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Research Article

Stability of an n-Dimensional Mixed-Type Additive and Quadratic Functional Equation in Random Normed Spaces

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We investigate the stability problems for the n-dimensional mixed-type additive and quadratic functional equation $2f(\sum_{j=1}^{n} x_j) + \sum_{1\le i,j\le n,\, i\ne j} f(x_i - x_j) = (n+1)\sum_{j=1}^{n} f(x_j) + (n-1)\sum_{j=1}^{n} f(-x_j)$ in random normed spaces by applying the fixed point method.

1. Introduction

In 1940, Ulam [1] gave a wide-ranging talk before a mathematical colloquium at the University of Wisconsin, in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms.

Let G_1 be a group, and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

If the answer is affirmative, we say that the functional equation for homomorphisms is stable. Hyers [2] was the first mathematician to present the result concerning the stability of functional equations. He answered the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces. This result of Hyers is stated as follows.

Let $f : E_1 \to E_2$ be a function between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \le \delta \tag{1.1}$$

for some $\delta > 0$ and for all $x, y \in E_1$. Then the limit $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for each $x \in E_1$, and $A : E_1 \to E_2$ is the unique additive function such that $\|f(x) - A(x)\| \le \delta$ for every $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then function A is linear.

We remark that the additive function A is directly constructed from the given function f, and this method is called the *direct method*. The direct method is a very powerful method for studying the stability problems of various functional equations. Taking this famous result into consideration, the additive Cauchy equation $f(x + y) = f(x) + f(y)$ is said to have the *Hyers-Ulam stability* on (E_1, E_2) if for every function $f : E_1 \to E_2$ satisfying the inequality (1.1) for some $\delta \ge 0$ and for all $x, y \in E_1$, there exists an additive function $A : E_1 \to E_2$ such that $f - A$ is bounded on E_1.

In 1950, Aoki [3] generalized the theorem of Hyers for additive functions, and in the following year, Bourgin [4] extended the theorem without proof. Unfortunately, it seems that their results failed to receive attention from mathematicians at that time. No one has made use of these results for a long time.

In 1978, Rassias [5] addressed the Hyers's stability theorem and attempted to weaken the condition for the bound of the norm of Cauchy difference and generalized the theorem of Hyers for linear functions.

Let $f : E_1 \to E_2$ be a function between Banach spaces. If f satisfies the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \le \theta\left(\|x\|^p + \|y\|^p\right) \tag{1.2}$$

for some $\theta \ge 0$, p with $0 \le p < 1$ and for all $x, y \in E_1$, then there exists a unique additive function $A : E_1 \to E_2$ such that $\|f(x) - A(x)\| \le (2\theta/(2 - 2^p))\|x\|^p$ for each $x \in E_1$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

This result of Rassias attracted a number of mathematicians who began to be stimulated to investigate the stability problems of functional equations. By regarding a large influence of Ulam, Hyers, and Rassias on the study of stability problems of functional equations, the stability phenomenon proved by Rassias is called the *Hyers-Ulam-Rassias stability*. For the last thirty years, many results concerning the Hyers-Ulam-Rassias stability of various functional equations have been obtained (see [6–17]).

In this paper, applying the fixed point method, we prove the Hyers-Ulam-Rassias stability of the *n-dimensional mixed-type additive and quadratic functional equation*

$$2f\left(\sum_{j=1}^{n} x_j\right) + \sum_{1\le i,j\le n,\ i\ne j} f(x_i - x_j) = (n+1)\sum_{j=1}^{n} f(x_j) + (n-1)\sum_{j=1}^{n} f(-x_j) \tag{1.3}$$

in random normed spaces. Every solution of (1.3) is called a *quadratic-additive function*.

Throughout this paper, let n be an integer larger than 1.

2. Preliminaries

We introduce some terminologies, notations, and conventions usually used in the theory of random normed spaces (see [18, 19]). The set of all probability distribution functions is denoted by

$$\Delta^+ := \{F : [0,\infty] \longrightarrow [0,1] \mid F \text{ is left-continuous and nondecreasing on } [0,\infty), \\ F(0) = 0, \text{ and } F(\infty) = 1\}. \tag{2.1}$$

Let us define $D^+ := \{F \in \Delta^+ \mid \lim_{t\to\infty} F(t) = 1\}$. The set Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \geq 0$. The maximal element for Δ^+ in this order is the distribution function $\varepsilon_0 : [0,\infty] \longrightarrow [0,1]$ given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } t > 0. \end{cases} \tag{2.2}$$

Definition 2.1 (See [18]). A function $\tau : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a *continuous triangular norm* (briefly, *continuous t-norm*) if τ satisfies the following conditions:

(a) τ is commutative and associative;

(b) τ is continuous;

(c) $\tau(a,1) = a$ for all $a \in [0,1]$;

(d) $\tau(a,b) \leq \tau(c,d)$ for all $a,b,c,d \in [0,1]$ with $a \leq c$ and $b \leq d$.

