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Research Article

Fractional Black-Scholes Model and Technical Analysis of Stock Price

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In the stock market, some popular technical analysis indicators (e.g., Bollinger bands, RSI, ROC, etc.) are widely used to forecast the direction of prices. The validity is shown by observed relative frequency of certain statistics, using the daily (hourly, weekly, etc.) stock prices as samples. However, those samples are not independent. In earlier research, the stationary property and the law of large numbers related to those observations under Black-Scholes stock price model and stochastic volatility model have been discussed. Since the fitness of both Black-Scholes model and short-range dependent process has been questioned, we extend the above results to fractional Black-Scholes model with Hurst parameter $H > 1/2$, under which the stock returns follow a kind of long-range dependent process. We also obtain the rate of convergence.

1. Introduction

Liu et al. discussed in [1] the Bollinger bands for the Black-Scholes model. They introduced the corresponding statistics $U_t^{(n)}$ calculated according to the formulation of the Bollinger bands, which is a stationary process, and then they gave the law of large numbers since $\{U_{t+kn}^{(n)}\}_{k=1,2,\ldots}$ are mutually independent for each fixed $t \geq 0$. Thus the Bollinger bands property which seems unthinkable at first glance was proved. The related results have been extended to stochastic volatility model in [2] and AR-ARCH model in [3].

It has been noted in [4] that "technical analysis is a security analysis discipline for forecasting the direction of prices through the study of past market data, primarily price and volume." Technical analysis has been widely used among traders and financial professionals in stock markets and foreign exchange markets in recent decades. However, technical analysis has not received the same level of academic scrutiny and acceptance as more traditional approaches such as fundamental analysis, since "a simulated sample is only one realization of geometric Brownian motion" and "it is difficult to draw general conclusions about the relative frequencies" (see [5]). However, given the stock price models, we show here that we can do statistics based on relative frequency of occurrence for some technical analysis indicators.

The fitness of both Black-Scholes model and short-range dependent process has been questioned. Since Willinger et al. [6] gave the empirical evidence of long-range dependence in stock price returns, there have been several empirical studies that lent further support to such property of long-range dependence (see, e.g., [7–10]). We consider the process of alternatives to short-range dependence, a model driven by the fractional Brownian motion (fBm) which is long-range dependent. In the following discussion, we assume that the stock price satisfies the fractional Black-Scholes model (see, e.g., [11]):

$$S_t = S_0 \exp\left\{\left(\mu + \nu\right) t + \sigma B_t^H\right\}, \quad t \in [0, T], \tag{1}$$

where $\mu, \nu \in \mathbb{R}$ are constants, σ is a positive real number, B_t^H is a fBm with Hurst parameter H, and $H \in (0, 1)$. The fractional Brownian motion is a continuous-time Gaussian process B_t^H on $[0, T]$, which starts at zero with expectation zero for all

$t \in [0, T]$, and has the following covariance function (see, e.g., [12, 13]):

$$E\left[B^H(t) B^H(s)\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad s, t \geq 0, \tag{2}$$

where H is a real number in (0,1), called the Hurst index or Hurst parameter associated with the fractional Brownian motion. In contrast to Brownian motion, the increments of the fBm are not independent if $H \neq 1/2$. The fBm is self-similar, that is, $B^H(at) \ (=) \ |a|^H B^H(t)$, and the increments are stationary, that is, $B^H(t) - B^H(s) \ (=) \ B^H(t-s)$, and the increments exhibit long-range dependence if $H > 1/2$, that is, $\sum_{n=1}^{\infty} E[B^H(1)(B^H(n+1) - B^H(n))] = \infty$, $H > 1/2$, where $X(=)Y$ denotes that X and Y have the same distribution. Note that the fBm is in fact a regular Brownian motion if $H = 1/2$.

In this paper, we discuss the statistical properties of some popular technical indicators such as Bollinger bands, Relative Strength Index (RSI), and Rate of Change (ROC). Under fractional Black-Scholes model (1), we show that the corresponding statistics are stationary and the law of large numbers holds for frequencies of stock prices falling out of normal scope of the technical indicators.

This paper is organized as follows. Section 2 introduces some technical indicators. Section 3 gives the ratios of Bollinger bands, RSI, and ROC falling in the corresponding sets. In Section 4, by constructing a statistic $U_t^{(n)}$, we investigate the stationary properties of corresponding statistics. In Section 5, we obtain the law of large numbers for frequencies of the statistics. And we give some comments of the results in Section 6.

