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A CONSTRUCTION OF SOME IDEALS IN AFFINE VERTEX ALGEBRAS

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We study ideals generated by singular vectors in vertex operator algebras associated with representations of affine Lie algebras of types A and C. We find new explicit formulas for singular vectors in these vertex operator algebras at integer and half-integer levels. These formulas generalize the expressions for singular vectors from Adamović (1994). As a consequence, we obtain a new family of vertex operator algebras for which we identify the associated Zhu's algebras. A connection with the representation theory of Weyl algebras is also discussed.

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1. Introduction. Let $\hat{\mathfrak{g}}$ be the affine Lie algebra associated with the finite-dimensional simple Lie algebra $\mathfrak{g}$. Then, on the generalized Verma module, $N_k(\mathfrak{g})$, $k \in \mathbb{C}$, exists the natural structure of a vertex operator algebra (VOA) (cf. [5, 10, 12, 14, 15]). The VOA $N_k(\mathfrak{g})$ provides a natural framework for studying many aspects of the representation theory of the affine Lie algebra $\hat{\mathfrak{g}}$. In particular, every highest-weight $\hat{\mathfrak{g}}$-module of level k can be treated as a module for the VOA $N_k(\mathfrak{g})$.

On the other hand, there are other VOAs associated with $\hat{\mathfrak{g}}$-modules of level k. In fact, every $\hat{\mathfrak{g}}$-submodule I of $N_k(\mathfrak{g})$ becomes an ideal in the VOA $N_k(\mathfrak{g})$, and on the quotient $N_k(\mathfrak{g})/I$, there exists the structure of a VOA (cf. [10]). Therefore, it is important to explicitly construct ideals in $N_k(\mathfrak{g})$. This can be done by constructing singular vectors in $N_k(\mathfrak{g})$. In this note, we present a new explicit construction of singular vectors in $N_k(\mathfrak{g})$.

Let $N_k^1(\mathfrak{g})$ be the maximal ideal in $N_k(\mathfrak{g})$. Then, the quotient $L_k(\mathfrak{g}) = N_k(\mathfrak{g})/N_k^1(\mathfrak{g})$ is a simple VOA. If k is a positive integer, then $N_k^1(\mathfrak{g})$ is generated by the singular vector $e_\theta(-1)^{k+1}\mathbf{1}$ (cf. [10, 11]). The similar situation is in the case when k is an admissible rational number (cf. [2, 4, 9, 13]), but the expressions for the singular vectors are much more complicated.

In order to study the annihilating ideals of highest-weight representations, it is very important to understand the ideal lattice of the VOA $N_k(\mathfrak{g})$. This problem was initiated in [9]. It is a known fact that in the case $\mathfrak{g} = sl_2$ and $k \neq -2$, $N_k^1(sl_2(\mathbb{C}))$ is the unique ideal in the VOA $N_k(sl_2(\mathbb{C}))$. A different situation is in the case of the critical level ($k = -h^\vee$, here $h^\vee$ denotes the dual Coxeter number). In this case, there exists a very rich structure of ideals of $N_{-h^\vee}(\mathfrak{g})$, which implies the existence of infinitely many nonisomorphic VOAs (cf. [5]).

In this note we introduce a family $J_{m,n}(\mathfrak{g})$ of ideals in $N_k(\mathfrak{g})$ and a family $V_{m,n}(\mathfrak{g})$ of corresponding quotient VOAs in the cases of affine Lie algebras $A_{\ell-1}^{(1)}$ and $C_\ell^{(1)}$. These families include VOAs associated with the integrable representations and VOAs associated with the admissible representations at half-integer levels investigated in [1]. The basic step in our construction is the construction of one infinite family of singular vectors in $N_k(\mathfrak{g})$. The expressions for these singular vectors provide a generalization of the results obtained from [1].

We also begin the study of the representation theory of these VOAs by identifying the corresponding Zhu's algebras explicitly. We demonstrate that the VOA $N_k(\mathfrak{g})$ for $k \in \mathbb{N}$ can have a nontrivial quotient which has infinitely many irreducible modules from the category $\mathbb{O}$. These representations are parameterized with certain algebraic curves.

2. Vertex operator algebra $N_k(\mathfrak{g})$. We make the assumption that the reader is familiar with the elementary theory of the VOAs and their representations (cf. [7, 8, 10, 12, 14, 16, 17]).

