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Research Article

Best Proximity Point Theorems for Some New Cyclic Mappings

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By using the stronger Meir-Keeler mapping, we introduce the concepts of the sMK-G-cyclic mappings, sMK-K-cyclic mappings, and sMK-C-cyclic mappings, and then we prove some best proximity point theorems for these various types of contractions. Our results generalize or improve many recent best proximity point theorems in the literature (e.g., Elderd and Veeramani, 2006; Sadiq Basha et al., 2011).

1. Introduction and Preliminaries

Let A and B be nonempty subsets of a metric space (X, d). Consider a mapping $T : A \cup B \to A \cup B$, T is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$. $x \in A$ is called a best proximity point of T in A if $d(x, Tx) = d(A, B)$ is satisfied, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. In 2005, Eldred et al. [1] proved the existence of a best proximity point for relatively nonexpansive mappings using the notion of proximal normal structure. In 2006, Eldred and Veeramani [2] proved the following existence theorem.

Theorem 1.1 (see Theorem 3.10 in [2]). *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Suppose $f : A \cup B \to A \cup B$ is a cyclic contraction, that is, $f(A) \subseteq B$ and $f(B) \subseteq A$, and there exists $k \in (0, 1)$ such that*

$$d(fx, fy) \leq kd(x, y) + (1 - k)d(A, B) \quad \text{for every } x \in A,\ y \in B. \tag{1.1}$$

Then there exists a unique best proximity point in A. Further, for each $x \in A$, $\{f^{2n}x\}$ converges to the best proximity point.

Later, best proximity point theorems for various types of contractions have been obtained in [3–7]. Particularly, in [8], the authors prove some best proximity point theorems for K-cyclic mappings and C-cyclic mappings in the frameworks of metric spaces and uniformly convex Banach spaces, thereby furnishing an optimal approximate solution to the equations of the form $Tx = x$, where T is a non-self-K-cyclic mapping or a non-self-C-cyclic mapping.

Definition 1.2 (see [8]). A pair of mappings $T : A \to B$ and $S : B \to A$ is said to form a K-cyclic mapping between A and B if there exists a nonnegative real number $k < 1/2$ such that

$$d(Tx, Sy) \le k\left[d(x, Tx) + d(y, Sy)\right] + (1 - 2k)d(A, B), \tag{1.2}$$

for $x \in A$ and $y \in B$.

Definition 1.3 (see [8]). A pair of mappings $T : A \to B$ and $S : B \to A$ is said to form a C-cyclic mapping between A and B if there exists a nonnegative real number $k < 1/2$ such that

$$d(Tx, Sy) \le k\left[d(x, Sy) + d(y, Tx)\right] + (1 - 2k)d(A, B), \tag{1.3}$$

for $x \in A$ and $y \in B$.

In this paper, we also recall the notion of Meir-Keeler mapping (see [9]). A function $\phi : [0,\infty) \to [0,\infty)$ is said to be a Meir-Keeler mapping if, for each $\eta > 0$, there exists $\delta > 0$ such that, for $t \in [0,\infty)$ with $\eta \le t < \eta + \delta$, we have $\phi(t) < \eta$. Generalization of the above function has been a heavily investigated branch of research. In this study, we introduce the below notion of the stronger Meir-Keeler function $\psi : [0,\infty) \to [0, 1/2)$.

Definition 1.4. We call $\psi : [0,\infty) \to [0, 1/2)$ a stronger Meir-Keeler mapping if the mapping ψ satisfies the following condition:

$$\forall \eta > 0 \quad \exists \delta > 0 \quad \exists \gamma_\eta \in \left[0, \frac{1}{2}\right) \quad \forall t \in [0,\infty) \quad \left(\eta \le t < \delta + \eta \Longrightarrow \psi(t) < \gamma_\eta\right). \tag{1.4}$$

The following provides two example of a stronger Meir-Keeler mapping.

Example 1.5. Let $\psi : [0,\infty) \to [0, 1/2)$ be defined by

$$\psi(t) = \begin{cases} 0, & \text{if } t \le 1, \\ \dfrac{t-1}{2}, & \text{if } 1 < t < 2, \\ \dfrac{1}{3}, & \text{if } t \ge 2. \end{cases} \tag{1.5}$$

Then ψ is a stronger Meir-Keeler mapping which is not a Meir-Keeler function.

