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Research Article

A Sharp Lower Bound for Toader-Qi Mean with Applications

Zhen-Hang Yang and Yu-Ming Chu

School of Mathematics and Computation Science, Hunan City University, Yiyang 413000, China

Correspondence should be addressed to Yu-Ming Chu; chuyuming2005@126.com

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We prove that the inequality $TQ(a,b) > L_p(a,b)$ holds for all $a,b > 0$ with $a \neq b$ if and only if $p \leq 3/2$, where $TQ(a,b) = (2/\pi)\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta$, $L_p(a,b) = [(b^p - a^p)/(p(b-a))]^{1/p}$ $(p \neq 0)$, and $L_0(a,b) = \sqrt{ab}$ are, respectively, the Toader-Qi and p-order logarithmic means of a and b. As applications, we find two fine inequalities chains for certain bivariate means.

1. Introduction

Let $p \in \mathbb{R}$ and $a,b > 0$ with $a \neq b$. Then the Toader-Qi mean $TQ(a,b)$ [1–3] and p-order logarithmic mean $L_p(a,b)$ are defined by

$$TQ(a,b) = \frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta, \quad (1)$$

$$L_p(a,b) = \left[\frac{b^p - a^p}{p(\log b - \log a)}\right]^{1/p} \quad (p \neq 0),$$
$$L_0(a,b) = \lim_{p\to 0} L_p(a,b) = \sqrt{ab}, \quad (2)$$

respectively. In particular, $L_1(a,b) = L(a,b)$ is the classical logarithmic mean of a and b.

It is well-known that the p-order logarithmic mean $L_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$. Recently, the Toader-Qi and p-order logarithmic means have been the subject of intensive research. In particular, many remarkable inequalities for the Toader-Qi and p-order logarithmic means can be found in the literature [2–7].

In [2], Qi et al. proved that the identity

$$TQ(a,b) = \sqrt{ab}\, I_0\left(\frac{1}{2}\log\frac{b}{a}\right) \quad (3)$$

and the inequalities

$$L(a,b) < TQ(a,b) < \frac{A(a,b) + G(a,b)}{2} < \frac{2A(a,b) + G(a,b)}{3} < I(a,b) \quad (4)$$

hold for all $a,b > 0$ with $a \neq b$, where

$$I_0(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^{2n}(n!)^2} \quad (5)$$

is the modified Bessel function of the first kind [8] and $A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$ and $I(a,b) = (b^b/a^a)^{1/(b-a)}/e$ are, respectively, the classical arithmetic, geometric, and identric means of a and b.

In [3], Yang proved that the double inequalities

$$\sqrt{\frac{2A(a,b)L(a,b)}{\pi}} < TQ(a,b) < \sqrt{A(a,b)L(a,b)}, \quad (6)$$

$$A^{1/4}(a,b)L^{3/4}(a,b) < TQ(a,b) < \frac{1}{4}A(a,b) + \frac{3}{4}L(a,b) \quad (7)$$

and conjectured that the inequalities

$$TQ(a,b) < I^{1/2}(a,b)L^{1/2}(a,b), \quad (8)$$

$$TQ(a,b) > L_{3/2}(a,b) \quad (9)$$

hold for all $a, b > 0$ with $a \neq b$. Inequality (8) was proved by Yang et al. in [9].

Let $b > a > 0$ and $t = (\log b - \log a)/2 > 0$. Then from (1)–(3) we clearly see that

$$L_p(a,b) = \sqrt{ab}\left[\frac{\sinh(pt)}{pt}\right]^{1/p} \quad (p \neq 0),$$
$$\begin{aligned} TQ(a,b) &= \frac{2\sqrt{ab}}{\pi}\int_0^{\pi/2} e^{t\cos(2\theta)}d\theta = \sqrt{ab}I_0(t) \\ &= \frac{2\sqrt{ab}}{\pi}\int_0^{\pi/2}\cosh(t\cos\theta)\,d\theta \\ &= \frac{2\sqrt{ab}}{\pi}\int_0^{\pi/2}\cosh(t\sin\theta)\,d\theta. \end{aligned} \tag{10}$$

The main purpose of this paper is to give a positive answer to the conjecture given by (9). As applications, we present two fine inequalities chains for certain bivariate means and a lower bound for the kernel function of the Szász-Mirakjan-Durrmeyer operator.