In this section, we recall some basic facts on affine VOAs. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $(\cdot,\cdot)$ be a nondegenerate symmetric bilinear form on $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ be a triangular decomposition for $\mathfrak{g}$. Let θ be the highest root for $\mathfrak{g}$, and e_θ the corresponding root vector. Define ρ as usual. The affine Lie algebra $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined as $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}d$, where c is the canonical central element [11] and the Lie algebra structure is given by

$$\begin{gathered}[x\otimes t^n, y\otimes t^m] = [x,y]\otimes t^{n+m} + n(x,y)\delta_{n+m,0}c,\\ [d, x\otimes t^n] = nx\otimes t^n,\end{gathered} \tag{2.1}$$

for $x,y\in\mathfrak{g}$. We write $x(n)$ for $x\otimes t^n$ and identify $\mathfrak{g}$ with $\mathfrak{g}\otimes t^0$.

The Cartan subalgebra $\hat{\mathfrak{h}}$ and the subalgebras $\hat{\mathfrak{n}}_\pm$, $\hat{\mathfrak{g}}_\pm$ of $\hat{\mathfrak{g}}$ are defined as

$$\hat{\mathfrak{h}} = \mathfrak{h}\oplus\mathbb{C}c\oplus\mathbb{C}d, \qquad \hat{\mathfrak{g}}_\pm = \mathfrak{g}\otimes t^{\pm1}\mathbb{C}[t^{\pm1}], \qquad \hat{\mathfrak{n}}_\pm = \mathfrak{n}_\pm\oplus\mathfrak{g}\otimes t^{\pm1}\mathbb{C}[t^{\pm1}]. \tag{2.2}$$

Let $P = \mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}c\oplus\mathbb{C}d$ be an upper parabolic subalgebra. For every $k\in\mathbb{C}$, $k\neq -h^\vee$, let $\mathbb{C}v_k$ be 1-dimensional P-module such that the subalgebra $\mathfrak{g}\otimes\mathbb{C}[t]+\mathbb{C}d$ acts trivially and the central element c acts as multiplication with $k\in\mathbb{C}$. Define the generalized Verma module $N_k(\mathfrak{g})$ as

$$N_k(\mathfrak{g}) = U(\hat{\mathfrak{g}})\otimes_{U(P)}\mathbb{C}v_k. \tag{2.3}$$

Then, $N_k(\mathfrak{g})$ has a unique structure of a VOA which is generated by the fields

$$x(z)=\sum_{n\in\mathbb{Z}}x(n)z^{-n-1},\quad x\in\mathfrak{g}. \tag{2.4}$$

The vacuum vector is $\mathbf{1}=1\otimes v_k$.

By construction, $N_k(\mathfrak{g})$ is a highest-weight $\hat{\mathfrak{g}}$-module. A weight vector $v\in N_k(\mathfrak{g})$ is called a singular vector if $\hat{\mathfrak{n}}_+v=0$. Every singular vector $v\in N_k(\mathfrak{g})$ generates the submodule $I=U(\hat{\mathfrak{g}})v$. It is an elementary fact from the vertex algebra theory that I is an ideal in $N_k(\mathfrak{g})$, and on the quotient $N_k(\mathfrak{g})/I$, there exists the structure of a VOA (cf. [10, 15]).

Recall that there is one-to-one correspondence between the irreducible modules of the VOA V and the irreducible modules for the corresponding Zhu's algebra $A(V)$ (cf. [10, 16, 17]). The Zhu's algebra of the VOA $N_k(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})$ (cf. [10]). Let $F:U(\hat{\mathfrak{g}}_-)\to U(\mathfrak{g})$ be the projection map defined as follows:

$$F\big(a_1(-i_1-1)\cdots a_n(-i_n-1)\big)=(-1)^{i_1+\cdots+i_n}a_na_{n-1}\cdots a_1 \tag{2.5}$$

for every $a_1,\dots,a_n\in\mathfrak{g}$, $i_1,\dots,i_n\in\mathbb{Z}_+$, and $n\in\mathbb{N}$. Assume that I is an ideal in the VOA $N_k(\mathfrak{g})$. Let $\langle F(I)\rangle$ be the two-sided ideal of $U(\mathfrak{g})$ generated by the set $\{F(w)\mid w\in U(\hat{\mathfrak{g}}_-),\ w\mathbf{1}\in I\}$. Then, the Zhu's algebra of the quotient VOA $N_k(\mathfrak{g})/I$ is isomorphic to the quotient algebra

$$\frac{U(\mathfrak{g})}{\langle F(I)\rangle} \tag{2.6}$$

(for more details, see [10]).