Example 1.6. Let $\psi : [0, \infty) \to [0, 1/2)$ be defined by

$$\psi(t) = \frac{t}{3t+1}. \tag{1.6}$$

Then ψ is a stronger Meir-Keeler mapping.

In this paper, by using the stronger Meir-Keeler mapping, we introduce the concepts of the sMK-G-cyclic mappings, sMK-K-cyclic mappings and sMK-C-cyclic mappings, and then we prove some best proximity point theorems for these various types of contractions. Our results generalize or improve many recent best proximity point theorems in the literature (e.g., [2, 8]).

2. sMK-*G*-Cyclic Mappings

In this section, we prove the best proximity point theorems for the sMK-G-cyclic non-self mappings.

Definition 2.1. Let (X, d) be a metric space, and let A and B be nonempty subsets of X. A pair of mappings $T : A \to B$ and $S : B \to A$ is said to form an sMK-G-cyclic mapping between A and B if there is a stronger Meir-Keeler function $\psi : \mathbb{R}^+ \to [0, 1/2)$ in X such that for $x \in A$ and $y \in B$,

$$d(Tx, Sy) - d(A, B) \le \psi(d(x, y)) \cdot [G(x, y) - 2d(A, B)], \tag{2.1}$$

where $G(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), d(x, Sy), d(y, Tx)\}$.

Lemma 2.2. *Let A and B be nonempty subsets of a metric space (X, d). Suppose that the mappings $T : A \to B$ and $S : B \to A$ form an sMK-G-cyclic mapping between A and B. Then there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B). \tag{2.2}$$

Proof. Let $x_0 \in A$ be given and let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Taking into account (2.1) and the definition of the stronger Meir-Keeler function $\psi : \mathbb{R}^+ \to [0, 1/2)$, we have that for each $n \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) - d(A, B) &= d(Tx_{2n}, Sx_{2n+1}) - d(A, B) \\ &\le \psi(d(x_{2n}, x_{2n+1})) \cdot [G(x_{2n}, x_{2n+1}) - 2d(A, B)], \end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
G(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1}), d(x_{2n}, Sx_{2n+1}),\\
&\qquad d(x_{2n+1}, Tx_{2n})\}\\
&= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})\}\\
&\leq \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1})\\
&\qquad + d(x_{2n+1}, x_{2n+2}), 0\}\\
&\leq 2 \cdot \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}.
\end{aligned} \tag{2.4}$$

Taking into account (2.3) and (2.4), we have that for each $n \in \mathbb{N} \cup \{0\}$

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) - d(A, B) &\leq \psi(d(x_{2n}, x_{2n+1})) \cdot 2 \cdot [\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} - d(A, B)]\\
&< \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} - d(A, B),
\end{aligned} \tag{2.5}$$

and so we conclude that

$$d(x_{2n+1}, x_{2n+2}) - d(A, B) < d(x_{2n}, x_{2n+1}) - d(A, B), \tag{2.6}$$

and, for each $n \in \mathbb{N}$,

$$\begin{aligned}
d(x_{2n}, x_{2n+1}) - d(A, B) &= d(Sx_{2n-1}, Tx_{2n}) - d(A, B)\\
&= d(Tx_{2n}, Sx_{2n-1}) - d(A, B)\\
&\leq \psi(d(x_{2n}, x_{2n-1})) \cdot [G(x_{2n}, x_{2n-1}) - 2d(A, B)],
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
G(x_{2n}, x_{2n-1}) &= \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, Tx_{2n}), d(x_{2n-1}, Sx_{2n-1}), d(x_{2n}, Sx_{2n-1}), d(x_{2n-1}, Tx_{2n})\}\\
&\leq \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n}), d(x_{2n-1}, x_{2n+1})\}\\
&\leq \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), 0, d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})\}\\
&\leq 2 \cdot \max\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\}.
\end{aligned} \tag{2.8}$$

Taking into account (2.7) and (2.8), we have that for each $n \in \mathbb{N}$

$$\begin{aligned}
d(x_{2n}, x_{2n+1}) - d(A, B) &\leq \psi(d(x_{2n-1}, x_{2n})) \cdot 2 \cdot [\max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\} - d(A, B)]\\
&< \max\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\} - d(A, B),
\end{aligned} \tag{2.9}$$

and so we conclude that

$$d(x_{2n+1}, x_{2n+2}) - d(A, B) < d(x_{2n}, x_{2n+1}) - d(A, B). \tag{2.10}$$

Generally, by (2.6) and (2.10), we have that for each $n \in \mathbb{N}$

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &< d(x_n, x_{n+1}), \\ d(x_{n+1}, x_{n+2}) - d(A, B) &\leq \psi(d(x_n, x_{n+1})) \cdot 2 \cdot [d(x_n, x_{n+1}) - d(A, B)]. \end{aligned} \tag{2.11}$$