2. Definitions of Technical Indicators

Let S_t be current stock price and $S_{t-i\delta}$ the stock price i periods ago, where δ is the length of the period between two observation spots (the period can be day, minute, etc.). We recall the definitions of technical indicators in the following:

(1) Bollinger Bands. Developed by John Bollinger in the 1980s, Bollinger Bands are volatility bands placed above and below a moving average denoted by

$$\overline{S}_t^{(n)} = \frac{1}{n}\sum_{i=0}^{n-1} S_{t-i\delta}, \qquad B_t^{n,\text{med}} = \frac{1}{\sum_{i=1}^{n} i}\sum_{i=0}^{n-1}(n-i) S_{t-i\delta},$$
$$s_t^{(n)} = \sqrt{\frac{1}{n-1}\sum_{i=0}^{n-1}\left(S_{t-i\delta} - \overline{S}_t^{(n)}\right)^2}, \quad t \geq (n-1)\delta. \tag{3}$$

The curve $B_t^{n,\text{med}}$ is called the middle Bollinger band, the curve $B_t^{n,\text{upp}} = B_t^{n,\text{med}} + 2s_t^{(n)}$ is called the upper Bollinger band, and $B_t^{n,\text{low}} = B_t^{n,\text{med}} - 2s_t^{(n)}$ is called the lower Bollinger band, where n is the number of selected periods. The Bollinger bands of S&P500 are shown in Figure 1. Usually we take $n = 12$ or 20, $\delta =$ one day. According to Bollinger [14] and Liu et al. [1], the bands contain more than 88-89% of price action, which makes a move outside the bands significant. Technically, prices are relatively high when above the upper band and relatively low when below the lower band. However, relatively high should not be regarded as bearish or as a sell signal. Likewise, relatively low should not be considered bullish or as a buy signal. As with other indicators, Bollinger bands are seldom used alone. Traders should combine Bollinger bands with basic trend analysis and other indicators for confirmation.

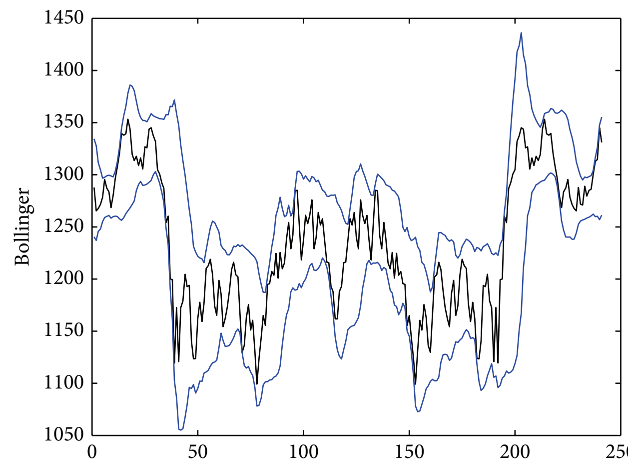

Figure 1: S&P500 annual Bollinger bands until March 27, 2012.

(2) Relative Strength Index (RSI). The RSI was developed by Wilder [15], and it is classified as a momentum oscillator, measuring the velocity and magnitude of directional price movements. If we denote

$$\Delta S_t^+ = \left(S_{t+\delta} - S_t\right) \vee 0, \qquad \Delta S_t^- = \left(S_t - S_{t+\delta}\right) \vee 0, \tag{4}$$

the n-period RSI is defined as

$$\text{RSI}_t^{(n)} = 100 \times \frac{\sum_{i=1}^{n} \Delta S_{t-i\delta}^+}{\sum_{i=1}^{n} \Delta S_{t-i\delta}^+ + \sum_{i=1}^{n} \Delta S_{t-i\delta}^-}, \quad t \geq n\delta. \tag{5}$$

The RSI of S&P500 is shown in Figure 2. Usually we take $n = 14$, $\delta =$ one day. RSI oscillates between zero and 100, with high and low levels marked at 70 and 30. More extreme high and low levels (80 and 20 or 90 and 10) occur less frequently but indicate stronger momentum. Traditionally, RSI readings greater than the 70 level are considered to be in overbought territory, and RSI readings lower than the 30 level are considered to be in oversold territory. If the RSI is below 50, it generally means that the market is in a week trend. When the RSI is above 50, it generally means that the market is in a strong trend. Zhu [16] discussed the statistical property and the forecasting ability of RSI.

(3) Rate of Change (ROC). The ROC is a pure momentum oscillator that measures the percent of change in price from one period to the next. The n-period ROC is defined as

$$\text{ROC}_t^{(n)} = 100 \times \frac{S_t - S_{t-n\delta}}{S_{t-n\delta}}, \quad t \geq n\delta. \tag{6}$$

Table 1: Ratio of SPY in 2008–2011.

Year	$S_t \in$ B-B	RSI $\in [20, 80]$	ROC $\in [-5, 5]$	ROC $\in [-10, 10]$	ROC $\in [-20, 20]$
2008	95.71%	97.91%	73.44%	90.04%	98.76%
2009	98.28%	91.60%	68.75%	90.00%	100%
2010	97.60%	91.59%	82.41%	100%	100%
2011	95.69%	97.06%	80.83%	97.50%	100%

Table 2: Ratio of QQQ in 2008–2011.