3. Lie algebras $sp_{2\ell}(\mathbb{C})$ and $sl_\ell(\mathbb{C})$. In this section, we recall the construction of the Lie algebras $sp_{2\ell}(\mathbb{C})$ and $sl_\ell(\mathbb{C})$ using Weyl algebras (cf. [3, 6]).

We consider the first two ℓ-dimensional vector spaces $A_1=\sum_{i=1}^{\ell}\mathbb{C}a_i$ and $A_2=\sum_{i=1}^{\ell}\mathbb{C}a_i^*$. Let $A=A_1+A_2$. The Weyl algebra $W(A)$ is defined as the associative algebra over $\mathbb{C}$ generated by A and the relations

$$[a_i,a_j]=[a_i^*,a_j^*]=0,\quad [a_i,a_j^*]=\delta_{i,j},\ i,j\in\{1,2,\dots,\ell\}. \tag{3.1}$$

The normal ordering on A is defined by

$$:xy:=\frac{1}{2}(xy+yx),\quad x,y\in A. \tag{3.2}$$

Then, all such elements $:xy:$ span a Lie algebra isomorphic to $\mathfrak{g}=sp_{2\ell}(\mathbb{C})$ with a Cartan subalgebra $\mathfrak{h}$ spanned by

$$h_i=-:a_ia_i^*:,\quad i=1,2,\dots,\ell. \tag{3.3}$$

Let $\{\epsilon_i \mid 1 \le i \le \ell\} \subset \mathfrak{h}^*$ be the dual basis such that $\epsilon_i(h_j) = \delta_{i,j}$. The root system of $\mathfrak{g}$ is given by

$$\Delta = \{\pm(\epsilon_i \pm \epsilon_j), \pm 2\epsilon_i \mid 1 \le i, j \le \ell,\ i < j\} \tag{3.4}$$

with $\alpha_1 = \epsilon_1 - \epsilon_2, \dots, \alpha_{\ell-1} = \epsilon_{\ell-1} - \epsilon_\ell$, and $\alpha_\ell = 2\epsilon_\ell$ being a set of simple roots. The highest root is $\theta = 2\epsilon_1$. We fix the root vectors

$$X_{\epsilon_i - \epsilon_j} =: a_i a_j^* :, \qquad X_{\epsilon_i + \epsilon_j} =: a_i a_j :, \qquad X_{-(\epsilon_i + \epsilon_j)} =: a_i^* a_j^* : . \tag{3.5}$$

Assume that $\ell \ge 2$. Then, the simple Lie algebra $sl_\ell(\mathbb{C})$ is a Lie subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$ generated by the set

$$\{X_{\epsilon_i - \epsilon_j} \mid i, j = 1, \dots, \ell;\ i \ne j\}. \tag{3.6}$$

The Cartan subalgebra $\mathfrak{h}_1$ is spanned by

$$\{h_i - h_j \mid i, j = 1, \dots, \ell;\ i \ne j\}. \tag{3.7}$$

From the above construction, we conclude that there are nonzero homomorphisms

$$\Phi : U(\mathfrak{g}) \longrightarrow W(A), \qquad \Phi_1 = \Phi|_{U(\mathfrak{g}_1)} : U(\mathfrak{g}_1) \longrightarrow W(A). \tag{3.8}$$

These homomorphisms are used in the following sections for establishing a connection between Zhu's algebras and Weyl algebras.

4. Ideals in the VOA $N_k(sp_{2\ell}(\mathbb{C}))$. In this section, let $\mathfrak{g}=sp_{2\ell}(\mathbb{C})$. We present one construction of singular vectors in $N_k(\mathfrak{g})$ for integer and half-integer values of k. This construction generalizes the construction of singular vectors at half-integer levels from [1]. At the end of this section, we give some detailed remarks and examples with applications of our results.