2. Lemmas

In order to prove our main result we need several lemmas, which we present in this section.

Lemma 1 (see [10]). *The double inequality*

$$\frac{1}{(x+a)^{1-a}} < \frac{\Gamma(x+a)}{\Gamma(x+1)} < \frac{1}{x^{1-a}} \tag{11}$$

holds for all $x > 0$ and $a \in (0,1)$, where $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ is the classical Euler gamma function.

Lemma 2 (see [3]). *Let $I_0(t)$ be defined by (5). Then the identity*

$$I_0^2(t) = \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^4}t^{2n} \tag{12}$$

holds for all $t \in \mathbb{R}$.

Lemma 3 (see [3]). *The Wallis ratio*

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)} \tag{13}$$

is strictly decreasing and log-convex with respect to all integers $n \geq 0$.

Lemma 4. *The identity*

$$\sum_{k=0}^{n}\frac{a^{2k}}{(2k)!(2n-2k)!} = \frac{(a+1)^{2n}+(a-1)^{2n}}{2(2n)!} \tag{14}$$

holds for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$.

Proof. Let $\binom{n}{k} = n!/k!(n-k)!$ be the number of combinations of n objects taken k at a time. Then from the well-known binomial theorem we have

$$\begin{aligned} (a+1)^{2n} &= \sum_{k=0}^{2n}\binom{2n}{k}a^k \\ &= \sum_{k=0}^{n}\binom{2n}{2k}a^{2k} + \sum_{k=1}^{n}\binom{2n}{2k-1}a^{2k-1}, \\ (a-1)^{2n} &= \sum_{k=0}^{2n}\binom{2n}{k}(-1)^{2n-k}a^k \\ &= \sum_{k=0}^{n}\binom{2n}{2k}a^{2k} - \sum_{k=1}^{n}\binom{2n}{2k-1}a^{2k-1}. \end{aligned} \tag{15}$$

Equation (15) leads to

$$\begin{aligned} \frac{(a+1)^{2n}+(a-1)^{2n}}{2} &= \sum_{k=0}^{n}\binom{2n}{2k}a^{2k} \\ &= \sum_{k=0}^{n}\frac{(2n)!a^{2k}}{(2k)!(2n-2k)!}. \end{aligned} \tag{16}$$

□

Lemma 5. *Let $k, n \in \mathbb{N}$ with $k \leq n$ and*

$$u_{k,n} = \frac{(2k)!}{2^{2n}(k!)^4[(n-k)!]^2}. \tag{17}$$

Then

$$u_{k,n} > \frac{2\sqrt{2}}{\pi\sqrt{\pi}(n+1)\sqrt{2n+1}}\frac{2^{2k}}{(2k)!(2n-2k)!} \tag{18}$$

for all $n \geq 8$.

Proof. Let W_n be defined by (13). Then it follows from Lemmas 1 and 3 together with (17) and $\Gamma(1/2) = \sqrt{\pi}$ that

$$\begin{aligned} u_{k,n} &= W_k^2 W_{n-k}\frac{2^{2k}}{(2k)!(2n-2k)!} \geq W_k W_{n/2}^2 \\ &\cdot \frac{2^{2k}}{(2k)!(2n-2k)!} \geq W_n W_{n/2}^2\frac{2^{2k}}{(2k)!(2n-2k)!} \\ &= \frac{1}{\pi\sqrt{\pi}}\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\left[\frac{\Gamma(n/2+1/2)}{\Gamma(n/2+1)}\right]^2 \\ &\cdot \frac{2^{2k}}{(2k)!(2n-2k)!} > \frac{1}{\pi\sqrt{\pi}} \\ &\cdot \frac{1}{\sqrt{n+1/2}}\left[\frac{1}{\sqrt{n/2+1/2}}\right]^2\frac{2^{2k}}{(2k)!(2n-2k)!} \\ &= \frac{2\sqrt{2}}{\pi\sqrt{\pi}(n+1)\sqrt{2n+1}}\frac{2^{2k}}{(2k)!(2n-2k)!} \end{aligned} \tag{19}$$

for all $n \geq 8$ and $0 \leq k \leq n$. □

3. Main Result

Theorem 6. *The inequality*

$$TQ(a,b) > L_p(a,b) \tag{20}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq 3/2$.