Thus the sequence $\{d(x_n, x_{n+1})\}_{n \in \mathbb{N} \cup \{0\}}$ is decreasing and bounded below and hence it is convergent. Let $\lim_{n \to \infty} d(x_n, x_{n+1}) = \eta \geq 0$. Then there exists $n_0 \in \mathbb{N}$ and $\delta > 0$ such that for all $n \in \mathbb{N}$ with $n \geq n_0$

$$\eta \leq d(x_n, x_{n+1}) < \eta + \delta. \tag{2.12}$$

Taking into account (2.12) and the definition of stronger Meir-Keeler function ψ, corresponding to η use, there exists $\gamma_\eta \in [0, 1/2)$ such that

$$\psi(d(x_n, x_{n+1})) < \gamma_\eta \quad \forall n \geq n_0. \tag{2.13}$$

Thus, we can deduce that for each $n \in \mathbb{N}$ with $n \geq n_0 + 1$

$$\begin{aligned} d(x_n, x_{n+1}) - d(A, B) &\leq \psi(d(x_{n-1}, x_n)) \cdot 2 \cdot [d(x_{n-1}, x_n) - d(A, B)] \\ &< \gamma_\eta \cdot 2 \cdot [d(x_{n-1}, x_n) - d(A, B)], \end{aligned} \tag{2.14}$$

and so

$$\begin{aligned} d(x_n, x_{n+1}) - d(A, B) &< \gamma_\eta \cdot 2 \cdot [d(x_{n-1}, x_n) - d(A, B)] \\ &< (2\gamma_\eta)^2 \cdot [d(x_{n-2}, x_{n-1}) - d(A, B)] \\ &< \cdots \\ &< (2\gamma_\eta)^{n-n_0} \cdot [d(x_{n_0}, x_{n_0+1}) - d(A, B)]. \end{aligned} \tag{2.15}$$

Since $\gamma_\eta \in [0, 1/2)$, we get

$$\lim_{n \to \infty} d(x_n, x_{n+1}) - d(A, B) = 0, \tag{2.16}$$

that is, $\lim_{n \to \infty} d(x_n, x_{n+1}) = d(A, B)$. □

Lemma 2.3. *Let A and B be nonempty closed subsets of a metric space (X, d). Suppose that the mappings $T : A \to B$ and $S : B \to A$ form an sMK-G-cyclic mapping between A and B. For a fixed point $x_0 \in A$, let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Then the sequence $\{x_n\}$ is bounded.*

Proof. It follows from Lemma 2.2 that $\{d(x_{2n-1}, x_{2n})\}$ is convergent and hence it is bounded. Since $T : A \to B$ and $S : B \to A$ form an sMK-G-cyclic mapping between A and B, there is a stronger Meir-Keeler function $\psi : \mathbb{R}^+ \to [0, 1/2)$ in X such that

$$\begin{aligned} d(x_{2n}, Tx_0) &= d(Sx_{2n-1}, Tx_0) \\ &= d(Tx_0, Sx_{2n-1}) \\ &\leq \psi(d(x_0, x_{2n-1})) \cdot [G(d(x_0, x_{2n-1})) - 2d(A, B)] + d(A, B), \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} G(d(x_0, x_{2n-1})) &= \max\{d(x_0, x_{2n-1}), d(x_0, Tx_0), d(x_{2n-1}, Sx_{2n-1}), d(x_0, Sx_{2n-1}), d(x_{2n-1}, Tx_0)\} \\ &= \max\{d(x_0, x_{2n-1}), d(x_0, Tx_0), d(x_{2n-1}, x_{2n}), d(x_0, x_{2n}), d(x_{2n-1}, Tx_0)\} \\ &\leq \max\{d(x_0, Tx_0) + d(Tx_0, x_{2n}) + d(x_{2n}, x_{2n-1}), d(x_0, Tx_0), d(x_{2n-1}, x_{2n}), \\ &\qquad d(x_0, Tx_0) + d(Tx_0, x_{2n}), d(x_{2n-1}, x_{2n}) + d(x_{2n}, Tx_0)\} \\ &= d(x_0, Tx_0) + d(Tx_0, x_{2n}) + d(x_{2n}, x_{2n-1}). \end{aligned} \tag{2.18}$$