Year	$S_t \in$ B-B	RSI $\in [20, 80]$	ROC $\in [-5, 5]$	ROC $\in [-10, 10]$	ROC $\in [-20, 20]$
2008	97.00%	95.82%	58.92%	87.97%	98.76%
2009	96.12%	91.60%	66.25%	91.25%	100%
2010	96.63%	81.78%	74.07%	99.54%	100%
2011	94.40%	96.64%	80.00%	96.67%	100%

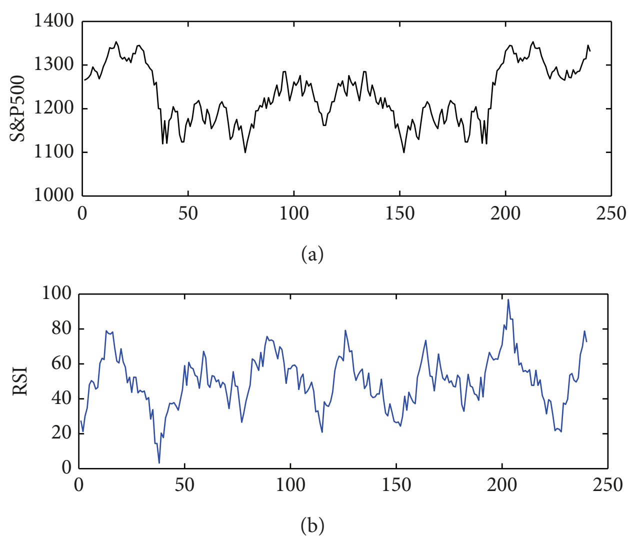

Figure 2: S&P500 annual RSI until March 27, 2012.

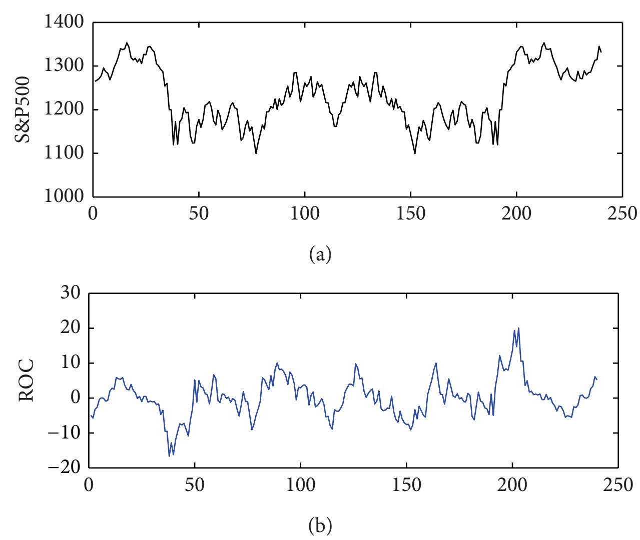

Figure 3: S&P500 annual ROC until March 27, 2012.

The ROC of S&P500 is shown in Figure 3. Usually we take $n = 12$, δ = one day. Prices are constantly increasing as long as the ROC remains positive. Conversely, prices are falling when the ROC is negative. The ROC has its antennas and grounds which are indefinite and can give identifiable extremes that signal overbought and oversold conditions. In general, it is time to sell out when the ROC rises to the first ultra-buy line (5), and then the rising trend mostly ends when it reaches the second ultra-buy line (10). It is time to buy in when ROC drops to the first ultra-sell line (−5), and then the dropping trend mostly ends when it reaches the second ultra-sell line (−10). Li [17] discussed the empirical evidence of ROC. Like all technical indicators, the ROC oscillator should be used in conjunction with other aspects of technical analysis.

3. Some Facts from the Stock Market

Liu et al. [1] traced 15 years of the DOW, S&P500, and NASDAQ daily closing prices and drew the conclusion that in every year more than 94% of daily closing prices are between the Bollinger bands. We give the ratios of Bollinger bands, RSI, and ROC falling in the corresponding sets from January, 2008, to December, 2011, in Tables 1, 2, and 3, where B-B denote the Bollinger bands. We can see that more than 95% of daily closing prices are between the Bollinger bands, more than 81% of RSI are in the interval [20,80], and more than 87% of ROC are in the interval $[-10, 10]$. So we show that the stationary of the indexes is still maintained even since the world economic crisis in 2008. In the following, we give a mathematical proof to this fact under the fractional Black-Scholes model.