Proof. Since both the Toader-Qi mean $TQ(a,b)$ and p-order logarithmic mean $L_p(a,b)$ are symmetric and homogeneous and $TQ(a,b) > L(a,b)$ and $L_p(a,b)$ is strictly increasing with respect to $p \in \mathbb{R}$ for all $a, b > 0$ with $a \neq b$, without loss of generality, we assume that $p > 1$ and $b > a > 0$. Let $t = (\log b - \log a)/2 > 0$. Then it follows from (10) that inequality (20) is equivalent to

$$I_0(t) > \left[\frac{\sinh(pt)}{pt}\right]^{1/p} \tag{21}$$

for all $t > 0$.

If inequality (21) holds for all $t > 0$. Then (5) and (21) lead to

$$\lim_{t\to 0^+} \frac{I_0(t) - [\sinh(pt)/pt]^{1/p}}{t^2} = -\frac{1}{6}\left(p - \frac{3}{2}\right) \geq 0, \tag{22}$$

which gives $p \leq 3/2$.

Next, we only need to prove that inequality (21) holds for $p = 3/2$ and all $t > 0$; that is

$$I_0^3(t) > \left[\frac{\sinh(3t/2)}{3t/2}\right]^2. \tag{23}$$

It follows from (5) and Lemma 2 that

$$\begin{aligned} I_0^3(t) &= \left(\sum_{n=0}^{\infty} \frac{t^{2n}}{2^{2n}(n!)^2}\right)^3 = \sum_{n=0}^{\infty} \frac{(2n)!t^{2n}}{2^{2n}(n!)^4} \sum_{n=0}^{\infty} \frac{t^{2n}}{2^{2n}(n!)^2} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{(2k)!}{2^{2k}(k!)^4} \frac{1}{2^{2(n-k)}[(n-k)!]^2}\right) t^{2n} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} u_{k,n}\right) t^{2n}, \end{aligned} \tag{24}$$

where $u_{k,n}$ is defined as in (17).

Note that

$$\left[\frac{\sinh(3t/2)}{3t/2}\right]^2 = \frac{2[\cosh(3t) - 1]}{9t^2} = 2\sum_{n=0}^{\infty} \frac{3^{2n}t^{2n}}{(2n+2)!}. \tag{25}$$

Let

$$v_n = \sum_{k=0}^{n} u_{k,n} - \frac{2 \times 3^{2n}}{(2n+2)!}. \tag{26}$$

Then simple computations lead to

$$\begin{aligned} v_0 &= v_1 = 0, \\ v_2 &= \frac{3}{320}, \\ v_3 &= \frac{113}{26880}, \\ v_4 &= \frac{2057}{2867200}, \\ v_5 &= \frac{1741}{25231360}, \\ v_6 &= \frac{4335377}{991895224320}, \\ v_7 &= \frac{2186227}{11021058048000}. \end{aligned} \tag{27}$$

From Lemmas 4 and 5 together with (24)–(26), we have

$$I_0^3(t) - \left[\frac{\sinh(3t/2)}{3t/2}\right]^2 = \sum_{n=0}^{\infty} v_n t^{2n},$$

$$\begin{aligned} v_n &> \frac{2\sqrt{2}}{\pi\sqrt{\pi}(n+1)\sqrt{2n+1}} \frac{3^{2n}+1}{2(2n)!} - \frac{2 \times 3^{2n}}{(2n+2)!} \\ &= \frac{\left[\sqrt{2(2n+1)} - \pi\sqrt{\pi}\right]3^{2n} + \sqrt{2(2n+1)}}{\pi\sqrt{\pi}(n+1)(2n+1)!} > 0 \end{aligned} \tag{28}$$

for all $n \geq 8$.

Therefore, inequality (23) follows from (27) and (28). □

Remark 7. Theorem 6 gives a positive answer to the conjecture given by (9).

Remark 8. It follows from (23) that the inequality

$$I_0^3(t) > \frac{2[\cosh(3t) - 1]}{9t^2} \tag{29}$$

holds for all $t \neq 0$.