Taking into account (2.17) and (2.18), we get

$$\begin{aligned} d(x_{2n}, Tx_0) \leq{}& \frac{\psi(d(x_0, x_{2n-1}))}{1 - \psi(d(x_0, x_{2n-1}))}[d(x_0, Tx_0) + d(x_{2n}, x_{2n-1})] \\ &+ \frac{1 - 2\psi(d(x_0, x_{2n-1}))}{1 - \psi(d(x_0, x_{2n-1}))} d(A, B). \end{aligned} \tag{2.19}$$

Therefore, the sequence $\{x_{2n}\}$ is bounded. Similarly, it can be shown that $\{x_{2n+1}\}$ is also bounded. So we complete the proof. □

Theorem 2.4. *Let A and B be nonempty closed subsets of a metric space. Let the mappings $T : A \to B$ and $S : B \to A$ form an sMK-G-cyclic mapping between A and B. For a fixed point $x_0 \in A$, let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then, x is a best proximity point of T.*

Proof. Suppose that a subsequence $\{x_{2n_k}\}$ converges to x in A. It follows from Lemma 2.2 that $d(x_{2n_k-1}, x_{2n_k})$ converges to $d(A, B)$. Since $T : A \to B$ and $S : B \to A$ form an sMK-G-cyclic mapping between A and B and taking into account (2.13), we have that for each $2n_k \in \mathbb{N}$ with $2n_k \geq n_0 + 1$

$$\begin{aligned} d(x_{2n_k}, Tx) &= d(Tx, x_{2n_k}) \\ &\leq \psi(d(x, x_{2n_k-1})) \cdot [G(x, x_{2n_k-1}) - 2d(A,B)] + d(A,B) \\ &< \gamma_\eta \cdot [G(x, x_{2n_k-1}) - 2d(A,B)] + d(A,B), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} G(x, x_{2n_k-1}) &= \max\{d(x, x_{2n_k-1}), d(x, Tx), d(x_{2n_k-1}, Sx_{2n_k-1}), d(x, Sx_{2n_k-1}), d(x_{2n_k-1}, Tx)\} \\ &= \max\{d(x, x_{2n_k-1}), d(x, Tx), d(x_{2n_k-1}, x_{2n_k}), d(x, x_{2n_k}), d(x_{2n_k-1}, Tx)\} \\ &= \max\{d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}), d(x, Tx), d(x_{2n_k-1}, x_{2n_k}), \\ &\qquad d(x, x_{2n_k}), d(x_{2n_k}, Tx) + d(x_{2n_k-1}, x_{2n_k})\} \\ &\leq d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}) + d(x_{2n_k}, Tx). \end{aligned} \tag{2.21}$$

Following from (2.20) and (2.21), we obtain that

$$d(x_{2n_k}, Tx) \leq \gamma_\eta [d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}) + d(x_{2n_k}, Tx) - 2d(A,B)] + d(A,B), \tag{2.22}$$

that is, we have that

$$d(A,B) \leq d(x_{2n_k}, Tx) \leq \frac{\gamma_\eta}{1-\gamma_\eta} \cdot [d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1})] + \left[1 - \frac{\gamma_\eta}{1-\gamma_\eta}\right] \cdot d(A,B), \tag{2.23}$$

letting $k \to \infty$. Then we conclude that

$$d(A,B) \leq d(x, Tx) \leq \frac{\gamma_\eta}{1-\gamma_\eta} \cdot [d(A,B) + 0] + \left[1 - \frac{\gamma_\eta}{1-\gamma_\eta}\right] \cdot d(A,B). \tag{2.24}$$

Therefore, $d(x, Tx) = d(A,B)$, that is, x is a best proximity point of T. □

3. sMK-*K*-Cyclic Mappings

In this section, we prove the best proximity point theorems for the sMK-K-cyclic non-self mappings.

