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Research Article

Soft Rough Approximation Operators on a Complete Atomic Boolean Lattice

Heba I. Mustafa

Mathematics Department, Faculty of Science, Zagazig University, Egypt

Correspondence should be addressed to Heba I. Mustafa; dr_heba_ibrahim@yahoo.com

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The concept of soft sets based on complete atomic Boolean lattice, which can be seen as a generalization of soft sets, is introduced. Some operations on these soft sets are discussed, and new types of soft sets such as full, keeping infimum, and keeping supremum are defined and supported by some illustrative examples. Two pairs of new soft rough approximation operators are proposed and the relationship among soft set is investigated, and their related properties are given. We show that Järvinen's approximations can be viewed as a special case of our approximation. If $B = \wp(U)$, then our soft approximations coincide with crisp soft rough approximations (Feng et al. 2011).

1. Introduction

Most of traditional methods for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. However, many practical problems within fields such as economics, engineering, environmental science, medical science, and social sciences involve data that contain uncertainties. We cannot use traditional methods because of various types of uncertainties present in these problems.

There are several theories probability theory, fuzzy set theory, theory of interval mathematics, and rough set theory [1], which we can be considered as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties (see [2]). For example, theory of probabilities can deal only with stochastically stable phenomena. To overcome these kinds of difficulties, Molodtsov [2] proposed a completely new approach, which is called soft set theory, for modelling uncertainty.

Presently, works on soft set theory are progressing rapidly. Maji et al. [3–5] further studied soft set theory, used this theory to solve some decision making problems, and devoted fuzzy soft sets combining soft sets with fuzzy sets. Roy and Maji [6] presented a fuzzy soft set theoretic approach towards decision making problems. Jiang et al. [7] extended soft sets with description logics. Aktas and Cagman [8] defined soft groups. Shabir and Naz [9] investigated soft topological spaces. Ge et al. [10] discussed relationships between soft sets and topological spaces.

Rough set theory was initiated by Pawlak [1] for dealing with vagueness and granularity in information systems. This theory handles the approximation of an arbitrary subset of a universe by two definable or observable subsets called lower and upper approximations. It has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems, and many other fields (see [1, 11]). Since many classes of information granules are lattice ordered [12, 13], lattice theory [14–16] has found renewed interest and applications in diverse areas such as mathematical morphology [17], fuzzy set theory [18, 19], computational intelligence [20], automated decision making [21], and formal concept analysis [22]. In [23, 24] Järvinen studied properties of approximations in a more general setting of complete atomic Boolean lattices. He defined in a lattice theoretical setting two maps which mimic the rough approximation operators and noted that this setting is suitable also for other operators based on binary relations.

It has been found that soft set and rough set are closely related concepts. Based on the equivalence relation on the universe of discourse, Feng et al. [25, 26] investigated the relationships among soft sets, rough sets, and fuzzy sets, obtaining three types of hybrid models: rough soft sets, soft rough sets, and soft rough fuzzy sets. They show that Pawlak's rough set can be viewed as a special case of soft rough sets. Soft rough sets, which could provide a better approximation than rough sets do, can be seen as a generalized rough set model, and defining soft rough sets and some related concepts needs using soft rough approximation operators based on soft sets. Thus, soft rough approximation operators deserve further research.

This paper is arranged as follows. In Section 2 we recall and develop some notions and notations concerning lattice, ordered set, and properties of maps. Also we discuss the generalization of rough sets in a more general setting of complete atomic Boolean lattices which was studied by Järvinen [23, 24]. The purpose of Section 3 is to introduce the new concept of soft sets on a complete atomic Boolean lattice as a generalization of soft sets, discuss some operations and define new types of theses soft sets. At the end of this section, we obtain the algebraic structure (i.e., the lattice structure) of our new soft sets. In Section 4, we consider two pairs of soft rough approximations based on a complete atomic Boolean lattice as a generalization of soft rough approximations and give their properties. In Section 5 another pair of soft rough approximations is investigated, and the fact that Järvinen's approximations can be viewed as a special case of our soft approximations is proved. The conclusion is in Section 6.

2. Preliminaries

We assume that the reader is familiar with the usual lattice-theoretical notation and conventions, which can be found in [27, 28].

First we recall some definitions and properties of maps. Let $\mathbf{B} = (B, \leq)$ be an ordered set. A mapping $f : B \to B$ is said to be extensive, if $x \leq f(x)$ for all $x \in B$. The map f is order preserving if $x \leq y$ implies $f(x) \leq f(y)$. Moreover, f is idempotent if $f(f(x)) = f(x)$ for all $x \in B$. A map $c : B \to B$ is said to be a closure operator on B, if c is extensive, order preserving, and idempotent. An element $x \in B$ is c closed if $c(x) = x$. Furthermore, if $i : B \to B$ is a closure operator on $\mathbf{B}^{\partial} = (B, \geq)$ then i is an interior operator on B. Let $\mathbf{B} = (B, \leq)$ and $\mathbf{Q} = (Q, \leq)$ be ordered sets. $f : B \to Q$ is an order embedding, if for any $a, b \in B$, $a \leq b$ in B if and only if $f(a) \leq f(b)$ in Q; note that an order embedding is always an injection. An order-embedding f onto Q is called an order-isomorphism between $\mathbf{B}$ and $\mathbf{Q}$; we say that $\mathbf{B}$ and $\mathbf{Q}$ are order-isomorphic and write $\mathbf{B} \cong \mathbf{Q}$. If $\mathbf{B} = (B, \leq)$ and $\mathbf{Q} = (Q, \leq)$ are order-isomorphic, then $\mathbf{B}$ and Q are said to be dually order-isomorphic. A pair $({}^{\triangledown}, {}^{\vartriangle})$ of maps ${}^{\triangledown} : B \to B$ and ${}^{\vartriangle} : B \to B$ is called a dual Galois connection on B if ${}^{\triangledown}$ and ${}^{\vartriangle}$ are order preserving and $x^{\triangledown\vartriangle} \leq x \leq x^{\vartriangle\triangledown}$ for all $x \in B$.

Before we consider the Boolean lattices, we present the following lemma, where $\wp(B)$ denotes the power set of B, that is, the set of all subsets of B.

Lemma 1 (see [23]). *Let $\mathbf{B} = (B, \leq)$ be a complete lattice, $S, T \subseteq B$, and $\{X_i : i \in I\} \subseteq \wp(B)$.*

(i) *If $S \subseteq T$, then $\bigvee S \subseteq \bigvee T$.*
(ii) *$\bigvee(S \cup T) = (\bigvee S) \bigvee (\bigvee T)$.*
(iii) *$\bigvee(\bigcup \{X_i : i \in I\}) = \bigvee\{\bigvee X_i \in I\}$.*

Next we recall the concept of Boolean lattices. They are bounded distributive lattices with a complementation operation.

Definition 2 (see [27]). A lattice $\mathbf{B} = (B, \leq)$ is called a Boolean lattice, if

(i) B is distributive;
(ii) B has a least element 0 and a greatest element 1, and;
(iii) each $x \in B$ has a complement $x' \in B$ such that $x \vee x' = 1$ and $x \wedge x' = 0$.

Lemma 3 (see [27]). *Let $\mathbf{B} = (B, \leq)$ be a Boolean lattice; then for all $x, y \in B$*

(i) *$0' = 1$ and $1' = 0$,*
(ii) *$x'' = x$,*
(iii) *$(x \vee y)' = x' \wedge y'$, and $(x \wedge y)' = x' \vee y'$,*
(iv) *$x \leq y$ iff $x \wedge y' = 0$.*

Let us recall some definitions and results that are useful in our consideration given in [23].

Lemma 4 (see [23]). *Let $\mathbf{B} = (B, \leq)$ be a complete Boolean lattice. Then for all $\{x_i : i \in I\} \subseteq B$ and $y \in B$*

$$y \wedge \left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I} (y \wedge x_i),$$
$$y \vee \left(\bigwedge_{i \in I} x_i\right) = \bigwedge_{i \in I} (y \vee x_i). \tag{1}$$

Definition 5 (see [23]). Let $\mathbf{B} = (B, \leq)$ be an ordered set and $x, y \in B$; we say that x is covered by y (or that y covers x), and write $x \prec y$ if $x < y$ and there is no element z in B with $x < z < y$.

