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Research Article

Commutative Pseudo Valuations on BCK-Algebras

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The notion of a commutative pseudo valuation on a BCK-algebra is introduced, and its characterizations are investigated. The relationship between a pseudo valuation and a commutative pseudo-valuation is examined.

1. Introduction

D. Buşneag [1] defined pseudo valuation on a Hilbert algebra and proved that every pseudo valuation induces a pseudometric on a Hilbert algebra. Also, D. Buşneag [2] provided several theorems on extensions of pseudo valuations. C. Buşneag [3] introduced the notions of pseudo valuations (valuations) on residuated lattices, and proved some theorems of extension for these (using the model of Hilbert algebras [2]). Using the Buşneag's model, Doh and Kang [4] introduced the notion of a pseudo valuation on BCK/BCI-algebras, and discussed several properties.

In this paper, we introduce the notion of a commutative pseudo valuation on a BCK-algebra, and investigate its characterizations. We discuss the relationship between a pseudo valuation and a commutative pseudo valuation. We provide conditions for a pseudo valuation to be a commutative pseudo valuation.

2. Preliminaries

A BCK-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type (2,0) is called a *BCI-algebra* if it satisfies the following axioms:

(i) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(ii) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(iii) $(\forall x \in X)\ (x * x = 0)$,

(iv) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(v) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following conditions:

(a1) $(\forall x \in X)\ (x * 0 = x)$,

(a2) $(\forall x, y, z \in X)\ (x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0)$,

(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,

(a4) $(\forall x, y, z \in X)\ (((x * z) * (y * z)) * (x * y) = 0)$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$.

A BCK-algebra X is said to be *commutative* if $x \wedge y = y \wedge x$ for all $x, y \in X$ where $x \wedge y = y * (y * x)$.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies the following conditions:

(b1) $0 \in A$,

(b2) $(\forall x, y \in X)\ (x * y \in A,\ y \in A \Rightarrow x \in A)$.

A subset A of a BCK-algebra X is called a *commutative ideal* of X (see [6]) if it satisfies (b1) and

(b3) $(\forall x, y, z \in X)\ ((x * y) * z \in A,\ z \in A \Rightarrow x * (y \wedge x) \in A)$.

We refer the reader to the book in [7] for further information regarding BCK-algebras.

3. Commutative Pseudo Valuations on BCK-Algebras

In what follows let X denote a BCK-algebra unless otherwise specified.

Definition 3.1 (see [4]). A real-valued function φ on X is called a *weak pseudo valuation* on X if it satisfies the following condition:

(c1) $(\forall x, y \in X)(\varphi(x * y) \leq \varphi(x) + \varphi(y))$.

Definition 3.2 (see [4]). A real-valued function φ on X is called a *pseudo valuation* on X if it satisfies the following two conditions:

(c2) $\varphi(0) = 0$,

(c3) $(\forall x, y \in X)(\varphi(x) \leq \varphi(x * y) + \varphi(y))$.

Table 1: $*$-operation.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Proposition 3.3 (see [4]). *For any pseudo valuation φ on X, one has the following assertions:*

(1) $\varphi(x) \geq 0$ *for all* $x \in X$.

(2) φ *is order preserving,*

(3) $\varphi(x * y) \leq \varphi(x * z) + \varphi(z * y)$ *for all* $x, y, z \in X$.

Definition 3.4. A real-valued function φ on X is called a *commutative pseudo valuation* on X if it satisfies (c2) and

(c4) $(\forall x, y, z \in X)$ $(\varphi(x * (y \wedge x)) \leq \varphi((x * y) * z) + \varphi(z))$.

Example 3.5. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the $*$-operation given by Table 1. Let ϑ be a real-valued function on X defined by

$$\vartheta = \begin{pmatrix} 0 & a & b & c \\ 0 & 7 & 9 & 9 \end{pmatrix}. \tag{3.1}$$

Routine calculations give that ϑ is a commutative pseudo valuation on X.