Definition 3.1. Let (X, d) be a metric space, and let A and B be nonempty subsets of X. A pair of mappings $T : A \to B$ and $S : B \to A$ is said to form an sMK-K-cyclic mapping between A

and B if there is a stronger Meir-Keeler function $\psi : \mathbb{R}^+ \to [0,1/2)$ in X such that, for $x \in A$ and $y \in B$,

$$d(Tx,Sy) - d(A,B) \leq \psi(d(x,y)) \cdot [K(x,y) - 2d(A,B)], \tag{3.1}$$

where $K(x,y) = d(x,Tx) + d(y,Sy)$.

Lemma 3.2. *Let A and B be nonempty subsets of a metric space (X,d). Suppose that the mappings $T : A \to B$ and $S : B \to A$ form an sMK-K-cyclic mapping between A and B. Then there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = d(A,B). \tag{3.2}$$

Proof. Let $x_0 \in A$ be given and let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Taking into account (3.1) and the definition of the stronger Meir-Keeler function $\psi : \mathbb{R}^+ \to [0,1/2)$, we have that $n \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) - d(A,B) &= d(Tx_{2n},Sx_{2n+1}) - d(A,B) \\ &\leq \psi(d(x_{2n},x_{2n+1})) \cdot [K(x_{2n},x_{2n+1}) - 2d(A,B)], \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} K(x_{2n},x_{2n+1}) &= d(x_{2n},Tx_{2n}) + d(x_{2n+1},Sx_{2n+1}) \\ &= d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2}). \end{aligned} \tag{3.4}$$

Taking into account (3.3) and (3.4), we have that

$$d(x_{2n+1},x_{2n+2}) < d(x_{2n},x_{2n+1}). \tag{3.5}$$

Similarly, we can conclude that

$$d(x_{2n},x_{2n+1}) < d(x_{2n-1},x_{2n}). \tag{3.6}$$

Generally, by (3.5) and (3.6), we have that for each $n \in \mathbb{N} \cup \{0\}$

$$d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1}). \tag{3.7}$$

Thus the sequence $\{d(x_n,x_{n+1})\}_{n\in\mathbb{N}\cup\{0\}}$ is decreasing and bounded below and hence it is convergent. Let $\lim_{n\to\infty} d(x_n,x_{n+1}) - d(A,B) = \eta \geq 0$. Then there exists $n_0 \in \mathbb{N}$ and $\delta > 0$ such that for all $n \in \mathbb{N}$ with $n \geq n_0$

$$\eta \leq d(x_n,x_{n+1}) < \eta + \delta. \tag{3.8}$$

Taking into account (3.8) and the definition of stronger Meir-Keeler function ψ, corresponding to η use, there exists $\gamma_\eta \in [0,1/2)$ such that

$$\psi(d(x_n, x_{n+1})) < \gamma_\eta \quad \forall n \geq n_0. \tag{3.9}$$

Thus, we can deduce that for each $n \in \mathbb{N}$ with $n \geq n_0 + 1$

$$\begin{aligned} d(x_n, x_{n+1}) - d(A,B) &\leq \psi(d(x_{n-1}, x_n)) \cdot [K(x_{n-1}, x_n) - 2d(A,B)] \\ &< \gamma_\eta \cdot [d(x_{n-1}, Tx_{n-1}) + d(x_n, Sx_n) - 2d(A,B)] \\ &= \gamma_\eta \cdot [d(x_{n-1}, x_n) + d(x_n, x_{n+1}) - 2d(A,B)], \end{aligned} \tag{3.10}$$

that is,

$$d(x_n, x_{n+1}) - d(A,B) < \frac{\gamma_\eta}{1-\gamma_\eta} \cdot [d(x_{n-1}, x_n) - d(A,B)], \tag{3.11}$$

since $\gamma_\eta \in [0,1/2)$. Therefore we get that for each $n \in \mathbb{N}$ with $n \geq n_0 + 1$

$$\begin{aligned} d(x_n, x_{n+1}) - d(A,B) &< \frac{\gamma_\eta}{1-\gamma_\eta} \cdot (d(x_{n-1}, x_n) - d(A,B)) \\ &< \left(\frac{\gamma_\eta}{1-\gamma_\eta}\right)^2 \cdot (d(x_{n-2}, x_{n-1}) - d(A,B)) \\ &< \cdots \\ &< \left(\frac{\gamma_\eta}{1-\gamma_\eta}\right)^{n-n_0} \cdot (d(x_{n_0}, x_{n_0+1}) - d(A,B)). \end{aligned} \tag{3.12}$$