Definition 6 (see [23]). Let $\mathbf{B} = (B, \leq)$ be a lattice with a least element 0. Then $a \in B$ is called an atom if $0 \prec a$. The set of atoms of B is denoted by $A(B)$. The lattice B is called atomic if every element of B is the supremum of the atoms below it; that is, $x = \bigvee\{a \in A(B) : a \leq x\}$.

It is obvious that in a lattice $\mathbf{B} = (B, \leq)$ with a least element 0,

$$a \wedge x \neq 0 \Longleftrightarrow a \leq x \tag{2}$$

for all $a \in A(B)$ and $x \in B$. This implies that $a \wedge b = 0$ for all $a, b \in A(B)$ s.t $a \neq b$. Furthermore, if B is atomic, then for all $x \neq 0$ there exists an atom $a \in A(B)$ s.t $a \leq x$. Namely, if $\{a \in A(B) : a \leq x\} = \phi$, then $x = \bigvee\{a \in A(B) : a \leq x\} = \bigvee \phi = 0$.

Definition 7 (see [23]). Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice. We say that $\varphi : A(B) \rightarrow B$ is

(i) extensive, if $a \leq \varphi(a)$ for all $a \in A(B)$,

(ii) symmetric, if $a \leq \varphi(b)$ implies $b \leq \varphi(a)$ for all $a, b \in A(B)$,

(iii) closed, if $b \leq \varphi(a)$ implies $\varphi(b) \leq \varphi(a)$ for all $a, b \in A(B)$.

Definition 8 (see [23]). Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \rightarrow B$ be any mapping. For any element $x \in B$, let

$$x^{\nabla} = \bigvee \{a \in A(B) : \varphi(a) \leq x\}, \quad x^{\Delta} = \bigvee \{a \in A(B) : \varphi(a) \wedge x \neq 0\}. \tag{3}$$

The elements x^{∇} and x^{Δ} are called the *lower* and the *upper* approximations of x with respect to φ, respectively. Two elements x and y are called equivalent if they have the same upper and lower approximations. The resulting equivalence classes are called rough sets.

The following results are shown in [23, 24]. The ordered sets $(B^{\Delta}, \leq)$ and $(B^{\Delta}, \leq)$ are always complete lattices. They are distributive sublattices of $(B, \leq)$ if φ is extensive and closed. If the map φ is extensive, symmetric, and closed, then the ordered sets $(B^{\Delta}, \leq)$ and $(B^{\Delta}, \leq)$ are mutually equal complete atomic Boolean lattices.

Proposition 9 (see [23]). *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \rightarrow B$ be any mapping. Then for all $a \in A(B)$ and $x \in B$,*

(i) $a \leq x^{\nabla} \Leftrightarrow \varphi(a) \leq x$;

(ii) $a \leq x^{\Delta} \Leftrightarrow \varphi(a) \wedge x \neq 0$.

Proposition 10 (see [23]). *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \rightarrow B$ be an extensive mapping. Then for all $x \in B$,*

(i) $x^{\nabla} \leq x$;

(ii) $x \leq x^{\Delta}$.

Proposition 11 (see [23]). *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \rightarrow B$ be extensive and closed mapping. Then for all $x \in B$,*

(i) $x^{\nabla} = x^{\nabla\nabla}$;

(ii) $x^{\Delta\Delta} = x^{\Delta}$.

Proposition 12 (see [23]). *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \rightarrow B$ be an extensive, symmetric and closed mapping. Then for all $x \in B$,*

(i) $x^{\nabla\Delta} = x^{\nabla}$;

(ii) $x^{\Delta\nabla} = x^{\Delta}$.

Next, we recall the definitions of Pawlak rough sets, soft sets, and soft rough approximation operators.

Definition 13 (see [29]). An information system (or a knowledge representation system) is a pair $\gamma = (U, A)$ of nonempty finite sets U and A, where U is a set of objects and A is a set of attributes; each attribute $a \in A$ is a function $a : U \rightarrow V_a$, where V_a is the set of values (called domain) of attribute a.

Let U be a non-empty finite universe and let R be an equivalence relation on U. The pair (U, R) is called a Pawlak approximation space. The equivalence relation R is often called an indiscernibility relation and related to an information system. Specifically, if $\gamma = (U, A)$ is an information system and $B \subseteq A$, then an indiscernibility relation $R = I(B)$ can be defined by

$$(x, y) \in I(B) \Longleftrightarrow a(x) = a(y) \quad \forall a \in B, \tag{4}$$

where $x, y \in U$ and $a(x)$ denotes the value of attribute a for object x.

Using the indiscernibility relation R, one can define the following two operations:

$$R_* X = \{x \in U : [x]_R \subseteq X\}, \quad R^* X = \{x \in U : [x]_R \cap X \neq \emptyset\} \tag{5}$$

assigning to every subset $X \subseteq U$ two sets $R_* X$ and $R^* X$ called the R-lower and the R-upper approximation of X, respectively.

If $R_* X = R^* X$, then X is said to be R-definable; otherwise, X is said to be R-rough.

Let us recall now the soft set notion, which is a newly emerging mathematical approach to vagueness.

Definition 14 (see [2]). Let U be a universal set and let E be a set of parameters. Let A be a nonempty subset of E. A soft set over A, with support A, denoted by f_A on U is defined by the set of ordered pairs

$$f_A = \{(e, f_A(e)) : e \in E, f_A(e) \in \wp(U)\} \tag{6}$$

or is a function $f_A : E \rightarrow \wp(U)$ s.t

$$f_A(e) \neq \phi, \quad \forall e \in A \subseteq E, \quad f_A(e) = \phi \quad \text{if } e \notin A. \tag{7}$$

Example 15. Suppose that U is the set of houses under consideration and A and B are both parameter sets. Let there be four houses in the universe U given by $U = \{h_1, h_2, h_3, h_4\}$. And $A = \{\text{expensive}, \text{modern}\}$ and $B = \{\text{modern}\}$. The soft sets f_A and g_B describe the "attractiveness of the houses." For the sake of ease of designation, we use e, instead of expensive and m instead of modern. The soft set f_A is defined as follows $f(e)$ means expensive houses, and $f(m)$ means modern houses. The soft set f_A is the collection of approximations as below:

$$f_A = \{(e, \{h_1, h_2\}), (m, \{h_4\})\}. \tag{8}$$

Table 1: An information table.

	u_1	u_2	u_3	u_4	u_5	u_6
Sex	Woman	Woman	Man	Man	Man	Man
Age category	Young	Young	Mature age	Old	Mature age	Baby
Living area	City	City	City	Village	City	Village
Habits	NSND	NSND	Smoke	SD	Smoke	NSND

The soft set g_B is defined as $g(m)$, which means the modern houses. The soft set g_B is the collection of approximations as below:

$$g_B = \{(m, \{h_1, h_4\})\}. \tag{9}$$

Definition 16 (see [25, 26]). Let U be a universal set and let f_A be a soft set over U. Then the pair $P = (U, f_A)$ is called soft approximation space. We define a pair of operators $\underline{\text{apr}}_P$, $\overline{\text{apr}}_P : \wp(U) \to \wp(U)$ as follows:

$$\begin{aligned} \underline{\text{apr}}_P(X) &= \{u \in U : \exists a \in A, \text{ s.t } u \in f(a) \subseteq X\}, \\ \overline{\text{apr}}_P(X) &= \{u \in U : \exists a \in A, \text{ s.t } u \in f(a), f(a) \cap X \neq \emptyset\}. \end{aligned} \tag{10}$$

The elements $\underline{\text{apr}}_P(X)$ and $\overline{\text{apr}}_P(X)$ are called the *soft P-lower* and the *soft P-upper* approximations of X.

If $\underline{\text{apr}}_P(X) = \overline{\text{apr}}_P(X)$, X is said to be soft P-definable; otherwise X is called a soft P-rough set.