Typical examples of continuous t-norms are $\tau_P(a,b) = ab$, $\tau_M(a,b) = \min\{a,b\}$, and $\tau_L(a,b) = \max\{a+b-1,0\}$.

Definition 2.2 (See [19]). Let X be a vector space, τ a continuous t-norm, and let $\Lambda : X \longrightarrow D^+$ be a function satisfying the following conditions:

(R_1) $\Lambda_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(R_2) $\Lambda_{\alpha x}(t) = \Lambda_x(t/|\alpha|)$ for all $x \in X$, $\alpha \neq 0$, and for all $t \geq 0$;

(R_3) $\Lambda_{x+y}(t+s) \geq \tau(\Lambda_x(t), \Lambda_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

A triple (X, Λ, τ) is called a *random normed space* (briefly, *RN-space*).

If $(X, \|\cdot\|)$ is a normed space, we can define a function $\Lambda : X \longrightarrow D^+$ by

$$\Lambda_x(t) = \frac{t}{t + \|x\|} \tag{2.3}$$

for all $x \in X$ and $t > 0$. Then (X, Λ, τ_M) is a random normed space, which is called the *induced random normed space*.

Definition 2.3. Let (X, Λ, τ) be an RN-space.

(i) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\Lambda_{x_n - x}(t) > 1 - \varepsilon$ whenever $n \geq N$.

(ii) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\Lambda_{x_n - x_m}(t) > 1 - \varepsilon$ whenever $n \geq m \geq N$.

(*iii*) An RN-space (X, Λ, τ) is called *complete* if and only if every Cauchy sequence in X converges to a point in X.

Definition 2.4. Let X be a nonempty set. A function $d : X^2 \to [0, \infty]$ is called a *generalized metric* on X if and only if d satisfies

(M_1) $d(x, y) = 0$ if and only if $x = y$;

(M_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(M_3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We now introduce one of the fundamental results of the fixed point theory. For the proof, we refer to [20] or [21].

Theorem 2.5 (See [20, 21]). *Let (X, d) be a complete generalized metric space. Assume that $\Lambda : X \to X$ is a strict contraction with the Lipschitz constant $L < 1$. If there exists a nonnegative integer n_0 such that $d(\Lambda^{n_0+1}x, \Lambda^{n_0}x) < \infty$ for some $x \in X$, then the following statements are true:*

(*i*) *the sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;*

(*ii*) *x^* is the unique fixed point of Λ in $X^* = \{y \in X \mid d(\Lambda^{n_0}x, y) < \infty\}$;*

(*iii*) *if $y \in X^*$, then*

$$d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y). \tag{2.4}$$

In 2003, Radu [22] noticed that many theorems concerning the Hyers-Ulam stability of various functional equations follow from the fixed point alternative (Theorem 2.5). Indeed, he applied the fixed point method to prove the existence of a solution of the inequality (1.1) and investigated the Hyers-Ulam stability of the additive Cauchy equation (see also [23–26]). Furthermore, Miheţ and Radu [27] applied the fixed point method to prove the stability theorems of the additive Cauchy equation in random normed spaces.

In 2009, Towanlong and Nakmahachalasint [28] established the general solution and the stability of the n-dimensional mixed-type additive and quadratic functional equation (1.3) by using the direct method. According to [28], a function $f : E_1 \to E_2$ is a quadratic-additive function, where E_1 and E_2 are vector spaces, if and only if there exist an additive function $a : E_1 \to E_2$ and a quadratic function $q : E_1 \to E_2$ such that $f(x) = a(x) + q(x)$ for all $x \in E_1$.

3. Hyers-Ulam-Rassias Stability

Throughout this paper, let X be a real vector space and let (Y, Λ, τ_M) be a complete RN-space. For a given function $f : X \to Y$, we use the following abbreviation:

$$\begin{aligned} &Df(x_1, x_2, \ldots, x_n) \\ &\quad := 2f\left(\sum_{j=1}^{n} x_j\right) + \sum_{1\leq i,j\leq n,\ i\neq j} f(x_i - x_j) - (n+1)\sum_{j=1}^{n} f(x_j) - (n-1)\sum_{j=1}^{n} f(-x_j) \end{aligned} \tag{3.1}$$

for all $x_1, x_2, \ldots, x_n \in X$.

We will now prove the stability of the functional equation (1.3) in random normed spaces by using fixed point method.