4. Applications

For $a, b > 0$, the Toader mean $T(a,b)$ [1] and arithmetic-geometric mean $\mathrm{AGM}(a,b)$ [11] are, respectively, defined by

$$\begin{aligned} T(a,b) &= \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\, d\theta, \\ \mathrm{AGM}(a,b) &= \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n, \end{aligned} \tag{30}$$

where a_n and b_n are given by

$$\begin{aligned} a_0 &= a, \\ b_0 &= b, \\ a_{n+1} &= \frac{(a_n + b_n)}{2} = A(a_n, b_n), \\ b_{n+1} &= \sqrt{a_n b_n} = G(a_n, b_n). \end{aligned} \tag{31}$$

Let $T_p(a,b) = T^{1/p}(a^p, b^p)$ and $I_q(a,b) = I^{1/q}(a^q, b^q)$ be the p-order Toader and q-order identric means of a and b, respectively. Then Theorem 6 leads to two fine inequalities chains for certain bivariate means.

Theorem 9. *The inequalities*

$$\begin{aligned} L(a,b) &< AGM(a,b) < A^{1/4}(a,b) L^{3/4}(a,b) \\ &< L_{3/2}(a,b) < TQ(a,b) \\ &< \frac{1}{4}A(a,b) + \frac{3}{4}L(a,b) \\ &< \frac{1}{2}L(a,b) + \frac{1}{2}I(a,b) \\ &< \frac{1}{2}A(a,b) + \frac{1}{2}G(a,b) < T_{1/3}(a,b) \\ &< I_{3/4}(a,b), \end{aligned} \tag{32}$$

$$\begin{aligned} L(a,b) &< AGM(a,b) < A^{1/4}(a,b) L^{3/4}(a,b) \\ &< L_{3/2}(a,b) < TQ(a,b) \\ &< L^{1/2}(a,b) I^{1/2}(a,b) < \frac{1}{2}L(a,b) + \frac{1}{2}I(a,b) \\ &< \frac{1}{2}A(a,b) + \frac{1}{2}G(a,b) < T_{1/3}(a,b) \\ &< I_{3/4}(a,b) \end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$.

Proof. The following inequalities can be found in the literature [3, 4, 7, 12–14]:

$$\frac{A(a,b) + G(a,b)}{2} < T_{1/3}(a,b) < I_{3/4}(a,b), \tag{33}$$

$$\begin{aligned} L(a,b) &< \mathrm{AGM}(a,b) \\ &< L^{3/4}(a,b) A^{1/4}(a,b) \\ &< L_{3/2}(a,b), \end{aligned} \tag{34}$$

$$\begin{aligned} I(a,b) &> \frac{L(a,b) + A(a,b)}{2}, \\ L(a,b) + I(a,b) &< A(a,b) + G(a,b) \end{aligned} \tag{35}$$

for all $a, b > 0$ with $a \neq b$.

It follows from (35) that

$$\begin{aligned} \frac{A(a,b) + G(a,b)}{2} &> \frac{I(a,b) + L(a,b)}{2} \\ &> \frac{3}{4}L(a,b) + \frac{1}{4}A(a,b) \end{aligned} \tag{36}$$

for all $a, b > 0$ with $a \neq b$.

Therefore, inequality (32) follows easily from (7), (8), (33), (34), (36), and Theorem 6. □

Remark 10. Let $b > a > 0$ and $t = (\log b - \log a)/2 > 0$. Then simple computations lead to

$$\begin{aligned} \frac{L(a,b)}{\sqrt{ab}} &= \frac{\sinh(t)}{t}, \\ \frac{I(a,b)}{\sqrt{ab}} &= e^{t\cosh(t)/\sinh(t) - 1}, \\ \frac{A(a,b)}{\sqrt{ab}} &= \cosh(t). \end{aligned} \tag{37}$$

Note that

$$\begin{aligned} &\lim_{t \to 0^+} \frac{(\sinh(t)/t)\, e^{t\cosh(t)/\sinh(t)-1} - (3\sinh(t)/4t + \cosh(t)/4)^2}{t^4} = \frac{1}{720}, \\ &\lim_{t \to \infty} \left[\frac{\sinh(t)}{t} e^{t\cosh(t)/\sinh(t)-1} - \left(\frac{3\sinh(t)}{4t} + \frac{\cosh(t)}{4} \right)^2 \right] = -\infty. \end{aligned} \tag{38}$$

Inequalities (37) and (38) imply that there exist small enough $\delta > 0$ and large enough $M > 1$ such that

$$I^{1/2}(a,b) L^{1/2}(a,b) > \frac{1}{4}A(a,b) + \frac{3}{4}L(a,b) \tag{39}$$

for all $b > a > 0$ with $b/a \in (1, 1+\delta)$ and

$$I^{1/2}(a,b) L^{1/2}(a,b) < \frac{1}{4}A(a,b) + \frac{3}{4}L(a,b) \tag{40}$$

for all $b > a > 0$ with $b/a \in (M, \infty)$.