Example 17. Let us consider the following soft set $S = f_E$ which describes "life expectancy". Suppose that the universe $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ consists of six persons and $E = \{e_1, e_2, e_3, e_4\}$ is a set of decision parameters. The e_i ($i = 1, 2, 3, 4$) stands for "under stress," "young," "drug addict" and "healthy." Set $f(e_1) = \{u_5\}$, $f(e_2) = \{u_1, u_2\}$, $f(e_3) = \emptyset$; and $f(e_4) = \{u_1, u_2, u_3, u_6\}$. The soft set f_E can be viewed as the following collection of approximations:

$$\begin{aligned} f_E = \{&(\text{under stress}, \{u_5\}); (\text{young}, \{u_1, u_2\}); \\ &(\text{drugaddict}, \emptyset); (\text{healthy}; \{u_1, u_2, u_3, u_6\})\}. \end{aligned} \tag{11}$$

On the other hand, "life expectancy" topic can also be described using rough sets as follows: the evaluation will be done in terms of attributes: "sex", "age category", "living area", and "habits", characterized by the value sets "{man, woman}", "{baby, young, mature age, old}", "{village, city}", and "{smoke, drinking, smoke and drinking, no smoke and no drinking}". We denote "smoke and drinking" by SD and "no smoke and no drinking" by NSND. The information will be given by Table 1, where the rows are labeled by attributes and the table entries are the attribute values for each person. From here we obtain the following equivalence classes, induced by the above mentioned attributes:

$$\begin{gathered} [u_1]_R = [u_2]_R = \{u_1, u_2\}, \\ [u_3]_R = [u_5]_R = \{u_3, u_5\}, \\ [u_4]_R = \{u_4\}, \qquad [u_6]_R = \{u_6\}. \end{gathered} \tag{12}$$

Let X be a target subset of U, that we wish to represent using the above equivalence classes. Hence we analyze the upper and lower approximations of X, in some particular cases.

(1) Set $X = \{u_1, u_2, u_3, u_6\}$. It follows that

$$\begin{aligned} R_* X &= \{u_1, u_2, u_6\}, \\ R^* X &= \{u_1, u_2, u_3, u_5, u_6\}. \end{aligned} \tag{13}$$

Let us calculate now the soft P-lower and P-upper approximations of X, where $P = (U, S)$. We obtain

$$\begin{aligned} \underline{\text{apr}}_P(X) &= \&\{u_1, u_2, u_3, u_6\} = X, \\ \overline{\text{apr}}_P(X) &= \{u_1, u_2, u_3, u_6\} = X; \end{aligned} \tag{14}$$

hence X is soft P-definable.

(2) Set $X = \{u_5\}$. It follows that $R_* X = \{u_3, u_5\}$. On the other hand, $\underline{\text{apr}}_P(X) = \overline{\text{apr}}_P(X) = X$, hence X; is soft P-definable.

The above results show that soft rough set approximation is a worth considering alternative to the rough set approximation. Soft rough sets could provide a better approximation than rough sets do, depending on the structure of the equivalence classes and of the subsets $F(e)$, where $e \in E$.

3. Soft Sets on a Complete Atomic Boolean Lattice

Definition 18. Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let E be a set of parameters. Let A be a non empty subset of E. A soft set over A, with support A, denoted by f_A on B is defined by the set of ordered pairs

$$f_A = \{(e, f_A(e)) : e \in E, f_A(e) \in B\}, \tag{15}$$

or is a function $F_A : E \to B$ s.t

$$\begin{aligned} f_A(e) &\neq 0, \quad \forall e \in A \subseteq E, \\ f_A(e) &= 0 \quad \text{if } e \notin A. \end{aligned} \tag{16}$$

In other words, a soft set over B is a parameterized family of elements of B. For each $e \in A$, $f(e)$ is considered as e-approximate element of f_A.

Definition 19. Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice. Let $A_1, A_2 \subseteq E$ and let f_{A_1} and g_{A_2} be two soft sets over B.

(i) f_{A_1} is a soft subset of g_{A_2}, denoted by $f_{A_1} \sqsubseteq g_{A_2}$ if $A_1 \subseteq A_2$ and $f(e) \leq g(e)$ for every $e \in A_1$.

(ii) f_{A_1} and g_{A_2} are called soft equal, denoted by $f_{A_1} = g_{A_2}$ if $f_{A_1} \sqsubseteq g_{A_2}$ and $g_{A_2} \sqsubseteq f_{A_1}$.

Definition 20. Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice. Let $A \subseteq E$ and let f_A be a soft set over B.

(i) f_A is called null, denoted by 0_A if $f(e) = 0$ for every $e \in A$.

(ii) f_A is called absolute, denoted by 1_A if $f(e) = 1$ for every $e \in A$.

We stipulate that 0_ϕ is also a soft set over B with $0 : \phi \to B$.

Let $A \subseteq E$ and let f_A be a soft set over B. Obviously,

$$0_A \sqsubseteq f_A \sqsubseteq 1_A. \tag{17}$$

Below, we introduce some operations on soft sets on B and investigate their properties.

Definition 21. Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice. Let $A_1, A_2 \subseteq E$ and let f_{A_1} and g_{A_2} be two soft sets over B.

(i) h_{A_3} is called the intersection of f_{A_1} and g_{A_2}, denoted by $f_{A_1} \sqcap g_{A_2} = h_{A_3}$ if $A_3 = A_1 \cap A_2$ and $h(e) = f(e) \wedge g(e)$ for every $e \in A_3$.

(ii) h_{A_3} is called the union of f_{A_1} and g_{A_2}, denoted by $f_{A_1} \sqcup g_{A_2} = h_{A_3}$ if $A_3 = A_1 \cup A_2$ and $h(e) = f(e)$ if $e \in A - B, h(e) = g(e)$ if $e \in B - A$ and $h(e) = f(e) \vee g(e)$ if $e \in A \cap B$.

Definition 22. Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice. Let $A \subseteq E$ and let f_A be a soft set over B. The complement of f_A, denoted by $(f_A)^c$ is defined by $(f_A)^c = (f^c, A)$, where $f^c : A \to B$ is a mapping given by $f^c(e) = f(e)'$ for every $e \in A$.

Proposition 23. *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice. Let $A_1, A_2, A_3 \subseteq E$ and let f_{A_1}, g_{A_2}, and h_{A_3} be three soft sets over B. Then*

(i) $f_{A_1} \sqcup f_{A_1} = f_{A_1}$,

(ii) $f_{A_1} \sqcup g_{A_2} = g_{A_2} \sqcup f_{A_1}$,

(iii) $(f_{A_1} \sqcup g_{A_2}) \sqcup h_{A_3} = f_{A_1} \sqcup (g_{A_2} \sqcup h_{A_3})$.

Proof. (i) and (ii) are obvious. We only prove (iii). Put

$$\begin{gathered} \left(f_{A_1} \sqcup g_{A_2}\right) \sqcup h_{A_3} = k_{A_1 \cup A_2 \cup A_3}, \\ f_{A_1} \sqcup \left(g_{A_2} \sqcup h_{A_3}\right) = l_{A_1 \cup A_2 \cup A_3}, \\ f_{A_1} \sqcup g_{A_2} = s_{A_1 \cup A_2}, \quad g_{A_2} \sqcup h_{A_3} = t_{A_2 \cup A_3}. \end{gathered} \tag{18}$$

For any $e \in A_1 \cup A_2 \cup A_3$ it follows that $e \in A_1$, or $e \in A_2$, or $e \in A_3$.

Case 1 ($e \in A_3$).

(a) If $e \notin A_1$ and $e \notin A_2$, then $k(e) = h(e) = t(e) = l(e)$.

(b) If $e \notin A_1$ and $e \in A_2$, then $k(e) = s(e) \vee h(e) = g(e) \vee h(e) = t(e) = l(e)$.

(c) If $e \in A_1$ and $e \notin A_2$, then $k(e) = s(e) \vee h(e) = f(e) \vee h(e) = f(e) \vee t(e) = l(e)$.

(d) If $e \in A_1$ and $e \in A_2$, then $k(e) = s(e) \vee h(e) = f(e) \vee g(e) \vee h(e) = f(e) \vee t(e) = l(e)$.

Case 2 ($e \notin A_3$).

(a) If $e \notin A_1$ and $e \in A_2$, then $k(e) = s(e) = g(e) = t(e) = l(e)$.

(b) If $e \in A_1$ and $e \notin A_2$, then $k(e) = s(e) = f(e) = l(e)$.