Theorem 3.1. *Let X be a real vector space, (Z, Λ', τ_M) an RN-space, (Y, Λ, τ_M) a complete RN-space, and let $\varphi : (X \setminus \{0\})^n \to Z$ be a function. Assume that φ satisfies one of the following conditions:*

(*i*) $\Lambda'_{\alpha\varphi(x_1,x_2,\ldots,x_n)}(t) \le \Lambda'_{\varphi(nx_1,nx_2,\ldots,nx_n)}(t)$ *for some* $0 < \alpha < n$;

(*ii*) $\Lambda'_{\varphi(nx_1,nx_2,\ldots,nx_n)}(t) \le \Lambda'_{\alpha\varphi(x_1,x_2,\ldots,x_n)}(t)$ *for some* $\alpha > n^2$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ *and* $t > 0$. *If a function* $f : X \to Y$ *satisfies* $f(0) = 0$ *and*

$$\Lambda_{Df(x_1,x_2,\ldots,x_n)}(t) \ge \Lambda'_{\varphi(x_1,x_2,\ldots,x_n)}(t) \tag{3.2}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ *and* $t > 0$, *then there exists a unique function* $F : X \to Y$ *such that*

$$DF(x_1, x_2, \ldots, x_n) = 0 \tag{3.3}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ *and*

$$\Lambda_{f(x)-F(x)}(t) \ge \begin{cases} M(x, 2(n-\alpha)t) & \text{if } \varphi \text{ satisfies } (i), \\ M(x, 2(\alpha - n^2)t) & \text{if } \varphi \text{ satisfies } (ii) \end{cases} \tag{3.4}$$

for all $x \in X \setminus \{0\}$ *and* $t > 0$, *where* $M(x,t) := \tau_M(\Lambda'_{\varphi(\hat{x})}(t),\ \Lambda'_{\varphi(\widehat{-x})}(t))$, *and* $\hat{x} = (x, x, \ldots, x)$.

Proof. We will first treat the case where φ satisfies the condition (i). Let S be the set of all functions $g : X \to Y$ with $g(0) = 0$, and let us define a generalized metric on S by

$$d(g,h) := \inf\{u \in [0,\infty] \mid \Lambda_{g(x)-h(x)}(ut) \ge M(x,t)\ \forall x \in X \setminus \{0\}, t > 0\}. \tag{3.5}$$

It is not difficult to show that (S, d) is a complete generalized metric space (see [29] or [30, 31]).

Consider the operator $J : S \to S$ defined by

$$Jf(x) := \frac{f(nx) - f(-nx)}{2n} + \frac{f(nx) + f(-nx)}{2n^2}. \tag{3.6}$$

Then we can apply induction on m to prove

$$J^m f(x) = \frac{f(n^m x) - f(-n^m x)}{2n^m} + \frac{f(n^m x) + f(-n^m x)}{2n^{2m}} \tag{3.7}$$

for all $x \in X$ and $m \in \mathbb{N}$.

Let $f, g \in S$ and let $u \in [0, \infty]$ be an arbitrary constant with $d(g, f) \le u$. For some $0 < \alpha < n$ satisfying the condition (i), it follows from the definition of d, (R_2), (R_3), and (i) that

$$\begin{aligned}
\Lambda_{Jg(x)-Jf(x)}\left(\frac{\alpha ut}{n}\right) &= \Lambda_{((n+1)(g(nx)-f(nx))/2n^2)-((n-1)(g(-nx)-f(-nx))/2n^2)}\left(\frac{\alpha ut}{n}\right) \\
&\ge \tau_M\left(\Lambda_{(n+1)(g(nx)-f(nx))/2n^2}\left(\frac{(n+1)\alpha ut}{(2n^2)}\right),\right. \\
&\qquad \left.\Lambda_{(n-1)(g(-nx)-f(-nx))/2n^2}\left(\frac{(n-1)\alpha ut}{(2n^2)}\right)\right) \\
&\ge \tau_M\left(\Lambda_{g(nx)-f(nx)}(\alpha ut),\ \Lambda_{g(-nx)-f(-nx)}(\alpha ut)\right) \\
&\ge \tau_M\left(\Lambda'_{\varphi(\widehat{nx})}(\alpha t),\ \Lambda'_{\varphi(\widehat{-nx})}(\alpha t)\right) \\
&\ge M(x, t)
\end{aligned} \tag{3.8}$$

for all $x \in X \setminus \{0\}$ and $t > 0$, which implies that

$$d(Jf, Jg) \le \frac{\alpha}{n} d(f, g). \tag{3.9}$$

That is, J is a strict contraction with the Lipschitz constant $0 < \alpha/n < 1$.