4. Stationary Property

Let S_t denote the observed stock price under the model (1). And let

$$\begin{aligned} h(t,i,n) &= \exp\left\{B^H_{t-i\delta} - B^H_{t-n\delta}\right\}, \quad i = 0,1,2,\ldots,n-1, \\ U_t^{(n)} &= f\left(h(t,0,n),\ldots,h(t,n-1,n)\right), \quad t \geq n\delta, \end{aligned} \tag{7}$$

Table 3: Ratio of DIA in 2008–2011.

Year	$S_t \in$ B-B	RSI $\in [20, 80]$	ROC $\in [-5, 5]$	ROC $\in [-10, 10]$	ROC $\in [-20, 20]$
2008	96.57%	96.23%	76.35%	92.12%	98.76%
2009	97.42%	91.63%	69.71%	90.87%	100%
2010	98.56%	92.56%	87.10%	100%	100%
2011	94.83%	94.96%	84.17%	98.33%	100%

where f is a measurable function: $\mathbb{R}^n \to \mathbb{R}$. Then we have the following results:

The process $\{U_t^{(n)}\}_{t\geq n\delta}$ is stationary.

Remark 1. Let S_t be the stock price generated by the model (1), $L_t^{(n)} = (S_t - B_t^{n,\text{med}})/s_t^{(n)} (t \geq n\delta)$. Then the process $\{L_t^{(n)}\}_{t\geq n\delta}$ is stationary.

Remark 2. Let S_t be the stock price generated by the model (1). Then the process

$$\text{RSI}_t^{(n)} = 100 \times \frac{\sum_{i=1}^n \Delta S_{t-i\delta}^+}{\sum_{i=1}^n \Delta S_{t-i\delta}^+ + \sum_{i=1}^n \Delta S_{t-i\delta}^-} \quad (t \geq n\delta) \tag{8}$$

is stationary.

Remark 3. Let S_t be the stock price generated by the model (1). Then the process $\text{ROC}_t^{(n)} = 100\times(S_t - S_{t-n\delta})/S_{t-n\delta} (t \geq n\delta)$ is stationary.

5. Law of Large Numbers

Let $K_{\Gamma,i}^{(n)} = I_{\{U_{i\delta}^{(n)}\in\Gamma\}}$, $i \geq n$, where Γ is a subset of $\mathbb{R}$. And let

$$V_{N,\Gamma}^{(n)} = \frac{1}{N+1}\sum_{i=0}^{N} K_{\Gamma,n+i}^{(n)}, \tag{9}$$

which is the observed frequency of the events $[U_{(n+i)\delta}^{(n)} \in \Gamma]$ $(i = 0, 1, \ldots, N)$.

It is natural to assume $\delta < 1$; that is, the length between two observation spots is less than one year. From the above discussion, we can let $p = P(U_t^{(n)} \in \Gamma), t \geq n\delta$. We denote $U_{i\delta}^{(n)}$ by $U_i^{(n)}$, $i \geq n$, in the following discussion for convenience. Denote by $\mathbb{R}^{m\times n}$ the set of $m \times n$ real matrices, and set

$$Z_{k,j} = K_{\Gamma,(k+1)n+j}^{(n)} - P\left(U_n^{(n)} \in \Gamma\right) \tag{10}$$

for each fixed j and k; we give the following lemma.

Lemma 4. *For all $\Gamma \subset \mathbb{R}$, there exist $\alpha \in (0, 1)$ and a constant $C > 0$; when N is large enough and $|k_1 - k_2| - n > N^\alpha$, one has*

$$E\left(Z_{k_1,j} Z_{k_2,j}\right) \leq C\,|2H-1|\,N^{\alpha(H-1)/2}. \tag{11}$$

Proof. Let $W_{k,i} = B^H_{[(k+1)n+j-i]\delta} - B^H_{[(k+1)n+j-(i+1)]\delta}$, $k \in \mathbb{Z}^+$, $i = 0, 1, 2, \ldots, n-1$. And set $X = (W_{k_1,0}, \ldots, W_{k_1,n-1})^T \triangleq (X_1, \ldots, X_n)^T$, $Y = (W_{k_2,0}, \ldots, W_{k_2,n-1})^T \triangleq (Y_1, \ldots, Y_n)^T$, $Z = (X^T, Y^T)^T$, where X^T is the transpose of X; then $EX = EY = \mathbf{0}$, $EZ = \mathbf{0}$. We denote by $A = (a_{ij})$ the covariance matrix of X, denote by $D = (d_{ij})$ the covariance matrix of Y, and denote by $B = (b_{ij})$ the covariance matrix of X and Y. Let Σ be the covariance matrix of Z; then $\Sigma = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$, $|A| > 0$, $|D| > 0$. Since the fBm has stationary increments, we can get $A = D$, and

$$\begin{aligned} a_{ij} = d_{ij} = & \frac{1}{2}\left|\left(|i-j|-1\right)\delta\right|^{2H} \\ & + \frac{1}{2}\left|\left(|i-j|+1\right)\delta\right|^{2H} - \left|(i-j)\delta\right|^{2H}, \quad H \in (0,1). \end{aligned} \tag{12}$$