(c) If $e \in A_1$ and $e \in A_2$, then $k(e) = s(e) = f(e) \vee g(e) = f(e) \vee t(e) = l(e)$.

Thus $(f_{A_1} \sqcup g_{A_2}) \sqcup h_{A_3} = f_{A_1} \sqcup (g_{A_2} \sqcup h_{A_3})$. □

Proposition 24. *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice. Let $A_1, A_2, A_3 \subseteq E$ and let f_{A_1}, g_{A_2}; and h_{A_3} be three soft sets over B. Then*

(i) $f_{A_1} \sqcap f_{A_1} = f_{A_1}$,

(ii) $f_{A_1} \sqcap g_{A_2} = g_{A_2} \sqcap f_{A_1}$,

(iii) $(f_{A_1} \sqcap g_{A_2}) \sqcap h_{A_3} = f_{A_1} \sqcap (g_{A_2} \sqcap h_{A_3})$.

Proof. (i) and (ii) are obvious. We only prove (iii). Put

$$\begin{gathered} \left(f_{A_1} \sqcap g_{A_2}\right) \sqcap h_{A_3} = k_{A_1 \cap A_2 \cap A_3}, \\ f_{A_1} \sqcap \left(g_{A_2} \sqcup h_{A_3}\right) = l_{A_1 \cap A_2 \cap A_3}. \end{gathered} \tag{19}$$

For any $e \in A_1 \cap A_2 \cap A_3$, it follows that $e \in A_1$, $e \in A_2$, and $e \in A_3$. Since $k(e) = (f(e) \wedge g(e)) \wedge h(e) = f(e) \wedge (g(e) \wedge h(e)) = l(e)$, then $(f_{A_1} \sqcap g_{A_2}) \sqcap h_{A_3} = f_{A_1} \sqcap (g_{A_2} \sqcap h_{A_3})$. □

Proposition 25. *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice. Let $A_1, A_2, A_3 \subseteq E$ and let f_{A_1}, g_{A_2}, and h_{A_3} be three soft sets over B. Then*

(i) $(f_{A_1} \sqcup g_{A_2}) \sqcap h_{A_3} = (f_{A_1} \sqcap h_{A_3}) \sqcup (g_{A_2} \sqcap h_{A_3})$,

(ii) $(f_{A_1} \sqcap g_{A_2}) \sqcup h_{A_3} = (f_{A_1} \sqcup h_{A_3}) \sqcap (g_{A_2} \sqcup h_{A_3})$.

Proof. (i) Put $(f_{A_1} \sqcup g_{A_2}) \sqcap h_{A_3} = k_{(A_1 \cup A_2) \cap A_3}$, $(f_{A_1} \sqcup h_{A_3}) \sqcap (g_{A_2} \sqcup h_{A_3}) = l_{(A_1 \cap A_3) \cup (A_2 \cap A_3)}$. Obviously, $(A_1 \cup A_2) \cap A_3 = (A_1 \cap A_3) \cup (A_2 \cap A_3)$. For any $e \in (A_1 \cup A_2) \cap A_3$, it follows that $e \in A_1 \cap A_3$ or $e \in A_2 \cap A_3$.

(a) If $e \notin A_1 \cap A_3$ and $e \in A_2 \cap A_3$, then $e \notin A_1$, $e \in A_2$, and $e \in A_3$. So $k(e) = g(e) \wedge h(e) = l(e)$.

(b) If $e \in A_1 \cap A_3$ and $e \notin A_2 \cap A_3$, then $e \in A_1$, $e \notin A_2$, and $e \in A_3$. So $k(e) = f(e) \wedge h(e) = l(e)$.

(c) If $e \in A_1 \cap A_3$ and $e \in A_2 \cap A_3$, then $e \in A_1, e \in A_2$, and $e \in A_3$. So $k(e) = (f(e) \vee g(e)) \wedge h(e) = (f(e) \wedge h(e)) \vee (g(e) \wedge h(e)) = l(e)$.

Thus $(f_{A_1} \sqcup g_{A_2}) \sqcap h_{A_3} = (f_{A_1} \sqcap h_{A_3}) \sqcup (g_{A_2} \sqcap h_{A_3})$.

(ii) This is similar to the proof of (i). □

Proposition 26. *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice. Let $A_1, A_2 \subseteq E$ and let f_A and g_A be two soft sets over B.*

(i) $((f_A)^c)^c = f_A$.

(ii) $f_A \sqcup (f_A)^c = 1_A$.

(iii) $f_A \sqcap (f_A)^c = 0_A$.

(iv) $(f_A \sqcup g_A)^c = (f_A)^c \sqcap (g_A)^c$.

(v) $(f_A \sqcap g_A)^c = (f_A)^c \sqcup (g_A)^c$.

Proof. (i) Put $(f_A)^c = g_A, (g_A)^c = h_A$.

For any $e \in A, h(e) = g^c(e) = g(e)', g(e) = f^c(e) = f(e)'$. So, $h(e) = g(e)' = (f(e)')' = f(e)$ (by Lemma 3). This Shows that $h_A = f_A$; that is $((f_A)^c)^c = f_A$.

(ii) Put $f_A \sqcup (f_A)^c = h_A$.

For any $e \in A, h(e) = f(e) \vee f^c(e) = f(e) \vee f(e)' = 1$. Hence $f_A \sqcup (f_A)^c = 1_A$.

(iii) This is similar to the proof of (ii).

(iv) Put $(f_A \sqcup g_A)^c = h_A, (f_A)^c \sqcap (g_A)^c = l_A$.

For any $e \in A, h(e) = (f(e) \vee g(e))', l(e) = f(e)' \wedge g(e)'$. Hence $h(e) = l(e)$ by Lemma 3.

(v) This is similar to the proof of (iv). □

Definition 27. Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and Let f_E be a soft set over B.

(i) f_E is called full, if $\bigvee_{e \in E} f(e) = 1$;

(ii) f_E is keeping infimum, if for any $e_1, e_2 \in E$, there exists $e_3 \in E$ such that $f(e_1) \wedge f(e_2) = f(e_3)$;

(iii) f_E is keeping supremum, if for any $e_1, e_2 \in E$, there exists $e_3 \in E$ such that $f(e_1) \vee f(e_2) = f(e_3)$;

(iv) f_E is called partition of B if

(1) $\bigvee_{e \in E} f(e) = 1$,

(2) for every $e \in E$, $f(e) \neq 0$,

(3) for every $e_1, e_2 \in E$ either $f(e_1) = f(e_2)$ or $f(e_1) \wedge f(e_2) = 0$.

Obviously, every partition soft set is full and f_E is keeping infimum (resp., keeping supremum) if and only if for every $E^* \subseteq E$, there exists $e^* \in E$ such that $\bigwedge_{e \in E^*} f(e) = f(e^*)$ (resp., $\bigvee_{e \in E^*} f(e) = f(e^*)$).

Example 28. Let $B = \{0, a, b, c, d, e, f, 1\}$ and let the order $\leq$ be defined as in Figure 1.

The set of atoms of a complete atomic Boolean lattice $\mathbf{B} = (B, \leq)$ is $\{a, b, c\}$. Let $A = \{e_1, e_2, e_3, e_4\}$ and let f_A be a soft set over B defined as follows:

$$f(e_1) = e, \qquad f(e_2) = b,$$
$$f(e_3) = c, \qquad f(e_4) = 0. \tag{20}$$

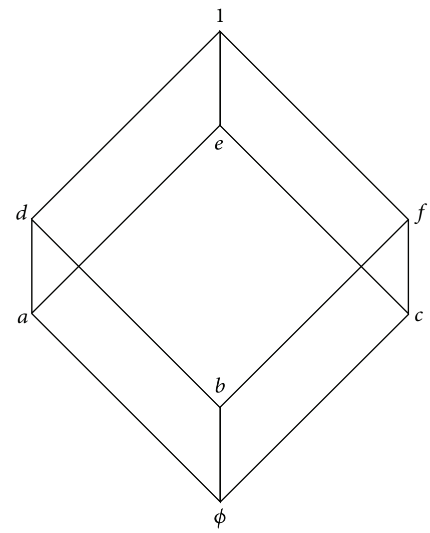

Figure 1

Obviously, f_A is not a partition since $f(e_4) = 0$. Also, f_A is full since $\bigvee_{e \in A} f(e) = e \vee b \vee c = 1$. Also, f_A is keeping infimum. In fact $f(e_1) \wedge f(e_2) = f(e_1) \wedge f(e_4) = f(e_3) \wedge f(e_4) = f(e_2) \wedge f(e_4) = f(e_4) = 0$.