Moreover, by (R_2), (R_3), and (3.2), we see that

$$\begin{aligned}
\Lambda_{f(x)-Jf(x)}\left(\frac{t}{2n}\right) &= \Lambda_{(-(n+1)Df(\widehat{x})+(n-1)Df(\widehat{-x}))/4n^2}\left(\frac{t}{2n}\right) \\
&\ge \tau_M\left(\Lambda_{(n+1)Df(\widehat{x})/4n^2}\left(\frac{(n+1)t}{4n^2}\right),\ \Lambda_{(n-1)Df(\widehat{-x})/4n^2}\left(\frac{(n-1)t}{4n^2}\right)\right) \\
&\ge \tau_M\left(\Lambda_{Df(\widehat{x})}(t),\ \Lambda_{Df(\widehat{-x})}(t)\right) \\
&\ge M(x, t)
\end{aligned} \tag{3.10}$$

for all $x \in X \setminus \{0\}$ and $t > 0$. Hence, it follows from the definition of d that

$$d(f, Jf) \le \frac{1}{2n} < \infty. \tag{3.11}$$

Now, in view of Theorem 2.5, the sequence $\{J^m f\}$ converges to the unique "fixed point" $F : X \to Y$ of J in the set $T = \{g \in S \mid d(f, g) < \infty\}$ and F is represented by

$$F(x) = \lim_{m \to \infty}\left(\frac{f(n^m x) - f(-n^m x)}{2n^m} + \frac{f(n^m x) + f(-n^m x)}{2n^{2m}}\right) \tag{3.12}$$

for all $x \in X$.

By Theorem 2.5, (3.11), and the definition of d, we have

$$d(f,F) \leq \frac{1}{1-\alpha/n} d(f,Jf) \leq \frac{1}{2(n-\alpha)}, \tag{3.13}$$

that is, the first inequality in (3.4) holds true.

We will now show that F is a quadratic-additive function. It follows from (R_3) and the definition of τ_M that

$$\begin{aligned}
\Lambda_{DF(x_1,x_2,\ldots,x_n)}(t) \geq \min\Bigg\{ &\Lambda_{2(F-J^m f)(\sum_{j=1}^{n} x_j)}\left(\frac{t}{5}\right), \\
&\min\left\{ \Lambda_{(F-J^m f)(x_i-x_j)}\left(\frac{t}{(5n(n-1))}\right) \;\Big|\; 1 \leq i,j \leq n,\ i \neq j \right\}, \\
&\min\left\{ \Lambda_{(n+1)(J^m f-F)(x_j)}\left(\frac{t}{(5n)}\right) \mid j = 1,\ldots,n \right\}, \\
&\min\left\{ \Lambda_{(n-1)(J^m f-F)(-x_j)}\left(\frac{t}{(5n)}\right) \mid j = 1,\ldots,n \right\}, \\
&\Lambda_{DJ^m f(x_1,x_2,\ldots,x_n)}\left(\frac{t}{5}\right) \Bigg\}
\end{aligned} \tag{3.14}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$, $t > 0$, and $m \in \mathbb{N}$. Due to the definition of F, the first four terms on the right-hand side of the above inequality tend to 1 as $m \to \infty$.

By a somewhat tedious manipulation, we have

$$\begin{aligned}
DJ^m f(x_1,x_2,\ldots,x_n) = {} & \frac{1}{2n^{2m}} Df(n^m x_1,\ldots,n^m x_n) + \frac{1}{2n^{2m}} Df(-n^m x_1,\ldots,-n^m x_n) \\
&+ \frac{1}{2n^m} Df(n^m x_1,\ldots,n^m x_n) - \frac{1}{2n^m} Df(-n^m x_1,\ldots,-n^m x_n).
\end{aligned} \tag{3.15}$$

Hence, it follows from (R_2), (R_3), definition of τ_M, (3.2), and (i) that

$$\begin{aligned}
\Lambda_{DJ^m f(x_1,\ldots,x_n)}\left(\frac{t}{5}\right) &\geq \min\Bigg\{ \Lambda_{Df(n^m x_1,\ldots,n^m x_n)/2n^{2m}}\left(\frac{t}{20}\right),\ \Lambda_{Df(-n^m x_1,\ldots,-n^m x_n)/2n^{2m}}\left(\frac{t}{20}\right), \\
&\qquad \Lambda_{Df(n^m x_1,\ldots,n^m x_n)/2n^{m}}\left(\frac{t}{20}\right),\ \Lambda_{Df(-n^m x_1,\ldots,-n^m x_n)/2n^{m}}\left(\frac{t}{20}\right) \Bigg\} \\
&\geq \min\Bigg\{ \Lambda_{Df(n^m x_1,\ldots,n^m x_n)}\left(\frac{n^{2m} t}{10}\right),\ \Lambda_{Df(-n^m x_1,\ldots,-n^m x_n)}\left(\frac{n^{2m} t}{10}\right), \\
&\qquad \Lambda_{Df(n^m x_1,\ldots,n^m x_n)}\left(\frac{n^{m} t}{10}\right),\ \Lambda_{Df(-n^m x_1,\ldots,-n^m x_n)}\left(\frac{n^{m} t}{10}\right) \Bigg\}
\end{aligned}$$