We use the notation as in Section 3. For $m \in \mathbb{N}$, $m \le \ell$, we define the matrices C_m and $C_m(-1)$ by

$$C_m = \begin{bmatrix} X_{2\epsilon_1} & X_{\epsilon_1+\epsilon_2} & \cdots & X_{\epsilon_1+\epsilon_m} \\ X_{\epsilon_1+\epsilon_2} & X_{2\epsilon_2} & \cdots & X_{\epsilon_2+\epsilon_m} \\ \vdots & \vdots & \ddots & \vdots \\ X_{\epsilon_1+\epsilon_m} & \cdots & & X_{2\epsilon_m} \end{bmatrix},$$
$$C_m(-1) = \begin{bmatrix} X_{2\epsilon_1}(-1) & X_{\epsilon_1+\epsilon_2}(-1) & \cdots & X_{\epsilon_1+\epsilon_m}(-1) \\ X_{\epsilon_1+\epsilon_2}(-1) & X_{2\epsilon_2}(-1) & \cdots & X_{\epsilon_2+\epsilon_m}(-1) \\ \vdots & \vdots & \ddots & \vdots \\ X_{\epsilon_1+\epsilon_m}(-1) & \cdots & & X_{2\epsilon_m}(-1) \end{bmatrix}. \tag{4.1}$$

As usual, let $C_m^{i,j}$ (resp., $C_m^{i,j}(-1)$) be an $(m-1)\times(m-1)$ matrix obtained by deleting the ith row and the jth column of the matrix C_m (resp., $C_m(-1)$). Define next

$$\begin{aligned}\Delta_m(-1) = \det(C_m(-1)) &= \sum_{\sigma\in \mathrm{Sym}_m} (-1)^{\mathrm{sign}(\sigma)} \prod_{i=1}^{m} X_{\epsilon_i+\epsilon_{\sigma(i)}}(-1),\\ \Delta_m = \det(C_m) &= \sum_{\sigma\in \mathrm{Sym}_m} (-1)^{\mathrm{sign}(\sigma)} \prod_{i=1}^{m} X_{\epsilon_i+\epsilon_{\sigma(i)}}.\end{aligned} \tag{4.2}$$

Set $\Delta_m^{i,j}(-1) = \det(C_m^{i,j}(-1))$.

Using the definition and the properties of determinants, we can prove the following relations:

$$\begin{aligned}[X_{\alpha_i}(0), \Delta_m(-1)] &= 0 \quad \text{for } i = 1,\dots,\ell;\\ [X_{\epsilon_i-\epsilon_j}(0), \Delta_m(-1)] &= 0 \quad \text{for } i,j = 1,\dots,m,\ i\neq j;\\ X_{-2\epsilon_1}(1)(\Delta_m(-1))^n\mathbf{1} &= -4n(c-k_{m,n})(\Delta_m^{1,1}(-1))(\Delta_m(-1))^{n-1}\mathbf{1},\end{aligned} \tag{4.3}$$

where $k_{m,n} = n-(m+1)/2$.

These relations immediately give the following theorem.

Theorem 4.1. *For every $m,n\in\mathbb{N}$, $m\le \ell$, $(\Delta_m(-1))^n\mathbf{1}$ is a singular vector in $N_{k_{m,n}}(\mathfrak{g})$.*

Define the ideal $J_{m,n}(\mathfrak{g})$ in the VOA $N_{k_{m,n}}(\mathfrak{g})$ with

$$J_{m,n}(\mathfrak{g}) = U(\hat{\mathfrak{g}})(\Delta_m(-1))^n\mathbf{1}. \tag{4.4}$$

Let

$$V_{m,n}(\mathfrak{g}) = \frac{N_{k_{m,n}}(\mathfrak{g})}{J_{m,n}(\mathfrak{g})} \tag{4.5}$$

be the corresponding quotient VOA.

Remark 4.2. For $m=1$, Theorem 4.1 gives the known fact that $X_{2\epsilon_1}(-1)^s\mathbf{1}$ is a singular vector in $N_{s-1}(\mathfrak{g})$. Moreover, this vector generates the submodule $J_{1,s}(\mathfrak{g})$ which coincides with the maximal submodule of $N_{s-1}(\mathfrak{g})$.

For $m=2$, Theorem 4.1 reconstructs the result from [1, Theorem 3.1] that

$$(X_{2\epsilon_1}(-1)X_{2\epsilon_2}(-1) - X_{\epsilon_1+\epsilon_2}(-1)^2)^s\mathbf{1} \tag{4.6}$$

is a singular vector in $N_{s-3/2}(\mathfrak{g})$. Again, the corresponding submodule $J_{2,s}(\mathfrak{g})$ is the maximal submodule of $N_{s-3/2}(\mathfrak{g})$.