$f(e_1) \wedge f(e_3) = e \wedge c = c = f(e_3)$ and $f(e_2) \wedge f(e_3) = b \wedge c = 0 = f(e_4)$. Consequently, f_A is keeping infimum. On the other hand, f_A is not keeping supremum since $f(e_1) \vee f(e_2) = e \vee b = 1 \neq f(e)$ for every $e \in A$.

Let g_A be a soft set over B defined as follows:

$g(e_1) = d$, $g(e_2) = a$, $g(e_3) = e$, and $g(e_4) = 1$; then g_A is a partition, keeping infimum, and keeping supremum.

Next, we investigate the lattice structure of soft sets on a complete atomic Boolean Lattice B. We denote

$$S(B, E) = \{f_E : f_E \text{ is soft set over } B\},$$
$$S_1(B, E) = \{f_A : A \subseteq E \text{ and } f_A \text{ is soft set over } B\}. \tag{21}$$

Obviously,

$$S_1(B, E) \subseteq S(B, E). \tag{22}$$

Theorem 29. *For any $f_A, g_B \in S(B, E)$, define*

$$f_A \leq g_B \Longleftrightarrow f_A \sqsubseteq g_B, \qquad f_A \vee g_B = f_A \sqcup g_B,$$
$$f_A \wedge g_B = f_A \sqcap g_B. \tag{23}$$

Then $S(B, E)$ is a distributive lattice with smallest element $0_\Sigma = 0_\phi$ and greatest element $1_\Sigma = 1_E$.

Proof. Denote $\Sigma = S(B, E)$. It is easily proved that

$$0_\Sigma = 0_\phi, \qquad 1_\Sigma = 1_E. \tag{24}$$

By Proposition 25 $S(X, E)$ is a distributive lattice with 1_Σ and 0_Σ. □

Theorem 30. *For any $f_A, g_B \in S_1(B,E)$, define*

$$f_A \le g_B \Longleftrightarrow f_A \sqsubseteq g_B, \qquad f_A \vee g_B = f_A \sqcup g_B, \\ f_A \wedge g_B = f_A \sqcap g_B. \tag{25}$$

Then $S_1(B,E)$ is a Boolean lattice.

Proof. Denote $\Sigma_1 = S_1(B,E)$. It is easily proved that $S_1(B,E)$ is a distributive lattice with $0_{\Sigma_1} = 0_E$ and $1_{\Sigma_1} = 1_E$.

Let $f_E \in \Sigma_1$. Put $h_E = f_E \vee f_E^c$. Since $h_E = f_E \sqcup f_E^c$, then for any $e \in E$,

$$h(e) = f(e) \vee f^c(e) = f(e) \vee f(e)' = 1. \tag{26}$$

So, $h_E = 1_E = 1_{\Sigma_1}$. This shows that $f_E \vee f_E' = 1_{\Sigma_1}$. Similarly, we can prove that $f_E \wedge f_E' = 0_{\Sigma_1}$. Hence $(f_E)' = f_E^c$ and therefore $S_1(B,E)$ is a Boolean lattice. □

4. Soft Rough Approximation Operators on a Complete Atomic Boolean Lattice

Definition 31. Let $\mathbf{B} = (B,\le)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. For any element $x \in B$, we define a pair of operators $x^\vee, x^\wedge : B \to B$ as follows:

$$x^\vee = \bigvee \{b \in A(B) : \exists e \in A \text{ s.t } b \le f(e),\ f(e) \le x\}, \\ x^\wedge = \bigvee \{b \in A(B) : \exists e \in A \text{ s.t } b \le f(e),\ f(e) \wedge x \ne 0\}. \tag{27}$$

The elements $x^\vee$ and $x^\wedge$ are called the *soft lower* and the *soft upper* approximations of x over B. Two elements x and y are called soft equivalent if they have the same soft upper and soft lower approximations over B. The resulting equivalence classes are called soft rough sets over B.

Lemma 32. *Let $\mathbf{B} = (B,\le)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Then for all $c \in A(B)$ and $x \in B$*

(i) *$c \le x^\vee \Leftrightarrow \exists e \in A$ s.t $c \le f(e)$ and $f(e) \le x$;*

(ii) *$c \le x^\wedge \Leftrightarrow \exists e \in A$ s.t $c \le f(e)$ and $f(e) \wedge x \ne 0$.*

Proof. (i) ($\Rightarrow$) Suppose that $c \le x^\vee = \bigvee\{b \in A(B) : \exists e \in A \text{ s.t } b \le f(e) \text{ and } f(e) \le x\}$. Assume that for all $e \in A$ either $c \nleq f(e)$ or $f(e) \nleq x$. If $\forall e \in A$, $f(e) \nleq x$, then $c \nleq x^\vee$, a contradiction. If $\forall e \in A$, $c \nleq f(e)$, then $c \wedge x^\vee = c \wedge \bigvee\{b \in A(B) : \exists e \in A \text{ s.t } b \le f(e) \text{ and } f(e) \le x\} = \bigvee\{c \wedge b : b \in A(B), \exists e \in A \text{ s.t } b \le f(e) \text{ and } f(e) \le x\}$. Since $c \nleq f(e)$, then, $c \ne b$. So $c \wedge b = 0$ because c, $b \in A(B)$. Hence $c \wedge x^\vee = 0$. This implies that $c \le (x^\vee)'$, which is a contradiction.

($\Leftarrow$) Suppose that $\exists e \in A$ s.t $c \le f(e)$ and $f(e) \le x$; then $c \le \bigvee\{b \in A(B) : \exists e \in A \text{ s.t } b \le f(e) \text{ and } f(e) \le x\} = x^\vee$.

Condition (ii) can be proved similarly. □

Proposition 33. *Let $\mathbf{B} = (B,\le)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Then for all $x \in B$*

(i) *$x^\vee = \bigvee\{f(e) : e \in A \text{ and } f(e) \le x\} \le x$;*

(ii) *$x^\wedge = \bigvee\{f(e) : e \in A \text{ and } f(e) \wedge x \ne 0\}$.*

Proof. (i) Let $c \in A(B)$, s.t $c \le x^\vee$; then $\exists e \in A$ s.t $c \le f(e)$ and $f(e) \le x$. So, $c \le \bigvee\{f(e) : e \in A \text{ and } f(e) \le x\} \le x$. On the other hand, let $c \in A(B)$, s.t $c \le \bigvee\{f(e) : e \in A \text{ and } f(e) \le x\}$. Hence, $\exists e \in A$ s.t $c \le f(e)$ and $f(e) \le x$. In fact, if $e \in A$ and $f(e) \le x$ implies $c \nleq f(e)$, then $c \wedge f(e)' \ne 0$. Therefore $c \le f(e)'$ because $c \in A(B)$. Thus $c \le \bigvee\{f(e)' : e \in A \text{ and } f(e) \le x\}$. So, $c \le \bigvee\{f(e) \wedge f(e)' : e \in A \text{ and } f(e) \le x\} = 0$, a contradiction. So, $\exists e \in A$ s.t $c \le f(e)$ and $f(e) \le x$ and consequently, $c \le x^\vee$.

Condition (ii) can be proved similarly. □

Proposition 34. *Let $\mathbf{B} = (B,\le)$ be a complete atomic Boolean lattice and let f_A be a soft set over B.*

(i) *$0^\vee = 0^\wedge = 0$ and $1^\vee = 1^\wedge = \bigvee_{e\in A} f(e)$;*

(ii) *$x \le y$ implies $x^\vee \le y^\vee$ and $x^\wedge \le y^\wedge$.*

Proof. Obvious.