Similar to the conclusion given by Deng and Barkai [18], we can get from simple calculation

$$\begin{aligned} b_{ij} = EX_iY_j \sim \delta^2 H(2H-1)\left|k_1 - k_2 + j - i\right|^{2H-2}, \\ k_2 - k_1 \longrightarrow +\infty, \end{aligned} \tag{13}$$

where $p_k \sim q_k$ means $\lim_{k\to\infty}(p_k/q_k) = 1$. So we have $B = (b_{ij}) \to \mathbf{0}$.

When $H = 1/2$, we can easily derive that the conclusion of Lemma 4 holds. We assume $H \neq 1/2$ in the following proof. Let $p(z)$ be the probability density function of Z and $F(z)$ the distribution function of Z, and let the marginal distributions for Z be $F_i(z_i)$, $i = 1, 2, \ldots, 2n$, where $z \in \mathbb{R}^{2n\times 1}$, $z = (z_1, z_2, \ldots, z_{2n})^T$. Take the notation $\Theta = [-M, M], M > 0$, and $\Gamma' = \Theta\times\Theta\times\cdots\times\Theta \triangleq \Theta^n$. Furthermore, we put

$$g(Z) = Z_{k_1,j} Z_{k_2,j}. \tag{14}$$

First we will consider the integral of $g(Z)$ on $\Gamma' \times \Gamma'$. Referring to Bernstein [19], we have $\Sigma^{-1} = \Sigma_1 + \Sigma_2$, where $\widetilde{\Sigma} = D - B^TA^{-1}B$,

$$\begin{aligned} \Sigma_1 &= \begin{pmatrix} A^{-1} & \mathbf{0} \\ \mathbf{0} & D^{-1} \end{pmatrix}, \\ \Sigma_2 &= \begin{pmatrix} A^{-1}B\widetilde{\Sigma}^{-1}B^TA^{-1} & -A^{-1}B\widetilde{\Sigma}^{-1} \\ -\widetilde{\Sigma}^{-1}B^TA^{-1} & D^{-1}B^T\left(A - BD^{-1}B^T\right)^{-1}BD^{-1} \end{pmatrix}. \end{aligned} \tag{15}$$

We take the notation $d\widetilde{z} = dz_1, dz_2, \ldots, dz_{2n}$. Then we obtain

$$\begin{aligned}
&\overbrace{\int\cdots\int_{\Gamma'\times\Gamma'}}^{2n} g(z)\,p(z)\,d\widetilde{z} \\
&= \overbrace{\int\cdots\int_{\Gamma'\times\Gamma'}}^{2n} g(z)\,(2\pi)^{-n}(|A|\,|D|)^{-1/2} \\
&\quad\times \exp\left\{-\frac{1}{2}z^T\Sigma_1 z\right\} \\
&\quad\times\left(\frac{\left|D-B^TA^{-1}B\right|^{-1/2}}{|D|^{-1/2}}\exp\left\{-\frac{1}{2}z^T\Sigma_2 z\right\}-1\right)d\widetilde{z},
\end{aligned} \tag{16}$$

where $\lim_{|k_1-k_2|\to\infty}(|D - B^TA^{-1}B|/|D|) = (1/|D|)(|D - \lim_{|k_1-k_2|\to\infty}(B^TA^{-1}B)|) = 1$. Assume $\lim_{k_2-k_1\to\infty}(M/|k_1 - k_2|^{1-H}) = 0$; then $z^T\Sigma_2 z \to 0$, so we get $\forall\epsilon > 0$, $\exists N_1$, $\forall|k_1 - k_2| > N_1$,

$$\begin{aligned}
\left|\exp\left\{-\frac{1}{2}z^T\Sigma_2 z\right\}-1\right| &= \left|1-\frac{1}{2}z^T\Sigma_2 z + o\left(z^T\Sigma_2 z\right)-1\right| \\
&= \left|\frac{1}{2}z^T\Sigma_2 z + o\left(z^T\Sigma_2 z\right)\right|.
\end{aligned} \tag{17}$$

Choose $\alpha \in (0, 1)$ satisfying $N^\alpha > N_1$, where $N < (N + 1)/n$; then if $|k_1 - k_2| - n > N^\alpha$ and N is large enough, there exist $C_0 > 0$, and C_0 has relation with n, such that

$$\left|\frac{1}{2}z^T\Sigma_2 z + o\left(z^T\Sigma_2 z\right)\right| \le C_0 M^2\,|H(2H-1)|\,N^{2\alpha(H-1)}. \tag{18}$$