Since $\gamma_\eta \in [0,1/2)$, we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) - d(A,B) = 0, \tag{3.13}$$

that is, $\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A,B)$. □

Lemma 3.3. *Let A and B be nonempty closed subsets of a metric space (X,d). Suppose that the mappings $T : A \to B$ and $S : B \to A$ form an sMK-K-cyclic mapping between A and B. For a fixed point $x_0 \in A$, let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Then the sequence $\{x_n\}$ is bounded.*

Proof. It follows from Lemma 3.2 that $\{d(x_{2n-1}, x_{2n})\}$ is convergent and hence it is bounded. Since $T : A \to B$ and $S : B \to A$ form an sMK-K-cyclic mapping between A and B, there is a stronger Meir-Keeler function $\psi : \mathbb{R}^+ \to [0, 1/2)$ in X such that, for $x_0 \in A$ and $x_{2n-1} \in B$,

$$\begin{aligned} d(x_{2n}, Tx_0) - d(A,B) &= d(Sx_{2n-1}, Tx_0) - d(A,B) \\ &= d(Tx_0, Sx_{2n-1}) - d(A,B) \\ &\leq \psi(d(x_0, x_{2n-1})) \cdot [K(x_0, x_{2n-1}) - 2d(A,B)], \end{aligned} \tag{3.14}$$

where $K(x_0, x_{2n-1}) = d(x_0, Tx_0) + d(x_{2n-1}, Sx_{2n-1})$. So we get that

$$\begin{aligned} d(x_{2n}, Tx_0) \leq{}& \psi(d(x_0, x_{2n-1}))[d(x_0, Tx_0) + d(x_{2n-1}, x_{2n})] \\ &+ [1 - 2\psi(d(x_0, x_{2n-1}))]d(A,B). \end{aligned} \tag{3.15}$$

Therefore, the sequence $\{x_{2n}\}$ is bounded. Similarly, it can be shown that $\{x_{2n+1}\}$ is also bounded. So we complete the proof. □

Theorem 3.4. *Let A and B be nonempty closed subsets of a metric space. Let the mappings $T : A \to B$ and $S : B \to A$ form an sMK-K-cyclic mapping between A and B. For a fixed point $x_0 \in A$, let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then, x is a best proximity point of T.*

Proof. Suppose that a subsequence $\{x_{2n_k}\}$ converges to x in A. It follows from Lemma 2.2 that $d(x_{2n_k-1}, x_{2n_k})$ converges to $d(A,B)$. Since $T : A \to B$ and $S : B \to A$ form an sMK-K-cyclic mapping between A and B and taking into account (3.9), we have that for each $2n_k \in \mathbb{N}$ with $2n_k \geq n_0 + 1$

$$\begin{aligned} d(x_{2n_k}, Tx) &= d(Tx, x_{2n_k}) \\ &\leq \psi(d(x, x_{2n_k-1})) \cdot [K(x, x_{2n_k-1}) - 2d(A,B)] + d(A,B) \\ &< \gamma_\eta \cdot [K(x, x_{2n_k-1}) - 2d(A,B)] + d(A,B), \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} K(x, x_{2n_k-1}) &= d(x, Tx) + d(x_{2n_k-1}, Sx_{2n_k-1}) \\ &= d(x, Tx) + d(x_{2n_k-1}, x_{2n_k}). \end{aligned} \tag{3.17}$$

Following from (3.16) and (3.17), we obtain that for each $2n_k \in \mathbb{N}$ with $2n_k \geq n_0 + 1$

$$d(A,B) \leq d(x_{2n_k}, Tx) \leq \gamma_\eta[d(x, Tx) + d(x_{2n_k}, x_{2n_k-1})] + (1 - 2\gamma_\eta)d(A,B), \tag{3.18}$$

Letting $k \to \infty$. Then we conclude that $d(x, Tx) = d(A,B)$, that is, x is a best proximity point of T. □

4. sMK-C-Cyclic Mappings

In this section, we prove the best proximity point theorems for the sMK-C-cyclic non-self mappings.

Definition 4.1. Let (X, d) be a metric space, and let A and B be nonempty subsets of X. A pair of mappings $T : A \to B$ and $S : B \to A$ is said to form an sMK-C-cyclic mapping between A and B if there is a stronger Meir-Keeler function $\psi : \mathbb{R}^+ \to [0, 1/2)$ in X such that, for $x \in A$ and $y \in B$,

$$d(Tx, Sy) - d(A, B) \leq \psi(d(x, y)) \cdot [C(x, y) - 2d(A, B)], \tag{4.1}$$

where $C(x, y) = d(x, Sy) + d(y, Tx)$.