For all $S \subseteq B$, we denote $S^\vee = \{x^\vee : x \in S\}$ and $S^\wedge = \{x^\wedge : x \in S\}$. □

Proposition 35. *Let $\mathbf{B} = (B,\le)$ be a complete atomic Boolean lattice and let f_A be a soft set over B; then*

(i) *for all $S \subseteq B$, $\vee S^\wedge = (\vee S)^\wedge$;*

(ii) *if f_A is keeping infimum, then for all $S \subseteq B$, $\wedge S^\vee = (\wedge S)^\vee$;*

(iii) *$(B^\wedge,\le)$ is a complete lattice; 0 is the least element and $1^\wedge$ is the greatest element of $(B^\wedge,\le)$;*

(iv) *if f_A is keeping infimum, then $(B^\vee,\le)$ is a complete lattice; 0 is the least element and $1^\vee$ is the greatest element of $(B^\vee,\le)$;*

(v) *if f_A is keeping infimum, the kernal $\Theta_\vee = \{(x,y) : x^\vee = y^\vee\}$ of the map ${}^\vee : B \to B$ is a congruence on the semi lattice $(B,\wedge)$ such that the $\Theta_\vee$-class of any x has a least element;*

(vi) *the kernal $\Theta_\wedge = \{(x,y) : x^\wedge = y^\wedge\}$ of the map ${}^\wedge : B \to B$ is a congruence on the semilattice $(B,\vee)$ such that the $\Theta_\wedge$-class of any x has a least element.*

Proof. (i) Let $S \subseteq B$. The map ${}^\wedge : B \to B$ is order preserving, which implies that $\vee S^\wedge \le (\vee S)^\wedge$. Let $b \in A(B)$ and assume that $b \le (\vee S)^\wedge$. So, $\exists e \in A$ s.t $b \le f(e)$ and $f(e) \wedge \vee S \ne 0$. Then $0 \ne f(e) \wedge \bigvee S = \bigvee\{f(e) \wedge x : x \in S\}$, which implies that $f(e) \wedge x \ne 0$ for some $x \in S$. Thus $\{b \in A(B) : \exists e \in A \text{ s.t } b \le f(e) \text{ and } f(e) \wedge \bigvee S \ne 0\} \subseteq \cup_{x\in S}\{b \in A(B) : \exists e \in A \text{ s.t } b \le f(e) \text{ and } f(e) \wedge x \ne 0\}$. Then

$$(\vee S)^\wedge \\ = \bigvee \left\{b \in A(B) : \exists e \in A \text{ s.t } b \le f(e), f(e) \wedge \bigvee S \ne 0\right\}$$

$$\begin{aligned}
&\leq \bigvee \left(\cup_{x\in S}\{b \in A(B) : \exists e \in A \text{ s.t } b \leq f(e), f(e) \wedge x \neq 0\}\right) \\
&= \bigvee_{x\in S}\left(\bigvee \{b \in A(B) : \exists e \in E \text{ s.t } b \leq f(e), f(e) \wedge x \neq 0\}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 1)} \\
&= \bigvee \{x^{\wedge} : x \in S\} = \bigvee S^{\wedge}.
\end{aligned} \tag{28}$$

(ii) Let $S \subseteq B$. The map ${}^{\vee} : B \to B$ is order preserving, which implies that $(\wedge S)^{\vee} \leq \wedge S^{\vee}$. Let $b \in A(B)$ s.t $b \leq \wedge S^{\vee} = \wedge\{x^{\vee} : x \in S\}$. So, $\exists e \in A$ s.t $b \leq f(e)$ and $f(e) \leq x$ for every $x \in S$. Hence $\bigwedge\{f(e) : b \leq f(e)$ and $f(e) \leq x\} \leq x$ for every $x \in S$. This implies that $\bigwedge_{e\in A}\{f(e) : b \leq f(e)$ and $f(e) \leq x\} \leq \wedge\{x : x \in S\} = \wedge S$. Since f_A is keeping infimum, then $\wedge_{e\in A} f(e) = f(e_1)$ for $e_1 \in A$. So we show that $\exists e_1 \in A$ s.t $b \leq f(e_1)$ and $f(e_1) \leq \wedge S$. Therefore $b \leq (\wedge S)^{\vee}$. Consequently, $\wedge S^{\vee} \leq (\wedge S)^{\vee}$. Assertions (iii), and (iv) follow easily from (i), (ii) and Proposition 23(i). The proof of (v) and (vi) follows by (i) and (ii). □

In the following example, we show that in general $(B^{\wedge}, \leq)$ and $(B^{\vee}, \geq)$ are not dually order-isomorphic.

Example 36. Let $B = \{0, a, b, c, d, e, f, 1\}$ and let the order $\leq$ be defined as in Figure 1.

Let $A = \{e_1, e_2, e_3, e_4\}$ and let f_A be a soft set over B defined as follows:

$$\begin{aligned} f(e_1) &= e, & f(e_2) &= b, \\ f(e_3) &= c, & f(e_4) &= 0. \end{aligned} \tag{29}$$

Then f_A is not a partition since $f(e_4) = 0$. Let $x = c$ and $y = d$; then $x^{\wedge} = a \vee c = e$ and $y^{\wedge} = b \vee a \vee c = 1$. Therefore $x^{\wedge} \leq y^{\wedge}$. On the other hand $y'^{\vee} = c^{\vee} = c \nleq x'^{\vee} = d^{\vee} = b$.

Next, we show that $(B^{\wedge}, \leq)$ and $(B^{\vee}, \geq)$ are dually order-isomorphic if f_A is a partition.

Proposition 37. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* f_A *be a soft set over B. If* f_A *is a partition, then* $(B^{\vee}, \geq) \cong (B^{\wedge}, \leq)$.

Proof. We show that $x^{\wedge} \to (x')^{\vee}$ is the required dual order isomorphism. It is obvious that $x^{\wedge} \to (x')^{\vee}$ is onto $(B^{\vee}, \geq)$. We show that $x^{\wedge} \to (x')^{\vee}$ is order embedding. Suppose that $x^{\wedge} \leq y^{\wedge}$. Then for all $b \in A(B)$, $b \leq x^{\wedge}$ implies $b \leq y^{\wedge}$. So, $b \in A(B)$ such that $\exists e_1 \in A$, $b \leq f(e_1)$ and $f(e_1) \wedge x \neq 0$, implies $\exists e_2 \in A$, s.t $b \leq f(e_2)$ and $f(e_2) \wedge y \neq 0$. Since f_A is a partition and $b \leq f(e_1) \wedge f(e_2)$, then $f(e_1) = f(e_2)$. Hence if $\exists e \in A$, s.t $b \leq f(e)$ and $f(e) \wedge x \neq 0$, then $f(e) \wedge y \neq 0$. Suppose that $(y')^{\vee} \nleq (x')^{\vee}$. So there exists $b \in A(B)$ such that $b \leq (y')^{\vee}$ and $b \nleq (x')^{\vee}$. Since $b \leq (y')^{\vee}$, then $\exists e \in A$, s.t $b \leq f(e)$ and $f(e) \leq y'$. Since $b \leq f(e)$ and $b \nleq (x')^{\vee}$, then $f(e) \nleq x'$. Since $f(e) \nleq x'$ is equivalent to $f(e) \wedge x \neq 0$, then by hypothesis $f(e) \wedge y \neq 0$. But this means that $f(e) \nleq y'$, a contradiction. Hence $(y')^{\vee} \leq (x')^{\vee}$. On the other hand, assume that $(y')^{\vee} \leq (x')^{\vee}$. Since f_A is a partition, then $b \in A(B)$ s.t $\exists e \in A$, $b \leq f(e)$ and $f(e) \leq y'$ implies $f(e) \leq x'$. Suppose that $x^{\wedge} \nleq y^{\wedge}$. So there exists $b \in A(B)$ such that $b \leq x^{\wedge}$ and $b \nleq y^{\wedge}$. So $\exists e \in A$, $b \leq f(e)$, $f(e) \wedge x \neq 0$, and $f(e) \wedge y = 0$. But this implies that $f(e) \leq y'$. Since $x^{\wedge} \nleq y^{\wedge}$, then $f(e) \leq x'$. This is equivalent to $f(e) \wedge x = 0$, a contradiction. □

Next we study the properties of soft approximations more closely in cases when the soft set f_A is full, keeping union, keeping intersection, and partition.

Proposition 38. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* f_A *be a soft set over B. Then the following properties hold.*

(i) *If* f_A *is full, then*

(a) $x^{\vee} \leq x \leq x^{\wedge}$;

(b) $1^{\vee} = 1^{\wedge} = 1$.