Therefore, by (16), (17), and (18), we can derive that there exist $\widetilde{C}_0 > 0$ satisfying

$$\begin{aligned}
&\overbrace{\int\cdots\int_{\Gamma'\times\Gamma'}}^{2n} g(z)\,p(z)\,d\widetilde{z} \\
&\le p^2\left(\epsilon + C_0M^2\,|H(2H-1)|\,N^{2\alpha(H-1)}\right) \\
&\le \widetilde{C}_0\,|2H-1|\,N^{\alpha(H-1)}.
\end{aligned} \tag{19}$$

Then we will consider the integral of $g(Z)$ on the complementary set of $\Gamma' \times \Gamma'$ in the following. Let Ξ be a random variable satisfying $P(\Xi = (-\infty, -M)) = 1/2$, $P(\Xi = (M, \infty)) = 1/2$. Let Γ_i be the set that contains all elements of the following form:

$$\Xi\times\cdots\times\Theta\times\cdots\times\Xi\times\cdots\times\Theta \triangleq \Xi^{(i)}, \tag{20}$$

where Θ occurs i times and Ξ occurs $2n - i$ times in $\Xi^{(i)}$, $i = 0, 1, \ldots, 2n-1$. So we can see that Γ_i is composed of $\binom{2n}{i}\cdot 2^{2n-i}$ mutually disjoint elements. Therefore, the complementary set of $\Gamma' \times \Gamma'$ should be $(\Gamma' \times \Gamma')^c = \bigcup_{i=0}^{2n-1}\Gamma_i$ and $\Gamma_i \bigcap \Gamma_j = \Phi$, $i \ne j$; that is, the complementary set of $\Gamma' \times \Gamma'$ is the union of $(\sum_{i=0}^{2n-1}\binom{2n}{i}\cdot 2^{2n-i} = 3^{2n} - 1)$ mutually disjoint sets.

Since

$$\begin{aligned}
&\left|\overbrace{\int\cdots\int_{\Xi^{(i)}}}^{2n} g(z)\,p(z)\,d\widetilde{z}\right| \\
&\le \overbrace{\int\cdots\int_{\Xi^{(i)}}}^{2n} p(z)\,d\widetilde{z} \le F(z_1, z_2, \ldots, z_{2n}) \\
&\le \min_{1\le i\le 2n}\{F_i(z_i)\} \le F_1(-M) \le \frac{\delta^H e^{-M^2/2\delta^{2H}}}{M},
\end{aligned} \tag{21}$$

where M occurs i times and $-M$ occurs $2n - i$ times within $z_1, z_2, \ldots, z_{2n}$, the second inequality holds because $|g(z)| \le 1$, and the last inequality holds because $\int_M^{+\infty} e^{-x^2/2}dx = \int_{-\infty}^{-M} e^{-x^2/2}dx \le e^{-M^2/2}/M$.

Take $M = N^{\alpha(1-H)/2}$. We conclude from (19) and (21) that

$$\begin{aligned}
E\left(Z_{k_1,j}Z_{k_2,j}\right) &= \overbrace{\int\cdots\int_{\Gamma'\times\Gamma'}}^{2n} g(z)\,p(z)\,d\widetilde{z} \\
&\quad + \overbrace{\int\cdots\int_{(\Gamma'\times\Gamma')^c}}^{2n} g(z)\,p(z)\,d\widetilde{z} \\
&\le \widetilde{C}_0\,|2H-1|\,N^{\alpha(H-1)} \\
&\quad + \sum_{i=0}^{2n-1}\binom{2n}{i}\cdot 2^{2n-i}\frac{\delta^H e^{-(M)^2/2\delta^{2H}}}{M} \\
&\le \widetilde{C}_0\,|2H-1|\,N^{\alpha(H-1)} \\
&\quad + \left(3^{2n}-1\right)\delta^H N^{\alpha(H-1)/2}.
\end{aligned} \tag{22}$$

Take $C = \widetilde{C}_0 + (3^{2n} - 1)\delta^H/|2H - 1|$; then we have

$$\begin{aligned}
&\widetilde{C}_0\,|2H-1|\,N^{\alpha(H-1)} + \left(3^{2n}-1\right)\delta^H N^{\alpha(H-1)/2} \\
&\quad\le C\,|2H-1|\,N^{\alpha(H-1)/2}
\end{aligned} \tag{23}$$

from which the proof immediately follows. □

Then we obtain the law of large numbers.