Theorem 3.6. *In a BCK-algebra, every commutative pseudo valuation is a pseudo valuation.*

Proof. Let φ be a commutative pseudo valuation on X. For any $x, y, z \in X$, we have

$$\varphi(x) = \varphi(x * (0 \wedge x)) \leq \varphi((x * 0) * z) + \varphi(z) = \varphi(x * z) + \varphi(z). \tag{3.2}$$

This completes the proof. □

Combining Theorem 3.6 and [4, Theorem 3.9], we have the following corollary.

Corollary 3.7. *In a BCK-algebra, every commutative pseudo valuation is a weak pseudo valuation.*

The converse of Theorem 3.6 may not be true as seen in the following example.

Example 3.8. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the $*$-operation given by Table 2. Let ϑ be a real-valued function on X defined by

$$\vartheta = \begin{pmatrix} 0 & a & b & c & d \\ 0 & 5 & 8 & 8 & 8 \end{pmatrix}. \tag{3.3}$$

Table 2: $*$-operation.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Then ϑ is a pseudo valuation on X. Since

$$\vartheta(b * (c \wedge b)) = 8 \nleq 0 = \vartheta((b * c) * 0) + \vartheta(0), \tag{3.4}$$

ϑ is not a commutative pseudo valuation on X.

We provide conditions for a pseudo valuation to be a commutative pseudo valuation.

Theorem 3.9. *For a real-valued function φ on X, the following are equivalent:*

(1) *φ is a commutative pseudo valuation on X.*

(2) *φ is a pseudo valuation on X that satisfies the following condition:*

$$(\forall x, y \in X) \quad (\varphi(x * (y \wedge x)) \leq \varphi(x * y)). \tag{3.5}$$

Proof. Assume that φ is a commutative pseudo valuation on X. Then φ is a pseudo valuation on X by Theorem 3.6. Taking $z = 0$ in (c4) and using (a1) and (c2) induce the condition (3.5).

Conversely let φ be a pseudo valuation on X satisfying the condition (3.5). Then $\varphi(x * y) \leq \varphi((x * y) * z) + \varphi(z)$ for all $x, y, z \in X$. It follows from (3.5) that

$$\varphi(x * (y \wedge x)) \leq \varphi(x * y) \leq \varphi((x * y) * z) + \varphi(z) \tag{3.6}$$

for all $x, y, z \in X$ so that φ is a commutative pseudo valuation on X. □

Lemma 3.10 (see [8]). *Every pseudo valuation φ on X satisfies the following implication:*

$$(\forall x, y, z \in X) \quad ((x * y) * z = 0 \Longrightarrow \varphi(x) \leq \varphi(y) + \varphi(z)). \tag{3.7}$$

Theorem 3.11. *In a commutative BCK-algebra, every pseudo valuation is a commutative pseudo valuation.*

Proof. Let φ be a pseudo valuation on a commutative BCK-algebra X. Note that

$$\begin{aligned} ((x * (y \wedge x)) * ((x * y) * z)) * z &= ((x * (y \wedge x)) * z) * ((x * y) * z) \\ &\leq (x * (y \wedge x)) * (x * y) \\ &= (x \wedge y) * (y \wedge x) = 0 \end{aligned} \tag{3.8}$$

for all $x, y, z \in X$. Hence $((x*(y\wedge x))*((x*y)*z))*z = 0$ for all $x, y, z \in X$. It follows from Lemma 3.10 that $\varphi(x*(y\wedge x)) \leq \varphi((x*y)*z)+\varphi(z)$ for all $x, y, z \in X$. Therefore φ is a commutative pseudo valuation on X. □

For any real-valued function φ on X, we consider the set

$$I_\varphi := \{x \in X \mid \varphi(x) = 0\}. \tag{3.9}$$

Lemma 3.12 (see [4]). *If φ is a pseudo valuation on X, then the set I_φ is an ideal of X.*

Lemma 3.13 (see [7]). *For any nonempty subset I of X, the following are equivalent:*

