpha}}{(n-1)! \Gamma(n\alpha+1)} \right),
 \end{aligned} \tag{39}$$

$$\begin{aligned}
 v(x, t) &= e^x + e^x \frac{t^\alpha}{\Gamma(\alpha+1)} + e^x \frac{\Gamma(\alpha+1) t^{2\alpha}}{\Gamma(2\alpha+1)} \\
 &\quad + \frac{1}{2!} e^x \frac{\Gamma(2\alpha+1) t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{1}{3!} e^x \frac{\Gamma(3\alpha+1) t^{4\alpha}}{\Gamma(4\alpha+1)} + \dots \\
 &= e^x \left(1 + \sum_{n=1}^{\infty} \frac{\Gamma((n-1)\alpha+1) t^{n\alpha}}{(n-1)! \Gamma(n\alpha+1)} \right).
 \end{aligned}$$

If we put $\alpha \rightarrow 1$ in (39) or solve (30) with $\alpha = 1$, we obtain the exact solution

$$\begin{aligned}
 u(x, t) &= e^x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right) \\
 &= e^{x-t}, \\
 v(x, t) &= e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) \\
 &= e^{x+t}.
 \end{aligned} \tag{40}$$

5. Concluding Remarks

In this paper, we have used a new homotopy perturbation method for solving a system of two nonlinear time-fractional partial differential equations. The NHPM for solving system of variable coefficient coupled Burgers' equation with time-fractional derivative is based on two-component procedure and polynomial initial condition. The Computations finally lead to a set of nonlinear equations with one unspecified value in each equation. This set can be readily solved using Maple, and putting these values into the first approximate solution yields the analytical approximate solution. The present study has confirmed that NHPM offers significant advantages in terms of its straightforward applicability, computational efficiency, and accuracy. Thus, we conclude that the new method can be considered as an efficient method for solving linear and nonlinear problems.

Conflict of Interests

The authors of the paper do not have a direct financial relation that might lead to a "conflict of interests" for any of the authors.

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