Theorem 5. *There exist $\beta \in (0, 1)$ and a constant $\widetilde{C} > 0$ such that*

$$E\left|V_{N,\Gamma}^{(n)} - P\left(U_n^{(n)} \in \Gamma\right)\right|^2 \le \frac{\widetilde{C}}{N^\beta}. \tag{24}$$

Proof. To simplify notation, we put $\Lambda = \{k : 0 \le kn + j \le N\}$ and set for each fixed j and k,

$$Y_j = \sum_{k\in\Lambda}\left[K_{\Gamma,(k+1)n+j}^{(n)} - P\left(U_n^{(n)} \in \Gamma\right)\right]. \tag{25}$$

Since the process $\{U_t^{(n)}\}_{t\geq n\delta}$ is stationary, we have $P(U_t^{(n)} \in \Gamma) = E[I_{\{U_{(k+1)n+j}^{(n)}\in\Gamma\}}]$ holds for all $k > 0$. In addition, by C-r inequality and Lemma 4, it follows that

$$
\begin{aligned}
E\left|V_{N,\Gamma}^{(n)} - P\left(U_n^{(n)} \in \Gamma\right)\right|^2
&= E\left|\frac{1}{N+1}\sum_{j=0}^{n-1} Y_j\right|^2 \\
&\leq \frac{n}{(N+1)^2}\sum_{j=0}^{n-1} EY_j^2 \\
&= \frac{n}{(N+1)^2}\left[\sum_{j=0}^{n-1}\sum_{k\in\Lambda} EZ_{k,j}^2 + \sum_{j=0}^{n-1}\sum_{\substack{k_1\neq k_2\\ k_1,k_2\in\Lambda}} EZ_{k_1,j}Z_{k_2,j}\right] \\
&\leq \frac{n}{(N+1)^2}\left[N+1+\sum_{j=0}^{n-1}\sum_{\substack{0<|k_1-k_2|\leq N^\alpha\\ k_1,k_2\in\Lambda}} EZ_{k_1,j}Z_{k_2,j}\right. \\
&\qquad \left. +\sum_{j=0}^{n-1}\sum_{\substack{|k_1-k_2|>N^\alpha\\ k_1,k_2\in\Lambda}} EZ_{k_1,j}Z_{k_2,j}\right] \\
&\leq \frac{n}{(N+1)^2}\left[N+1+2n\frac{N+1}{n}N^\alpha\right. \\
&\qquad \left. +2n\frac{N+1}{n}\left(N-N^\alpha\right)C\left|2H-1\right|N^{\alpha(H-1)/2}\right].
\end{aligned}
\tag{26}
$$

Let $\beta = \min\{1-\alpha, \alpha(1-H)/2\}$ and $\widetilde{C} = 3n + 2nC|2H-1|$; then $E|V_{N,\Gamma}^{(n)} - P(U_n^{(n)} \in \Gamma)|^2 \leq \widetilde{C}/N^\beta$. □

Remark 6. From Theorem 5, it is reasonable to use the stationary distribution of $U_n^{(n)}$ to calculate $V_{N,\Gamma}^{(n)}$, which is the relative frequency of the technical indicators falling in the corresponding set.

Corollary 7. *Let $H_i^{(n)} = I_{\{L_{i\delta}^{(n)}\geq 2\}}$, $i \geq n$; then $EH_i^{(n)} = P(L_{i\delta}^{(n)} \geq 2)$. Let*

$$
J_N^{(n)} = \frac{1}{N+1}\sum_{i=0}^{N} H_{n+i}^{(n)}, \tag{27}
$$

which is the observed frequency of the events $[L_{(n+i)\delta}^{(n)} \geq 2]$ $(i = 0, 1, \ldots, N)$, that is, the frequency of stock falling out of the Bollinger bands. Then there exist $\beta \in (0,1)$ and a constant $\widetilde{C} > 0$ such that

$$
E\left|J_N^{(n)} - P\left(L_{n\delta}^{(n)} \geq 2\right)\right|^2 \leq \frac{\widetilde{C}}{N^\beta}, \tag{28}
$$

Corollary 8. *Let $H_i^{(n)} = I_{\{RSI_{i\delta}^{(n)}\in\Gamma\}}$, $i \geq n$, where $\Gamma = [0, 20] \cup [80, 100]$. Then $EH_i^{(n)} = P(RSI_{i\delta}^{(n)} \in \Gamma)$. Let*

$$
J_N^{(n)} = \frac{1}{N+1}\sum_{i=0}^{N} H_{n+i}^{(n)}, \tag{29}
$$

which is the observed frequency of the events $[RSI_{(n+i)\delta}^{(n)} \in \Gamma]$ $(i = 0, 1, \ldots, N)$. Then there exist $\beta \in (0,1)$ and a constant $\widetilde{C} > 0$ such that

$$
E\left|J_N^{(n)} - P(RSI_{n\delta}^{(n)} \in \Gamma)\right|^2 \leq \frac{\widetilde{C}}{N^\beta}. \tag{30}
$$