Lemma 4.2. *Let A and B be nonempty subsets of a metric space (X, d). Suppose that the mappings $T : A \to B$ and $S : B \to A$ form an sMK-C-cyclic mapping between A and B. Then there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = d(A, B). \tag{4.2}$$

Proof. Let $x_0 \in A$ be given and let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Taking into account (4.1) and the definition of the stronger Meir-Keeler function $\psi : \mathbb{R}^+ \to [0, 1/2)$, we have that $n \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) - d(A, B) &= d(Tx_{2n}, Sx_{2n+1}) - d(A, B) \\ &\leq \psi(d(x_{2n}, x_{2n+1})) \cdot [C(x_{2n}, x_{2n+1}) - 2d(A, B)], \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} C(x_{2n}, x_{2n+1}) &= d(x_{2n}, Sx_{2n+1}) + d(x_{2n+1}, Tx_{2n}) \\ &= d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}) \\ &\leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}). \end{aligned} \tag{4.4}$$

Taking into account (4.3) and (4.4), we conclude that

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}). \tag{4.5}$$

Similarly, we can conclude that

$$d(x_{2n}, x_{2n+1}) < d(x_{2n-1}, x_{2n}). \tag{4.6}$$

Generally, by (4.5) and (4.6), we have that for each $n \in \mathbb{N} \cup \{0\}$

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}). \tag{4.7}$$

Thus the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}\cup\{0\}}$ is decreasing and bounded below and hence it is convergent. Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = \eta \geq 0$. Then there exists $n_0 \in \mathbb{N}$ and $\delta > 0$ such that for all $n \in \mathbb{N}$ with $n \geq n_0$

$$\eta \leq d(x_n, x_{n+1}) < \eta + \delta. \tag{4.8}$$

Taking into account (4.5) and the definition of stronger Meir-Keeler function ψ, corresponding to η use, there exists $\gamma_\eta \in [0, 1/2)$ such that

$$\psi(d(x_n, x_{n+1})) < \gamma_\eta \quad \forall n \geq n_0. \tag{4.9}$$

Thus, we can deduce that for each $n \in \mathbb{N}$ with $n \geq n_0 + 1$

$$\begin{aligned} d(x_n, x_{n+1}) - d(A,B) &\leq \psi(d(x_{n-1}, x_n)) \cdot [C(x_{n-1}, x_n) - 2d(A,B)] \\ &< \gamma_\eta \cdot [d(x_{n-1}, Sx_n) + d(x_n, Tx_{n-1}) - 2d(A,B)] \\ &= \gamma_\eta \cdot [d(x_{n-1}, x_{n+1}) + d(x_n, x_n) - 2d(A,B)] \\ &\leq \gamma_\eta \cdot [d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 0 - 2d(A,B)], \end{aligned} \tag{4.10}$$

that is,

$$d(x_n, x_{n+1}) - d(A,B) < \frac{\gamma_\eta}{1-\gamma_\eta} \cdot [d(x_{n-1}, x_n) - d(A,B)], \tag{4.11}$$

since $\gamma_\eta \in [0, 1)$. Therefore we get that for each $n \in \mathbb{N}$ with $n \geq n_0 + 1$

$$\begin{aligned} d(x_n, x_{n+1}) - d(A,B) &< \frac{\gamma_\eta}{1-\gamma_\eta} \cdot (d(x_{n-1}, x_n) - d(A,B)) \\ &< \left(\frac{\gamma_\eta}{1-\gamma_\eta}\right)^2 \cdot (d(x_{n-2}, x_{n-1}) - d(A,B)) \\ &< \cdots \\ &< \left(\frac{\gamma_\eta}{1-\gamma_\eta}\right)^{n-n_0} \cdot (d(x_{n_0}, x_{n_0+1}) - d(A,B)). \end{aligned} \tag{4.12}$$

Since $\gamma_\eta \in [0, 1/2)$, we obtain that $\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A,B)$. □

Lemma 4.3. *Let A and B be nonempty closed subsets of a metric space (X, d). Suppose that the mappings $T : A \to B$ and $S : B \to A$ form an sMK-C-cyclic mapping between A and B. For a fixed point $x_0 \in A$, let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Then the sequence $\{x_n\}$ is bounded.*