(1) *I is a commutative ideal of X.*

(2) *I is an ideal of X that satisfies the following condition:*

$$(\forall x, y \in X) \quad (x*y \in I \Longrightarrow x*(y\wedge x) \in I). \tag{3.10}$$

Theorem 3.14. *If φ is a commutative pseudo valuation on X, then the set I_φ is a commutative ideal of X.*

Proof. Let φ be a commutative pseudo valuation on a BCK-algebra X. Using Theorem 3.6 and Lemma 3.12, we conclude that I_φ is an ideal of X. Let $x, y \in X$ be such that $x*y \in I_\varphi$. Then $\varphi(x*y) = 0$. It follows from (3.5) that $\varphi(x*(y\wedge x)) \leq \varphi(x*y) = 0$ so that $\varphi(x*(y\wedge x)) = 0$. Hence $x*(y\wedge x) \in I_\varphi$. Therefore I_φ is a commutative ideal of X by Lemma 3.13. □

The following example shows that the converse of Theorem 3.14 is not true.

Example 3.15. Consider a BCK-algebra $X = \{0, a, b, c\}$ with the $*$-operation given by Table 3. Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & a & b & c \\ 0 & 3 & 7 & 0 \end{pmatrix}. \tag{3.11}$$

Then $I_\varphi = \{0, c\}$ is a commutative ideal of X. Since

$$\varphi(b) = 7 > 6 = \varphi(b*a)+\varphi(a), \tag{3.12}$$

φ is not a pseudo valuation on X and so φ is not a commutative pseudo valuation on X.

Using an ideal, we establish a pseudo valuation.

Theorem 3.16. *For any ideal I of X, we define a real-valued function φ_I on X by*

$$\varphi_I(x) = \begin{cases} 0 & \text{if } x = 0, \\ t_1 & \text{if } x \in I \setminus \{0\}, \\ t_2 & \text{if } x \in X \setminus I \end{cases} \tag{3.13}$$

Table 3: $*$-operation.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

for all $x \in X$ *where* $0 < t_1 < t_2$. *Then* φ_I *is a pseudo valuation on* X.

Proof. Let $x, y \in X$. If $x = 0$, then clearly $\varphi_I(x) \leq \varphi_I(x * y) + \varphi_I(y)$. Assume that $x \neq 0$. If $y = 0$, then $\varphi_I(x) \leq \varphi_I(x * y) + \varphi_I(y)$. If $y \neq 0$, we consider the following four cases:

(i) $x * y \in I$ and $y \in I$,

(ii) $x * y \notin I$ and $y \notin I$,

(iii) $x * y \in I$ and $y \notin I$,

(iv) $x * y \notin I$ and $y \in I$.

Case (i) implies that $x \in I$ because I is an ideal of X. If $x * y = 0$, then $\varphi_I(x * y) = 0$ and so $\varphi_I(x) = t_1 = \varphi_I(x * y) + \varphi_I(y)$. If $x * y \neq 0$, then $\varphi_I(x * y) = t_1$ and thus $\varphi_I(x) = t_1 \leq \varphi_I(x * y) + \varphi_I(y)$. The second case implies that $\varphi_I(x * y) = t_2$ and $\varphi_I(y) = t_2$. Hence $\varphi_I(x) \leq t_2 < \varphi_I(x * y) + \varphi_I(y)$. Let us consider the third case. If $x * y = 0$, then $\varphi_I(x * y) = 0$ and thus $\varphi_I(x) \leq t_2 = \varphi_I(x * y) + \varphi_I(y)$. If $x * y \neq 0$, then $\varphi_I(x * y) = t_1$ and so $\varphi_I(x) \leq t_2 < t_1 + t_2 = \varphi_I(x * y) + \varphi_I(y)$. For the final case, the proof is similar to the third case. Therefore φ_I is a pseudo valuation on X. □

Before ending our discussion, we pose a question.

Question 1. If I is commutative ideal of X, then is the function φ_I in Theorem 3.16 a commutative pseudo valuation on X?

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