Assume that $m_1\neq m$, $n_1\neq n$, and $k_{m_1,n_1}=k_{m,n}$. This implies that $(\Delta_{m_1}(-1))^{n_1}\mathbf{1}$ and $(\Delta_m(-1))^n\mathbf{1}$ are different singular vectors in $N_{k_{m,n}}(\mathfrak{g})$. Then using Theorem 4.1 and Remark 4.2, we get the following reducibility result.

Corollary 4.3. *Assume that $\ell, s \in \mathbb{N}$ and $\ell \geq 3$. The maximal submodule $J_{1,s}(\mathfrak{g})$ of $N_{s-1}(\mathfrak{g})$ is reducible. If $\ell \geq 4$, then the maximal submodule $J_{2,s}(\mathfrak{g})$ of $N_{s-3/2}(\mathfrak{g})$ is reducible.*

The following result identifies the Zhu's algebra of the VOA $V_{m,n}(\mathfrak{g})$.

Theorem 4.4. *(1) The Zhu's algebra $A(V_{m,n}(\mathfrak{g}))$ is isomorphic to $U(\mathfrak{g})/\langle(\Delta_m)^n\rangle$, where $\langle(\Delta_m)^n\rangle$ is a two-sided ideal in $U(\mathfrak{g})$ generated by the vector $(\Delta_m)^n$.*

(2) For $m \geq 2$, there is a nontrivial homomorphism

$$\bar{\Phi} : A(V_{m,n}(\mathfrak{g})) \longrightarrow W(A). \tag{4.7}$$

In particular, if $\pi : W(A) \to \mathrm{End}(M)$ is any nontrivial $W(A)$-module, then $\pi \circ \bar{\Phi}$ is a module for the Zhu's algebra $A(V_{m,n}(\mathfrak{g}))$.

Proof. The proof of the statement (1) follows from the fact that the projection map F maps $(\Delta_m(-1))^n$ to $(\Delta_m)^n$.

In order to prove (2), we consider the (nontrivial) homomorphism $\Phi : U(\mathfrak{g}) \to W(A)$ defined in Section 3. For $m \geq 2$, we have

$$\Phi(\Delta_m) = \det \begin{bmatrix} a_1^2 & a_1a_2 & \cdots & a_1a_m \\ a_1a_2 & a_2^2 & \cdots & a_2a_m \\ \cdots & \cdots & \ddots & \cdots \\ a_1a_m & \cdots & \cdots & a_m^2 \end{bmatrix} = 0 \tag{4.8}$$

and $\Phi((\Delta_m)^n) = 0$, which implies that there is a nontrivial homomorphism $\bar{\Phi} : A(V_{m,n}(\mathfrak{g})) \to W(A)$. □

Remark 4.5. Since the Weyl algebra $W(A)$ has a rich structure of irreducible representations, Theorem 4.4 implies that, for $m \geq 2$, the Zhu's algebra $A(V_{m,n}(\mathfrak{g}))$ has infinitely many irreducible representations. Then the Zhu's algebra theory (cf. [10, 16, 17]) implies that the VOA $V_{m,n}(\mathfrak{g})$ has also infinitely many irreducible representations.

A very interesting question is the classification of irreducible modules in the category $\mathbb{O}$. In the case $m = 2$, irreducible representations in the category $\mathbb{O}$ of the VOA $V_{2,n}(\mathfrak{g})$ were classified in [1]. It was proved that any $V_{2,n}$-module from the category $\mathbb{O}$ is completely reducible.

Remark 4.6. An application of the theory of the VOAs to the integrable highest-weight modules was made in [15]. It was proved that there is a bijection between the irreducible loop modules associated with certain subspaces of $N_k(\mathfrak{g})$ and the singular vectors in $N_k(\mathfrak{g})$. As a consequence of our results in this note, we get a new family of such loop modules. Let $R_{m,n} = U(\mathfrak{g})(\Delta_m(-1))^n\mathbf{1}$ be the top level of the ideal $J_{m,n}(\mathfrak{g})$. It is clear that $R_{m,n}$ is an irreducible finite-dimensional $\mathfrak{g}$-module with the highest weight $2n\omega_m$ (here, $\omega_1,\dots,\omega_\ell$ denote the fundamental weights for $\mathfrak{g}$). Then, $\overline{R}_{m,n} = R_{m,n}\otimes\mathbb{C}[t,t^{-1}]$ is a loop module which acts on the highest-weight representations of level $k = k_{m,n}$ (for definitions, see [15]). The loop modules $\overline{R}_{m,n}$ for $m = 1$ and for $m = 2$ were constructed in [15] and [1], respectively. The results of Section 5 also provide new examples of loop modules acting on integer-level highest-weight representations in the case of the affine Lie algebra $A_{\ell-1}^{(1)}$.