$$\geq \min\left\{\Lambda'_{\varphi(x_1,\ldots,x_n)}\left(\frac{n^{2m}t}{(10\alpha^m)}\right),\ \Lambda'_{\varphi(-x_1,\ldots,-x_n)}\left(\frac{n^{2m}t}{(10\alpha^m)}\right),\right.$$
$$\left.\Lambda'_{\varphi(x_1,\ldots,x_n)}\left(\frac{n^{m}t}{(10\alpha^m)}\right),\ \Lambda'_{\varphi(-x_1,\ldots,-x_n)}\left(\frac{n^{m}t}{(10\alpha^m)}\right)\right\}, \tag{3.16}$$

which tends to 1 as $m \to \infty$ for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and $t > 0$. Therefore, (3.14) implies that

$$\Lambda_{DF(x_1,x_2,\ldots,x_n)}(t) = 1 \tag{3.17}$$

for any $x_1, \ldots, x_n \in X \setminus \{0\}$ and $t > 0$. By (R_1), this implies that $DF(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in X \setminus \{0\}$, which ends the proof of the first part.

Now, assume that φ satisfies the condition (ii). Let (S, d) be the same as given in the first part. We now consider the operator $J : S \to S$ defined by

$$Jg(x) := \frac{n}{2}\left(g\left(\frac{x}{n}\right) - g\left(-\frac{x}{n}\right)\right) + \frac{n^2}{2}\left(g\left(\frac{x}{n}\right) + g\left(-\frac{x}{n}\right)\right) \tag{3.18}$$

for all $g \in S$ and $x \in X$. Notice that

$$J^m g(x) = \frac{n^m}{2}\left(g\left(\frac{x}{n^m}\right) - g\left(-\frac{x}{n^m}\right)\right) + \frac{n^{2m}}{2}\left(g\left(\frac{x}{n^m}\right) + g\left(-\frac{x}{n^m}\right)\right) \tag{3.19}$$

for all $x \in X$ and $m \in \mathbb{N}$.

Let $f, g \in S$ and let $u \in [0, \infty]$ be an arbitrary constant with $d(g, f) \leq u$. From (R_2), (R_3), the definition of d, and (ii), we have

$$\Lambda_{Jg(x)-Jf(x)}\left(\frac{n^2ut}{\alpha}\right) = \Lambda_{((n^2+n)/2)(g(x/n)-f(x/n))+((n^2-n)/2)(g(-x/n)-f(-x/n))}\left(\frac{n^2ut}{\alpha}\right)$$
$$\geq \tau_M\left(\Lambda_{((n^2+n)/2)(g(x/n)-f(x/n))}\left(\frac{(n^2+n)ut}{(2\alpha)}\right),\right.$$
$$\left.\Lambda_{((n^2-n)/2)(g(-x/n)-f(-x/n))}\left(\frac{(n^2-n)ut}{(2\alpha)}\right)\right)$$
$$= \tau_M\left(\Lambda_{g(x/n)-f(x/n)}\left(\frac{ut}{\alpha}\right),\ \Lambda_{g(-x/n)-f(-x/n)}\left(\frac{ut}{\alpha}\right)\right)$$
$$\geq \tau_M\left(M\left(\frac{x}{n}, \frac{t}{\alpha}\right),\ M\left(-\frac{x}{n}, \frac{t}{\alpha}\right)\right)$$
$$= \tau_M\left(\Lambda'_{\varphi(\widehat{x/n})}\left(\frac{t}{\alpha}\right),\ \Lambda'_{\varphi(\widehat{-x/n})}\left(\frac{t}{\alpha}\right)\right)$$

$$\begin{aligned}
&= \tau_M\left(\Lambda'_{\alpha\varphi(\widehat{x/n})}(t),\ \Lambda'_{\alpha\varphi(\widehat{-x/n})}(t)\right) \\
&\geq \tau_M\left(\Lambda'_{\varphi(\widehat{x})}(t),\ \Lambda'_{\varphi(\widehat{-x})}(t)\right) \\
&= M(x,t)
\end{aligned} \tag{3.20}$$

for all $x \in X \setminus \{0\}$, $t > 0$, and for some $\alpha > n^2$ satisfying (ii), which implies that

$$d(Jf, Jg) \leq \frac{n^2}{\alpha} d(f,g). \tag{3.21}$$

That is, J is a strict contraction with the Lipschitz constant $0 < n^2/\alpha < 1$.