Let $x \in [0, \infty)$, $n > 0$, $k \geq 0$, $p_{n,k}(x) = (nx)^k e^{-nx}/k!$, and $f \in L_p([0,\infty))$ $(1 \leq p \leq \infty)$. Then the kernel function $T_n(x,y)$ of the Szász-Mirakjan-Durrmeyer operator [15]

$$\begin{aligned} M_n(f;x) &= \sum_{k=0}^{\infty} \frac{\langle f, p_{n,k} \rangle}{\langle 1, p_{n,k} \rangle} p_{n,k}(x) \\ &= n \langle f, T_n(x,\cdot) \rangle \int_0^{\infty} T_n(x,y) f(y)\, dy \end{aligned} \tag{41}$$

is given by

$$T_n(x, y) = \sum_{k=0}^{\infty} p_{n,k}(x)\, p_{n,k}(y) = e^{-n(x+y)} I_0\left(2n\sqrt{xy}\right). \quad (42)$$

Berdysheva [16] proved that $T_n(x, y)$ is completely monotonic with respect to $n > 0$ for fixed $x, y \in [0, \infty)$ and

$$T_n(x, y) \le e^{-n(\sqrt{x}-\sqrt{y})^2} \quad (43)$$

for all $x, y \in [0, \infty)$.

From Remark 8 and (42), we get a lower bound for the kernel function $T_n(x, y)$ immediately.

Corollary 11. *The inequality*

$$T_n(x, y) > e^{-n(x+y)} \left[\frac{\cosh\left(6n\sqrt{xy}\right) - 1}{18n^2 xy}\right]^{1/3} \quad (44)$$

holds for all $x, y \in (0, \infty)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] G. Toader, "Some mean values related to the arithmetic-geometric mean," *Journal of Mathematical Analysis and Applications*, vol. 218, no. 2, pp. 358–368, 1998.

[2] F. Qi, X.-T. Shi, F.-F. Liu, and Z.-H. Yang, "A double inequality for an integral mean in terms of the exponential and logarithmic means," http://www.researchgate.net/publication/278968439.

[3] Z.-H. Yang, "Some sharpinequalities for the Toader-Qi mean," http://arxiv.org/abs/1507.05430.

[4] J. M. Borwein and P. B. Borwein, "Inequalities for compound mean iterations with logarithmic asymptotes," *Journal of Mathematical Analysis and Applications*, vol. 177, no. 2, pp. 572–582, 1993.

[5] Z.-H. Yang, "A new proof of inequalities for Gauss compound mean," *International Journal of Mathematical Analysis*, vol. 4, no. 21-24, pp. 1013–1018, 2010.

[6] Y.-M. Chu, B.-Y. Long, W.-M. Gong, and Y.-Q. Song, "Sharp bounds for Seiffert and Neuman-Sándor means in terms of generalized logarithmic means," *Journal of Inequalities and Applications*, vol. 2013, article 10, 13 pages, 2013.

[7] Z.-H. Yang, Y.-Q. Song, and Y.-M. Chu, "Sharp bounds for the arithmetic-geometric mean," *Journal of Inequalities and Applications*, vol. 2014, article 192, 13 pages, 2014.

[8] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, U. S. Government Printing Office, Washington, DC, USA, 1964.

[9] Z.-H. Yang, Y.-M. Chu, and Y.-Q. Song, "Sharp bounds for Toader-Qi mean in terms of logarithmic and identric means," *Mathematical Inequalities & Applications*, In press.

[10] F. Qi, "Bounds for the ratio of two gamma functions," *Journal of Inequalities and Applications*, vol. 2010, Article ID 493058, 84 pages, 2010.

[11] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, John Wiley & Sons, New York, NY, USA, 1987.