Example 4.7. Let $\mathfrak{g} = sp_6(\mathbb{C})$. Set $R = R_{3,1} = U(\mathfrak{g})\cdot\Delta_{3,1}$. Then, R is a finite-dimensional irreducible $\mathfrak{g}$-module with the highest weight $2\omega_3$. Using the arguments similar to those in [1, 2, 15], we get that an irreducible highest-weight $\mathfrak{g}$-module $V(\lambda)$ with the highest weight λ is a module for the Zhu's algebra $A(V_{3,1}(\mathfrak{g}))$ if and only if

$$R_0 v_\lambda = 0, \tag{4.9}$$

where v_λ is the highest-weight vector and R_0 is the zero-weight subspace of R. Moreover, for every $u\in R_0$, there is a unique polynomial $p_u\in U(\mathfrak{h})$ such that $uv_\lambda = p_u(h)v_\lambda$. Since $\dim R_0 = 4$, we get that the irreducible highest-weight $A(V_{3,1})$-modules are parameterized with the zeros of four polynomials $p_1(h)$, $p_2(h)$, $p_3(h)$, and $p_4(h)$. Using the considerations similar to those in [1, Section 5], we get that these polynomials are

$$\begin{aligned}
p_1(h_1,h_2,h_3) &= (h_1+1)\left(h_2+\frac{1}{2}\right)h_3,\\
p_2(h_1,h_2,h_3) &= (h_1+1)(4h_3+(h_2+h_3)(h_2+h_3-1)),\\
p_3(h_1,h_2,h_3) &= h_3(4(h_2+1)+(h_1+h_2+2)(h_1+h_2-1)),\\
p_4(h_1,h_2,h_3) &= 4h_3(h_2+1)+(h_1+h_3-1)(h_2+h_3+h_2(h_1+h_3)).
\end{aligned} \tag{4.10}$$

Now, it is easy to find the zeros of these polynomials. The classification of irreducible modules follows from the Zhu's algebra theory. Finally, we obtain the following complete list of irreducible modules from the category $\mathcal{O}$:

$$\begin{aligned}
&\{L((-x-1)\Lambda_0+x\Lambda_1)\mid x\in\mathbb{C}\}\cup\{L((-x-1)\Lambda_1+x\Lambda_2)\mid x\in\mathbb{C}\}\\
&\quad\cup\{L((-x-1)\Lambda_2+x\Lambda_3)\mid x\in\mathbb{C}\}\\
&\quad\cup\Big\{L(-2\Lambda_0+\Lambda_2),L(\Lambda_1-2\Lambda_3),L\left(-\frac{1}{2}\Lambda_0-\frac{1}{2}\Lambda_3\right),L\left(-\frac{1}{2}\Lambda_0+\Lambda_2-\frac{3}{2}\Lambda_3\right),\\
&\qquad L\left(-\frac{3}{2}\Lambda_0+\Lambda_1-\frac{1}{2}\Lambda_3\right),L\left(-\frac{3}{2}\Lambda_0+\Lambda_1+\Lambda_2-\frac{3}{2}\Lambda_3\right)\Big\}.
\end{aligned} \tag{4.11}$$

It is also important to see that the irreducible modules are parameterized with a union of one finite set and a union of three lines in $\mathbb{C}^4$.

We also notice that every module for the Zhu's algebra $A(V_{3,1}(\mathfrak{g}))$ is also a module for the Zhu's algebra $A(V_{3,n}(\mathfrak{g}))$ for every $n \in \mathbb{N}$. Using the previous arguments, we conclude that, for every $n \in \mathbb{N}$, the VOA $V_{3,n}$ has uncountably many irreducible modules from the category $\mathcal{O}$.