Moreover, by (R_2), (3.2), and (ii), we see that

$$\begin{aligned}
\Lambda_{f(x)-Jf(x)}\left(\frac{t}{(2\alpha)}\right) &= \Lambda_{(1/2)Df(\widehat{x/n})}\left(\frac{t}{(2\alpha)}\right) \\
&\geq \Lambda'_{\varphi(\widehat{x/n})}\left(\frac{t}{\alpha}\right) \\
&= \Lambda'_{\alpha\varphi(\widehat{x/n})}(t) \\
&\geq \Lambda'_{\varphi(\widehat{x})}(t) \\
&\geq M(x,t)
\end{aligned} \tag{3.22}$$

for all $x \in X \setminus \{0\}$ and $t > 0$. This implies that $d(f, Jf) \leq 1/(2\alpha) < \infty$ by the definition of d. Therefore, according to Theorem 2.5, the sequence $\{J^m f\}$ converges to the unique "fixed point" $F : X \to Y$ of J in the set $T = \{g \in S \mid d(f,g) < \infty\}$ and F is represented by

$$F(x) = \lim_{m\to\infty}\left(\frac{n^m}{2}\left(f\left(\frac{x}{n^{\mathrm{m}}}\right) - f\left(-\frac{x}{n^m}\right)\right) + \frac{n^{2m}}{2}\left(f\left(\frac{x}{n^m}\right) + f\left(-\frac{x}{n^m}\right)\right)\right) \tag{3.23}$$

for all $x \in X$. Since

$$d(f,F) \leq \frac{1}{1-n^2/\alpha} d(f,Jf) \leq \frac{1}{2(\alpha - n^2)}, \tag{3.24}$$

the second inequality in (3.4) holds true.

Next, we will show that F is a quadratic-additive function. As we did in the first part, we obtain the inequality (3.14). In view of the definition of F, the first four terms on the right-hand side of the inequality (3.14) tend to 1 as $m \to \infty$. Furthermore, a long manipulation yields

$$\begin{aligned} DJ^m f(x_1, x_2, \ldots, x_n) &= \frac{n^{2m}}{2} Df\left(\frac{x_1}{n^m}, \ldots, \frac{x_n}{n^m}\right) + \frac{n^{2m}}{2} Df\left(-\frac{x_1}{n^m}, \ldots, -\frac{x_n}{n^m}\right) \\ &\quad + \frac{n^m}{2} Df\left(\frac{x_1}{n^m}, \ldots, \frac{x_n}{n^m}\right) - \frac{n^m}{2} Df\left(-\frac{x_1}{n^m}, \ldots, -\frac{x_n}{n^m}\right). \end{aligned} \tag{3.25}$$

Thus, it follows from (R_2), (R_3), definition of τ_M, (3.2), and (ii) that

$$\begin{aligned} &\Lambda_{DJ^m f(x_1,\ldots,x_n)}\left(\frac{t}{5}\right) \\ &\geq \min\Big\{ \Lambda_{(n^{2m}/2)Df(x_1/n^m,\ldots,x_n/n^m)}\left(\frac{t}{20}\right), \Lambda_{(n^{2m}/2)Df(-x_1/n^m,\ldots,-x_n/n^m)}\left(\frac{t}{20}\right), \\ &\qquad \Lambda_{(n^m/2)Df(x_1/n^m,\ldots,x_n/n^m)}\left(\frac{t}{20}\right), \Lambda_{-(n^m/2)Df(-x_1/n^m,\ldots,-x_n/n^m)}\left(\frac{t}{20}\right)\Big\} \\ &\geq \min\Big\{ \Lambda'_{\varphi(x_1/n^m,\ldots,x_n/n^m)}\left(\frac{t}{(10n^{2m})}\right), \Lambda'_{\varphi(-x_1/n^m,\ldots,-x_n/n^m)}\left(\frac{t}{(10n^{2m})}\right), \\ &\qquad \Lambda'_{\varphi(x_1/n^m,\ldots,x_n/n^m)}\left(\frac{\mathrm{t}}{(10n^{m})}\right), \Lambda'_{\varphi(-x_1/n^m,\ldots,-x_n/n^m)}\left(\frac{t}{(10n^{m})}\right)\Big\} \\ &\geq \min\Big\{ \Lambda'_{\alpha^{-m}\varphi(x_1,\ldots,x_n)}\left(\frac{t}{(10n^{2m})}\right), \Lambda'_{\alpha^{-m}\varphi(-x_1,\ldots,-x_n)}\left(\frac{t}{(10n^{2m})}\right), \\ &\qquad \Lambda'_{\alpha^{-m}\varphi(x_1,\ldots,x_n)}\left(\frac{t}{(10n^{m})}\right), \Lambda'_{\alpha^{-m}\varphi(-x_1,\ldots,-x_n)}\left(\frac{t}{(10n^{m})}\right)\Big\} \\ &= \min\Big\{ \Lambda'_{\varphi(x_1,\ldots,x_n)}\left(\frac{\alpha^m t}{(10n^{2m})}\right), \Lambda'_{\varphi(-x_1,\ldots,-x_n)}\left(\frac{\alpha^m t}{(10n^{2m})}\right), \\ &\qquad \Lambda'_{\varphi(x_1,\ldots,x_n)}\left(\frac{\alpha^m t}{(10n^{m})}\right), \Lambda'_{\varphi(-x_1,\ldots,-x_n)}\left(\frac{\alpha^m t}{(10n^{m})}\right)\Big\}, \end{aligned} \tag{3.26}$$

which tends to 1 as $m \to \infty$ for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and $t > 0$. Therefore, it follows from (3.14) that

$$\Lambda_{DF(x_1,x_2,\ldots,x_n)}(t) = 1 \tag{3.27}$$

for any $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and $t > 0$. By (R_1), this implies that

$$DF(x_1, x_2, \ldots, x_n) = 0 \tag{3.28}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$, which ends the proof. □

By a similar way presented in the proof of Theorem 3.1, we can also prove the preceding theorem if the domains of relevant functions include 0.