Corollary 9. *Let $H_i^{(n)} = I_{\{ROC_{i\delta}^{(n)}\in\Gamma\}}$, $i \geq n$, where $\Gamma = [-\infty, \xi] \cup [\eta, \infty]$, and ξ and η are the indefinite ground and antenna of ROC. Then $EH_i^{(n)} = P(ROC_{i\delta}^{(n)} \in \Gamma)$. Let*

$$
J_N^{(n)} = \frac{1}{N+1}\sum_{i=0}^{N} H_{n+i}^{(n)}, \tag{31}
$$

which is the observed frequency of the events $[ROC_{(n+i)\delta}^{(n)} \in \Gamma]$ $(i = 0, 1, \ldots, N)$. Then there exist $\beta \in (0,1)$ and a constant $\widetilde{C} > 0$ such that

$$
E\left|J_N^{(n)} - P(ROC_{n\delta}^{(n)} \in \Gamma)\right|^2 \leq \frac{\widetilde{C}}{N^\beta}. \tag{32}
$$

6. Conclusion

In the above discussion, we considered a class of long-range dependent processes, of which the rate of decay is slower than the exponential one, typically a power-like decay. We derived the rate of convergence of the ergodic theorem for several stationary processes associated with the technical analysis in the security market and extended the previous results (see [1–3, 16, 17]). Thus, we established the theoretical foundation of technical analysis for fractional Black-Scholes model.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] W. Liu, X. Huang, and W. Zheng, "Black-Scholes' model and Bollinger bands," *Physica A*, vol. 371, no. 2, pp. 565–571, 2006.

[2] W. Liu and W. A. Zheng, "Stochastic volatility model and technical analysis of stock price," *Acta Mathematica Sinica*, vol. 27, no. 7, pp. 1283–1296, 2011.

[3] X. Huang and W. Liu, "Properties of some statistics for AR-ARCH model with application to technical analysis," *Journal of Computational and Applied Mathematics*, vol. 225, no. 2, pp. 522–530, 2009.

[4] C. D, I. I. Kirkpatrick, and J. R. Dahlquist, *Technical Analysis: The Complete Resource for Financial Market Technicians*, Pearson Education, Upper Saddle River, NJ, USA, 2nd edition, 2011.

[5] A. W. Lo, H. Mamaysky, and J. Wang, "Foundations of technical analysis: computational algorithms, statistical inference, and empirical implementation," *Journal of Finance*, vol. 55, no. 4, pp. 1705–1770, 2000.

[6] W. Willinger, M. S. Taqqu, and V. Teverovsky, "Stock market prices and long-range dependence," *Finance and Stochastics*, vol. 3, no. 1, pp. 1–13, 1999.

[7] A. W. Lo, "Fat tails, long memory, and the stock market since the 1960s," *Economic Notes*, vol. 26, no. 2, pp. 219–252, 1997.

[8] P. Robinson, Ed., *Time Series with Long Memory*, Advanced Texts in Econometrics, Oxford University Press, New York, NY, USA, 2003.

[9] R. Cont, "Long range dependence in financial markets," in *Fractals in Engineering*, E. Lutton and J. Vehel, Eds., Springer, London, UK, 2005.

[10] D. O. Cajueiro and B. M. Tabak, "Testing for time-varying long-range dependence in real state equity returns," *Chaos, Solitons and Fractals*, vol. 38, no. 1, pp. 293–307, 2008.

[11] P. Cheridito, "Arbitrage in fractional Brownian motion models," *Finance and Stochastics*, vol. 7, no. 4, pp. 533–553, 2003.

[12] T. E. Duncan, Y. Hu, and B. Pasik-Duncan, "Stochastic calculus for fractional Brownian motion. I. Theory," *SIAM Journal on Control and Optimization*, vol. 38, no. 2, pp. 582–612, 2000.

[13] Y. Hu, B. Øksendal, and A. Sulem, "Optimal consumption and portfolio in a Black-Scholes market driven by fractional Brownian motion," *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, vol. 6, no. 4, pp. 519–536, 2003.

[14] J. Bollinger, *Bollinger on Bollinger Bands*, McGraw-Hill, New York, NY, USA, 2002.

[15] J. W. Wilder, *New Concepts in Technical Trading Systems*, Trend Research, Greensboro, NC, USA, 1st edition, 1978.

[16] W. Zhu, *Statistical analysis of stock technical indicators [M.S. thesis]*, East China Normal University, Shanghai, China, 2006.

[17] W. Li, *A new standard of effectiveness of moving average technical indicators in stock markets [M.S. thesis]*, East China Normal University, Shanghai, China, 2006.

[18] W. Deng and E. Barkai, "Ergodic properties of fractional Brownian-Langevin motion," *Physical Review E*, vol. 79, no. 1, Article ID 011112, 7 pages, 2009.

[19] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas with Application to Linear Systems Theory*, Princeton University Press, Princeton, NJ, USA, 2005.