5. Ideals in the VOA $N_k(sl_\ell(\mathbb{C}))$. In this section, let $\mathfrak{g} = sl_\ell(\mathbb{C})$. We present one construction of singular vectors in $N_k(\mathfrak{g})$ for integer values of k. All results are completely analogous to those from Section 4.

As before, we use the notation from Section 3. For $m \in \mathbb{N}$, $2m \le \ell$, we define the matrices A_m and $A_m(-1)$ by

$$A_m = \begin{bmatrix} X_{\epsilon_1-\epsilon_\ell} & X_{\epsilon_1-\epsilon_{\ell-1}} & \cdots & X_{\epsilon_1-\epsilon_{\ell-m+1}} \\ X_{\epsilon_2-\epsilon_\ell} & X_{\epsilon_2-\epsilon_{\ell-1}} & \cdots & X_{\epsilon_2-\epsilon_{\ell-m+1}} \\ \vdots & \vdots & \ddots & \vdots \\ X_{\epsilon_m-\epsilon_\ell} & \cdots & & X_{\epsilon_m-\epsilon_{\ell-m+1}} \end{bmatrix},$$
$$A_m(-1) = \begin{bmatrix} X_{\epsilon_1-\epsilon_\ell}(-1) & X_{\epsilon_1-\epsilon_{\ell-1}}(-1) & \cdots & X_{\epsilon_1-\epsilon_{\ell-m+1}}(-1) \\ X_{\epsilon_2-\epsilon_\ell}(-1) & X_{\epsilon_2-\epsilon_{\ell-1}}(-1) & \cdots & X_{\epsilon_2-\epsilon_{\ell-m+1}}(-1) \\ \vdots & \vdots & \ddots & \vdots \\ X_{\epsilon_m-\epsilon_\ell}(-1) & \cdots & & X_{\epsilon_m-\epsilon_{\ell-m+1}}(-1) \end{bmatrix}. \tag{5.1}$$

Let $\Delta_m = \det(A_m)$ and $\Delta_m(-1) = \det(A_m(-1))$.

Theorem 5.1. *For every $m,n \in \mathbb{N}$, $2m \le \ell$, set $k_{m,n} = n - m$. Then, $(\Delta_m(-1))^n \mathbf{1}$ is a singular vector in $N_{k_{m,n}}(\mathfrak{g})$.*

Define the ideal $J_{m,n}(\mathfrak{g})$ in the VOA $N_{k_{m,n}}(\mathfrak{g})$ by

$$J_{m,n}(\mathfrak{g}) = U(\hat{\mathfrak{g}})(\Delta_m(-1))^n \mathbf{1}. \tag{5.2}$$

Let $V_{m,n}(\mathfrak{g}) = N_{k_{m,n}}(\mathfrak{g})/J_{m,n}(\mathfrak{g})$ be the quotient VOA.

Remark 5.2. For $m = 1$, Theorem 5.1 gives the known fact that $X_{\epsilon_1-\epsilon_\ell}(-1)^s \mathbf{1}$ is a singular vector in $N_{s-1}(\mathfrak{g})$. Moreover, this vector generates the submodule $J_{1,s}(\mathfrak{g})$, which is the maximal submodule of $N_{s-1}(\mathfrak{g})$.

We have the following corollary.

Corollary 5.3. *Assume that $\ell, s \in \mathbb{N}$, $\ell \ge 4$. Then, the maximal submodule $J_{1,s}(\mathfrak{g})$ of $N_{s-1}(\mathfrak{g})$ is reducible.*

Similarly as in Section 4, we can explicitly identify the Zhu's algebra $A(V_{m,n}(\mathfrak{g}))$ and find a connection with the Weyl algebra $W(A)$.

Theorem 5.4. *(1) The Zhu's algebra $A(V_{m,n}(\mathfrak{g}))$ is isomorphic to the quotient algebra*

$$\frac{U(\mathfrak{g})}{\langle(\Delta_m)^n\rangle}, \tag{5.3}$$

where $\langle(\Delta_m)^n\rangle$ is a two-sided ideal in $U(\mathfrak{g})$ generated by $(\Delta_m)^n$.

(2) For $m \geq 2$, there is a nontrivial homomorphism

$$\bar{\Phi}_1 : A(V_{m,n}(\mathfrak{g})) \longrightarrow W(A). \tag{5.4}$$

In particular, every module for the Weyl algebra $W(A)$ can be lifted to the module for the Zhu's algebra $A(V_{m,n}(\mathfrak{g}))$.

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