Theorem 3.2. *Let X be a real vector space, (Z, Λ', τ_M) an RN-space, (Y, Λ, τ_M) a complete RN-space, and let $\varphi : X^n \to Z$ be a function. Assume that φ satisfies one of the conditions (i) and (ii) in Theorem 3.1 for all $x_1, x_2, \ldots, x_n \in X$ and $t > 0$. If a function $f : X \to Y$ satisfies $f(0) = 0$ and (3.2) for all $x_1, x_2, \ldots, x_n \in X$ and $t > 0$, then there exists a unique quadratic-additive function $F : X \to Y$ satisfying (3.4) for all $x \in X$ and $t > 0$.*

Now, we obtain general Hyers-Ulam stability results of (1.3) in normed spaces. If X is a normed space, then (X, Λ, τ_M) is an induced random normed space. We get the following result.

Corollary 3.3. *Let X be a real vector space, Y a complete normed space, and let $\varphi : (X \setminus \{0\})^n \to [0, \infty)$ be a function. Assume that φ satisfies one of the following conditions:*

(iii) $\varphi(nx_1, \ldots, nx_n) \le \alpha\varphi(x_1, \ldots, x_n)$ for some $1 < \alpha < n$;

(iv) $\varphi(nx_1, \ldots, nx_n) \ge \alpha\varphi(x_1, \ldots, x_n)$ for some $\alpha > n^2$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$. If a function $f : X \to Y$ satisfies $f(0) = 0$ and

$$\|Df(x_1, x_2, \ldots, x_n)\| \le \varphi(x_1, x_2, \ldots, x_n) \tag{3.29}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$, then there exists a unique function $F : X \to Y$ such that

$$DF(x_1, x_2, \ldots, x_n) = 0 \tag{3.30}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and

$$\|f(x) - F(x)\| \le \begin{cases} \dfrac{\max\{\varphi(\widehat{x}),\ \varphi(\widehat{-x})\}}{2(n-\alpha)} & \text{if } \varphi \text{ satisfies } (iii), \\ \dfrac{\max\{\varphi(\widehat{x}),\ \varphi(\widehat{-x})\}}{2(\alpha - n^2)} & \text{if } \varphi \text{ satisfies } (iv) \end{cases} \tag{3.31}$$

for all $x \in X \setminus \{0\}$.

Proof. Let us put

$$Z := \mathbb{R}, \qquad \Lambda_x(t) := \frac{t}{t + \|x\|}, \qquad \Lambda'_z(t) := \frac{t}{t + |z|} \tag{3.32}$$

for all $x, x_1, x_2, \ldots, x_n \in X \setminus \{0\}$, $z \in \mathbb{R} \setminus \{0\}$, and $t \ge 0$. If φ satisfies the condition (iii) for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and for some $1 < \alpha < n$, then

$$\Lambda'_{\alpha\varphi(x_1,\ldots,x_n)}(t) = \frac{t}{t + \alpha\varphi(x_1, \ldots, x_n)} \le \frac{t}{t + \varphi(nx_1, \ldots, nx_n)} = \Lambda'_{\varphi(nx_1,\ldots,nx_n)}(t) \tag{3.33}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and $t > 0$, that is, φ satisfies the condition (i). In a similar way, we can show that if φ satisfies (iv), then it satisfies the condition (ii).

Moreover, we get

$$\Lambda_{Df(x_1,\ldots,x_n)}(t) = \frac{t}{t + \left\|Df(x_1,\ldots,x_n)\right\|} \geq \frac{t}{t + \varphi(x_1,\ldots,x_n)} = \Lambda'_{\varphi(x_1,\ldots,x_n)}(t) \tag{3.34}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and $t > 0$, that is, f satisfies the inequality (3.2) for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$.

According to Theorem 3.1, there exists a unique function $F : X \to Y$ such that

$$DF(x_1, x_2, \ldots, x_n) = 0 \tag{3.35}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and

$$\Lambda_{f(x)-F(x)}(t) \geq \begin{cases} \tau_M\left(\Lambda'_{\varphi(\widehat{x})}(2(n-\alpha)t), \Lambda'_{\varphi(\widehat{-x})}(2(n-\alpha)t)\right) & \text{if } \varphi \text{ satisfies } (iii), \\ \tau_M\left(\Lambda'_{\varphi(\widehat{x})}(2(\alpha-n^2)t), \Lambda'_{\varphi(\widehat{-x})}(2(\alpha-n^2)t)\right) & \text{if } \varphi \text{ satisfies } (iv) \end{cases} \tag{3.36}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and $t > 0$, which ends the proof. □

We now prove the Hyers-Ulam-Rassias stability of (1.3) in the framework of normed spaces.