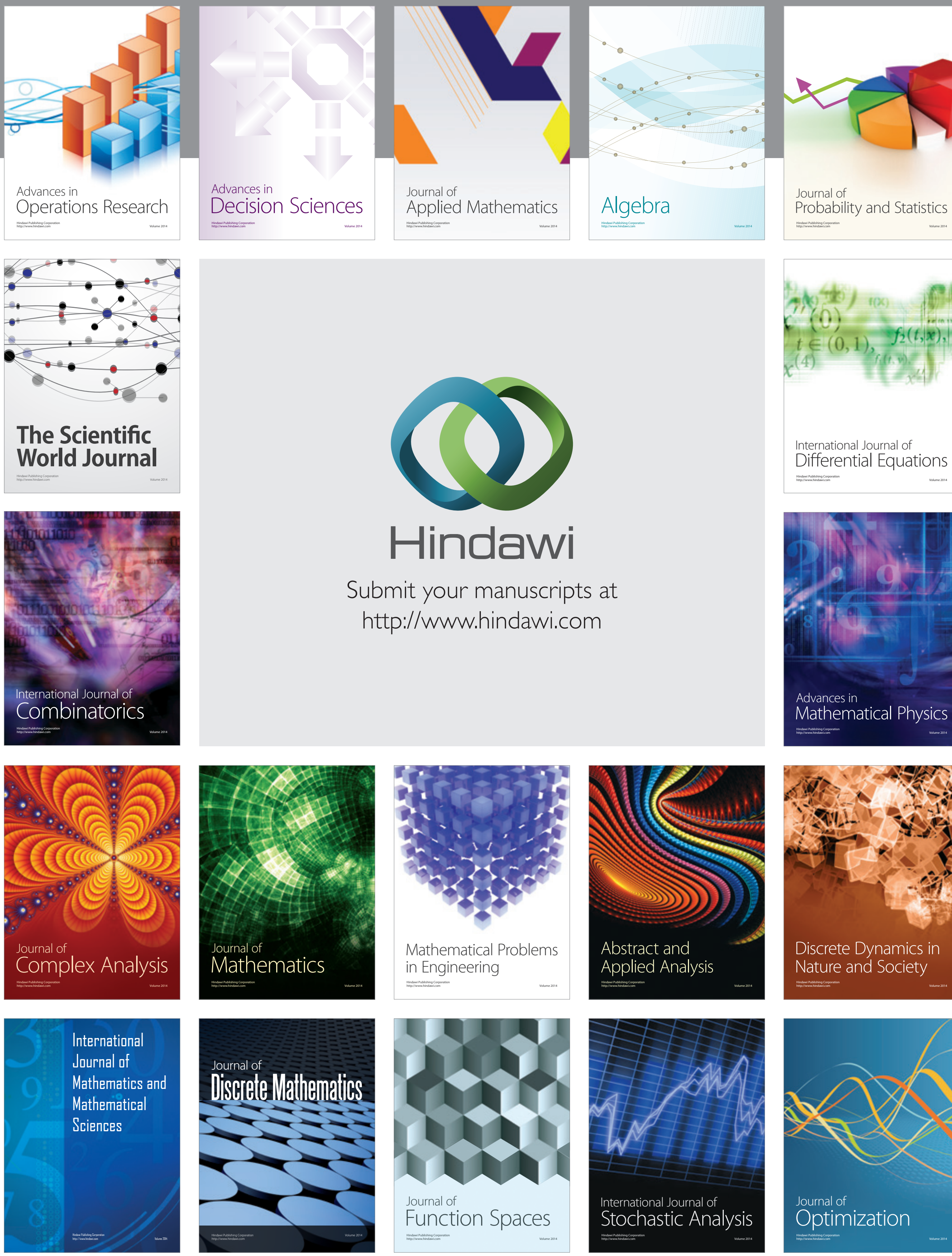
[12] H. Alzer, "Ungleichungen für Mittelwerte," *Archiv der Mathematik*, vol. 47, no. 5, pp. 422–426, 1986.

[13] J. Sándor, "On the identric and logarithmic means," *Aequationes Mathematicae*, vol. 40, no. 1, pp. 261–270, 1990.

[14] B. C. Carlson and M. Vuorinen, "Inequality of the AGM and the logarithmic mean," *SIAM Review*, vol. 33, no. 4, pp. 655–655, 1991.

[15] S. M. Mazhar and V. Totik, "Approximation by modified Szász operators," *Acta Universitatis Szegediensis (Szeged)*, vol. 49, no. 1–4, pp. 257–269, 1985.

[16] E. E. Berdysheva, "Studying Baskakov-Durrmeyer operators and quasi-interpolants via special functions," *Journal of Approximation Theory*, vol. 149, no. 2, pp. 131–150, 2007.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