(ii) *If* f_A *is keeping supremum, then*

(a) *for all* $x \in B$, $\exists e \in A$, *s.t* $x^{\vee} = f(e)$;

(b) *for all* $x \in B$, $\exists e \in A$, *s.t* $x^{\wedge} = f(e)$.

(iii) *If* f_A *is full and keeping supremum, then*

$$x^{\wedge} = 1 \quad \textit{for every } x \in B \textit{ and } x \neq 0.$$

Proof. (a) By Proposition 41 $x^{\vee} \leq x$. Suppose that $b \in A(B)$ and $b \leq x$. Since f_A is full, then $\exists e \in A$, s.t $b \leq f(e)$. So, $b \leq f(e) \wedge x$ and therefore $f(e) \wedge x \neq 0$. Consequently, $b \leq x^{\wedge}$. (b) Obvious.

(ii) It follows by Proposition 33.

(iii) Let $x \in B$, then in general $x \leq 1$. Since f_A is full and keeping supremum, then $\exists e^* \in A$, s.t $\bigvee_{e\in A} f(e) = f(e^*) = 1$. So, $b \leq f(e^*)$ and $f(e^*) \wedge x \neq 0$. Consequently, $b \leq x^{\wedge}$ and so $x^{\wedge} = 1$. □

Proposition 39. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* f_A *be a soft set over B. If* f_A *is a partition, then,*

(i) $x^{\vee\wedge} = x^{\vee}$;

(ii) $x^{\wedge\wedge} = x^{\wedge}$;

(iii) *the map* ${}^{\wedge} : B \to B$ *is a closure operator.*

Proof. (i) Since f_A is full, then $x^{\vee} \leq x^{\vee\wedge}$ by Proposition 38(1). Let $b \in A(B)$ and $b \leq x^{\vee\wedge}$; then $\exists e_1 \in A$, s.t $b \leq f(e_1)$ and $f(e_1) \wedge x^{\vee} \neq 0$. Hence, $\exists c \in A(B)$, s.t $c \leq f(e_1)$ and $c \leq x^{\vee}$. But $c \leq x^{\vee}$ implies $\exists e_2 \in A$, s.t $c \leq f(e_2)$ and $f(e_2) \leq x$. Since f_A is a partition and $c \leq f(e_1) \wedge f(e_2)$, then $f(e_1) = f(e_2)$. So, $\exists e_1 \in A$, s.t $b \leq f(e_1)$ and $f(e_1) \leq x$ and therefore, $b \leq x^{\vee}$. Consequently, $x^{\vee\wedge} \leq x^{\vee}$. Claim (ii) can be proved similarly.

(iii) Since f_A is full, then by Proposition 38 the map ${}^{\wedge} : B \to B$ is extensive, and it is order preserving by Proposition 34. By (ii), $x^{\wedge\wedge} = x^{\wedge}$. □

5. Another Soft Rough Approximation Operators on a Complete Atomic Boolean Lattice

Definition 40. Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Define a mapping $\varphi_f : A(B) \to B$ by

$$c \leq \varphi_f(b) \Longleftrightarrow \exists e \in A, \text{ s.t } c \leq f(e), \ b \leq f(e) \tag{30}$$

for every $c, b \in A(B)$. Then φ_f is called the mapping induced by f_A on B.

Proposition 41. *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $\varphi_f : A(B) \to B$ be the mapping induced by f_A on B. Then the following properties hold.*

(i) *φ_f is symmetric.*

(ii) *If f_A is full, then φ_f is extensive.*

(iii) *If f_A is a partition, then φ_f is extensive, symmetric, and closed.*

Proof. (i) Obvious.

(ii) Let $b \in A(B)$. Since f_A is full, then $\exists e \in A$, s.t $b \leq f(e)$. Hence $b \leq \varphi_f(b)$.

(iii) If f_A is a partition, then f_A is full and hence φ_f is extensive. Since φ_f is symmetric, it remains to show that φ_f is closed. Let $c, b \in A(B)$ s.t $c \leq \varphi_f(b)$. We show that $\varphi_f(c) \leq \varphi_f(b)$. Since $c \leq \varphi_f(b)$, then $\exists e_1 \in A$, s.t $c \leq f(e_1)$ and $b \leq f(e_1)$. Suppose that $\varphi_f(c) \nleq \varphi_f(b)$. So, $\exists d \in A(B)$, s.t $d \leq \varphi_f(c)$ and $d \nleq \varphi_f(b)$. But $d \nleq \varphi_f(b)$ implies that for every $e \in A$, either $d \nleq f(e)$ or $b \nleq f(e)$. Since $d \leq \varphi_f(c)$, then $\exists e_2 \in A$, s.t $d \leq f(e_2)$ and $c \leq f(e_2)$. Since f_A is a partition and $c \leq f(e_1) \wedge f(e_2)$, then $f(e_1) = f(e_2)$. Hence we show that $\exists e_1 \in A$, s.t $d \leq f(e_1)$ and $b \leq f(e_1)$, a contradiction. Consequently, $\varphi_f(c) \leq \varphi_f(b)$ and thus φ_f is closed. □

Proposition 42. *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $\varphi_f : A(B) \to B$ be the mapping induced by f_A on B. Then the following properties hold.*

(i) *If $b \leq f(e)$ for $e \in A$ and $b \in A(B)$, then $f(e) \leq \varphi_f(b)$.*

(ii) *If f_A is a partition and $b \leq f(e)$ for $e \in A$ and $b \in A(B)$, then $f(e) = \varphi_f(b)$.*

(iii) *If f_A is keeping supremum, then for all $b \in A(B) \exists e \in A$, s.t $\varphi_f(b) = f(e)$.*

Proof. (i) Let $c \in A(B)$ s.t $c \leq f(e)$. Since $b \leq f(e)$, then $c \leq \varphi_f(b)$. Hence $f(e) \leq \varphi_f(b)$.

(ii) Suppose that f_A is a partition and assume that $b \leq f(e)$ for $e \in A$ and $b \in A(B)$. By (i) $f(e) \leq \varphi_f(b)$. On the other hand, let $c \in A(B)$ s.t $c \leq \varphi_f(b)$. Then $\exists e_1 \in A$, s.t $c \leq f(e_1)$ and $b \leq f(e_1)$. So, $b \leq f(e) \wedge f(e_1)$, and since f_A is a partition, then $f(e) = f(e_2)$. Hence $c \leq f(e)$ and therefore $\varphi_f(b) \leq f(e)$. Consequently, $\varphi_f(b) = f(e)$.

(iii) Suppose that f_A is keeping supremum and $b \in A(B)$. Let $c \in A(B)$ s.t $c \leq \varphi_f(b)$. Then $\exists e_c \in A$, s.t $c \leq f(e_c)$ and $b \leq f(e_c)$. So $f(e_c) \leq \varphi_f(b)$ by (i). Hence, $\varphi_f(b) = \bigvee_{c \in A(B)} \{f(e_c) : c \leq \varphi_f(b)\}$. Since f_A is keeping supremum, then $\bigvee_{c \in A(B)} \{f(e_c) : c \leq \varphi_f(b)\} = f(e)$ for $e \in A$. Therefore $\varphi_f(b) = f(e)$. □

Definition 43. Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $\varphi_f : A(B) \to B$ be the mapping induced by f_A on B. We define a pair of soft approximation operators ${}^{\nabla_f}, {}^{\Delta_f} : B \to B$ as follows:

$$\begin{aligned} x^{\nabla_f} &= \bigvee \left\{ b \in A(B) : \varphi_f(b) \leq x \right\}, \\ x^{\Delta_f} &= \bigvee \left\{ b \in A(B) : \varphi_f(b) \wedge x \neq 0 \right\}. \end{aligned} \tag{31}$$

The elements x^{∇_f} and x^{Δ_f} are called the *soft lower* and the *soft upper* approximations of x with respect to the mapping φ_f induced by f_A, respectively. Two elements x and y are called equivalent if they have the same soft upper and lower approximations with respect to the mapping φ_f induced by f_A on B. The resulting equivalence classes are called soft rough sets with respect to the mapping φ_f induced by f_A on B.