Corollary 3.4. *Let X be a real normed space, $p \in [0, 1) \cup (2, \infty)$, and let Y be a complete normed space. If a function $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\left\|Df(x_1, x_2, \ldots, x_n)\right\| \leq \theta\left(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p\right) \tag{3.37}$$

for all $x_1, x_2, \ldots, x_n \in X$ and for some $\theta \geq 0$, then there exists a unique quadratic-additive function $F : X \to Y$ such that

$$\left\|f(x) - F(x)\right\| \leq \begin{cases} \dfrac{n\theta\|x\|^p}{2(n-n^p)} & \text{if } 0 \leq p < 1, \\ \dfrac{n\theta\|x\|^p}{2(n^p-n^2)} & \text{if } p > 2 \end{cases} \tag{3.38}$$

for all $x \in X$.

Proof. If we put

$$\varphi(x_1, x_2, \ldots, x_n) := \theta\left(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p\right), \tag{3.39}$$

then the induced random normed space (X, Λ_x, τ_M) satisfies the conditions stated in Theorem 3.2 with $\alpha = n^p$. □

Corollary 3.5. *Let X be a real normed space, $p \in (-\infty, 0)$, and let Y be a complete normed space. If a function $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\left\| Df(x_1, x_2, \ldots, x_n) \right\| \leq \theta \sum_{1 \leq i \leq n,\ x_i \neq 0} \|x_i\|^p \tag{3.40}$$

for all $x_1, x_2, \ldots, x_n \in X$ and for some $\theta \geq 0$, then there exists a unique quadratic-additive function $F : X \to Y$ satisfying

$$\left\| f(x) - F(x) \right\| \leq \begin{cases} \dfrac{n\theta\|x\|^p}{2(n - n^p)} & \text{if } x \in X \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases} \tag{3.41}$$

Proof. If we put $Z := \mathbb{R}$, $\alpha := n^p$, and define

$$\begin{gathered} \Lambda_x(t) := \frac{t}{t + \|x\|}, \qquad {\Lambda'}_z(t) := \frac{t}{t + |z|}, \\ \varphi(x_1, x_2, \ldots, x_n) := \theta \sum_{1 \leq i \leq n,\ x_i \neq 0} \|x_i\|^p \end{gathered} \tag{3.42}$$

for all $x, x_1, x_2, \ldots, x_n \in X$ and $z \in Z$, then we have

$$\begin{aligned} {\Lambda'}_{\alpha\varphi(x_1, x_2, \ldots, x_n)}(t) &= \frac{t}{t + \alpha\varphi(x_1, \ldots, x_n)} \\ &= \frac{t}{t + \varphi(nx_1, \ldots, nx_n)} \\ &= {\Lambda'}_{\varphi(nx_1, nx_2, \ldots, nx_n)}(t), \end{aligned} \tag{3.43}$$

that is, φ satisfies condition (i) given in Theorem 3.1 for all $x_1, x_2, \ldots, x_n \in X$ and $t > 0$. We moreover get

$$\begin{aligned} \Lambda_{Df(x_1, x_2, \ldots, x_n)}(t) &= \frac{t}{t + \left\| Df(x_1, \ldots, x_n) \right\|} \\ &\geq \frac{t}{t + \theta\sum_{1 \leq i \leq n,\ x_i \neq 0} \|x_i\|^p} \\ &= \frac{t}{t + \varphi(x_1, \ldots, x_n)} \\ &= {\Lambda'}_{\varphi(x_1, x_2, \ldots, x_n)}(t), \end{aligned} \tag{3.44}$$

that is, f satisfies the inequality (3.2) for all $x_1, x_2, \ldots, x_n \in X$ and $t > 0$.

According to Theorem 3.2, there exists a unique quadratic-additive function $F : X \to Y$ satisfying

$$\frac{t}{t + \|f(x) - F(x)\|} = \Lambda_{f(x)-F(x)}(t) \geq M(x, 2(n - n^p)t) = \begin{cases} \dfrac{2(n-n^p)t}{2(n-n^p)t + n\theta\|x\|^p} & \text{if } x \in X \setminus \{0\}, \\ 1 & \text{if } x = 0 \end{cases} \tag{3.45}$$

for all $t > 0$, or equivalently

$$\frac{\|f(x) - F(x)\|}{t} \leq \begin{cases} \dfrac{n\theta\|x\|^p}{2(n-n^p)t} & \text{if } x \in X \setminus \{0\}, \\ 0 & \text{if } x = 0 \end{cases} \tag{3.46}$$

for all $t > 0$, which ends the proof. □

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