Proof. It follows from Lemma 4.2 that $\{d(x_{2n-1}, x_{2n})\}$ is convergent and hence it is bounded. Since $T : A \to B$ and $S : B \to A$ form an sMK-C-cyclic mapping between A and B, there is a stronger Meir-Keeler function $\psi : \mathbb{R}^+ \to [0, 1/2)$ in X such that for $x_0 \in A$ and $x_{2n-1} \in B$,

$$\begin{aligned} d(x_{2n}, Tx_0) - d(A, B) &= d(Sx_{2n-1}, Tx_0) - d(A, B) \\ &= d(Tx_0, Sx_{2n-1}) - d(A, B) \\ &\leq \psi(d(x_0, x_{2n-1})) \cdot [C(x_0, x_{2n-1}) - 2d(A, B)], \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} C(x_0, x_{2n-1}) &= d(x_0, Sx_{2n-1}) + d(x_{2n-1}, Tx_0) \\ &= d(x_0, x_{2n}) + d(x_{2n-1}, Tx_0). \end{aligned} \tag{4.14}$$

So we get that

$$\begin{aligned} d(x_{2n}, Tx_0) &\leq \psi(d(x_0, x_{2n-1}))[d(x_0, x_{2n}) + d(x_{2n-1}, Tx_0)] \\ &\quad + [1 - 2\psi(d(x_0, x_{2n-1}))]d(A, B) \\ &\leq \psi(d(x_0, x_{2n-1}))[d(x_{2n-1}, x_{2n}) + 2d(x_{2n}, Tx_0) + d(x_0, Tx_0)] \\ &\quad + [1 - 2\psi(d(x_0, x_{2n-1}))]d(A, B). \end{aligned} \tag{4.15}$$

Therefore, the sequence $\{x_{2n}\}$ is bounded. Similarly, it can be shown that $\{x_{2n+1}\}$ is also bounded. So we complete the proof. □

Theorem 4.4. *Let A and B be nonempty closed subsets of a metric space. Let the mappings $T : A \to B$ and $S : B \to A$ form an sMK-C-cyclic mapping between A and B. For a fixed point $x_0 \in A$, let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then, x is a best proximity point of T.*

Proof. Suppose that a subsequence $\{x_{2n_k}\}$ converges to x in A. It follows from Lemma 2.2 that $d(x_{2n_k-1}, x_{2n_k})$ converges to $d(A, B)$. Since $T : A \to B$ and $S : B \to A$ form an sMK-C-cyclic mapping between A and B and taking into account (4.9), we have that, for each $2n_k \in \mathbb{N}$ with $2n_k \geq n_0 + 1$,

$$\begin{aligned} d(x_{2n_k}, Tx) &= d(Tx, x_{2n_k}) \\ &\leq \psi(d(x, x_{2n_k-1})) \cdot [C(x, x_{2n_k-1}) - 2d(A, B)] + d(A, B) \\ &< \gamma_\eta \cdot [C(x, x_{2n_k-1}) - 2d(A, B)] + d(A, B), \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} C(x, x_{2n_k-1}) &= d(x, Sx_{2n_k-1}) + d(x_{2n_k-1}, Tx) \\ &= d(x, x_{2n_k}) + d(x_{2n_k-1}, Tx). \end{aligned} \tag{4.17}$$

Following from (4.16) and (4.17), we obtain that

$$d(x_{2n_k}, Tx) \leq \gamma_\eta [d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}) + d(x_{2n_k}, Tx) - 2d(A,B)] + d(A,B), \tag{4.18}$$

that is, we have that

$$d(A,B) \leq d(x_{2n_k}, Tx) \leq \frac{\gamma_\eta}{1-\gamma_\eta} \cdot [d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1})] + \left[1 - \frac{\gamma_\eta}{1-\gamma_\eta}\right] \cdot d(A,B). \tag{4.19}$$

Letting $k \to \infty$. Then we conclude that

$$d(A,B) \leq d(x, Tx) \leq \frac{\gamma_\eta}{1-\gamma_\eta} \cdot [d(A,B) + 0] + \left[1 - \frac{\gamma_\eta}{1-\gamma_\eta}\right] \cdot d(A,B). \tag{4.20}$$

Therefore, $d(x, Tx) = d(A,B)$, that is, x is a best proximity point of T. □

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