Proposition 44. *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $\varphi_f : A(B) \to B$ be the mapping induced by f_A.*

(i) $b \leq x^{\nabla_f} \Leftrightarrow \varphi_f(b) \leq x$.

(ii) $b \leq x^{\Delta_f} \Leftrightarrow \varphi_f(b) \wedge x \neq 0$.

(iii) *If f_A is full, then $x^{\nabla_f} \leq x \leq x^{\Delta_f}$.*

(iv) *$0^{\Delta_f} = 0$ and $1^{\nabla_f} = 1$. If f_A is full, then $0^{\nabla_f} = 0^{\Delta_f} = 0$ and $1^{\nabla_f} = 1^{\Delta_f} = 1$.*

(v) *$x \leq y$ implies $x^{\nabla_f} \leq y^{\nabla_f}$ and $x^{\Delta_f} \leq y^{\Delta_f}$.*

(vi) *The mappings ${}^{\nabla_f} : B \to B$ and ${}^{\Delta_f} : B \to B$ are mutually dual.*

(vii) *For all $S \subseteq B$, $\vee S^{\Delta_f} = (\vee S)^{\Delta_f}$.*

(viii) *For all $S \subseteq B$, $\wedge S^{\nabla_f} = (\wedge S)^{\nabla_f}$.*

(ix) *$(B^{\Delta_f}, \leq)$ is a complete lattice; 0 is the least element and 1^{Δ_f} is the greatest element of $(B^{\Delta_f}, \leq)$.*

(x) *The pair $({}^{\nabla_f}, {}^{\Delta_f})$ is a dual Galois connection on B.*

(xi) $(B^{\nabla_f}, \geq) \cong (B^{\Delta_f}, \leq)$.

Proof. It follows by Propositions 41, 9, 10, 11, and 12; see [23]. □

In the following we study the relation between the above two pairs of soft rough approximation operators given in Definitions 31 and 40

Proposition 45. *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $\varphi_f : A(B) \to B$ be the mapping induced by f_A. Then the following properties hold.*

(i) *If f_A is full, then $x^{\nabla_f} \leq x^{\vee}$.*

(ii) *If f_A is full and keeping supremum, then $x^{\Delta_f} \leq x^{\wedge}$.*

(iii) *If* f_A *is a partition, then*

(a) $x^{\nabla_f} = x^{\vee}$,
(b) $x^{\Delta_f} = x^{\wedge}$.

Proof. (i) Let $b \in A(B)$ s.t $b \leq x^{\nabla_f}$. Then $\varphi_f(b) \leq x$. Since f_A is full, then $\exists e \in A$, s.t $b \leq f(e)$. By Proposition 42(i) $f(e) \leq \varphi_f(b)$. Thus $b \leq f(e) \leq x$ and hence $b \leq x^{\vee}$. Consequently, $x^{\nabla_f} \leq x^{\vee}$.

(ii) If $x = 0$, then $x^{\wedge} = 0^{\wedge} = 0 = x^{\Delta_f}$. If $x \neq 0$ and f_A is keeping supremum, then by Proposition 38(3) $x^{\wedge} = 1$. Hence $x^{\Delta_f} \leq x^{\wedge}$.

(iii) (a) If f_A is a partition, then it is full. So $x^{\nabla_f} \leq x^{\vee}$ by (i). On the other hand, let $b \in A(B)$ s.t $b \leq x^{\vee}$. So $\exists e \in A$, s.t $b \leq f(e) \leq x$. Since f_A is a partition and $b \leq f(e)$, then by Proposition 34(ii) $f(e) = \varphi_f(b)$. This implies that $b \leq x^{\nabla_f}$ and therefore $x^{\vee} \leq x^{\nabla_f}$. Consequently, $x^{\nabla_f} = x^{\vee}$.

(b) This is similar to the proof of (a). □

Example 46. Let $B = \{0, a, b, c, d, e, f, 1\}$ and let the order $\leq$ be defined as in Figure 1. Let $A = \{e_1, e_2, e_3, e_4\}$ and let f_A be a soft set over B defined as follows:

$$f(e_1) = a, \quad f(e_2) = b, \quad f(e_3) = d, \quad f(e_4) = 0. \tag{32}$$

Obviously, f_A is not full. Also $\varphi_f(a) = \bigvee\{b \in A(B) : b \leq \varphi_f(a)\} = a \vee b = d$, $\varphi_f(b) = a \vee b = d$, and $\varphi_f(c) = 0$.

Let $x = b$, $y = a$. So $x^{\nabla_f} = \bigvee\{d \in A(B) : \varphi_f(d) \leq b\} = c$, and $x^{\Delta_f} = \bigvee\{d \in A(B) : \varphi_f(d) \wedge b \neq 0\} = a \vee b = d$. On the other hand

$y^{\nabla_f} = c$ and $y^{\Delta_f} = d$. Hence $x^{\nabla_f} \leq y^{\nabla_f}$ and $x^{\Delta_f} \leq y^{\Delta_f}$, but $x \nleq y$.

Example 47. Let $B = \{0, a, b, c, d, e, f, 1\}$ and let the order $\leq$ be defined as in Figure 1. Let $A = \{e_1, e_2, e_3, e_4\}$ and let g_A be a soft set over B defined as follows:

$$g(e_1) = a, \quad g(e_2) = e, \quad g(e_3) = c, \quad g(e_4) = f. \tag{33}$$

Obviously. g_A is full. Also $\varphi_g(a) = a \vee c = e$, $\varphi_g(b) = b \vee c = f$, and $\varphi_g(c) = a \vee b \vee c = 1$. Let $x = f$, then $x^{\nabla_g} = b$ and $x^{\Delta_g} = a \vee b \vee c = 1$. Hence $x^{\nabla_g} \leq x \leq x^{\Delta_g}$.

In the following, we give a relation between soft rough approximation operators and Järvinen's approximation operators on a complete atomic Boolean lattice.

Definition 48. Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice. Let $\varphi : A(B) \to B$ be extensive, symmetric, and closed mapping. Define a mapping $f_\varphi : A \to B$ by $f_\varphi(e) = \varphi(e)$ for every $e \in A$, where $A = A(B)$. Then $(f_\varphi)_A$ is called the soft set induced by φ on B.

Theorem 49. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice. Let* $(f_\varphi)_A$ *be the soft set induced by* φ *on* B*. Then for every* $x \in B$, $x^{\vee_\varphi} = x^{\nabla}$ *and* $x^{\wedge_\varphi} = x^{\Delta}$*, where*

$$\begin{aligned} x^{\vee_\varphi} &= \bigvee\left\{b \in A(B) : \exists e \in A \text{ s.t } b \leq \left(f_\varphi\right)(e), \left(f_\varphi\right)(e) \leq x\right\}, \\ x^{\nabla} &= \bigvee\{b \in A(B) : \varphi(b) \leq x\}, \\ x^{\wedge_\varphi} &= \bigvee\left\{b \in A(B) : \exists e \in A \text{ s.t } b \leq \left(f_\varphi\right)(e), \left(f_\varphi\right)(e) \wedge x \neq 0\right\}, \\ x^{\Delta} &= \bigvee\{b \in A(B) : \varphi(b) \wedge x \neq 0\}. \end{aligned} \tag{34}$$

Proof. Obvious. □

Theorem 50. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice. Let* f_A *be a partition soft set over* B*. Then for every* $x \in B$, $x^{\nabla_f} = x^{\vee}$ *and* $x^{\Delta_f} = x^{\wedge}$.

Proof. It follows immediately by Propositions 41(iii) and 45(iii). □

Remark 51. Theorems 49 and 50 illustrate that Järvinen's approximations can be viewed as a special case of our soft rough approximations on a complete atomic Boolean lattice.

6. Conclusion

In this paper, we introduced the concept of soft sets on a complete atomic Boolean lattice as a generalization of soft sets and obtained the lattice structure of these soft sets. Two pairs of soft rough approximation operators on a complete atomic Boolean lattice were considered, and their properties were given. We show that Järvinen's approximations can be viewed as a special case of our soft rough approximations. We may mention that soft rough sets on a complete atomic Boolean lattice can be used in object evaluation and group decision making. It should be noted that the use of soft rough sets could, to some extent, automatically reduce the noise factor caused by the subjective nature of the expert's evaluation. We will investigate these problems in future papers.

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