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ON IDEALS OF IMPLICATIVE SEMIGROUPS

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ABSTRACT. We introduce the notion of ideals in implicative semigroups, and then state the characterizations of the ideals.

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1. Introduction. The notions of implicative semigroup and ordered filter were introduced by Chan and Shum [3]. The first is a generalization of implicative semilattice (see Nemitz [6] and Blyth [2]) and has a close relation with implication in mathematical logic and set theoretic difference (see Birkhoff [1] and Curry [4]). For the general development of implicative semilattice theory the ordered filters play an important role which is shown by Nemitz [6]. Motivated by this, Chan and Shum [3] established some elementary properties, and constructed quotient structure of implicative semigroups via ordered filters. Jun et al. [5] discussed ordered filters of implicative semigroups. In this paper, we introduce the notion of ideals in implicative semigroups. By introducing special subsets of an implicative semigroups, we provide a condition for the special subset to be an ideal. We establish two characterizations of ideals.

2. Preliminaries. We recall some definitions and results. By a *negatively partially ordered semigroup* (briefly, *n.p.o. semigroup*) we mean a set S with a partial ordering $\leq$ and a binary operation $\cdot$ such that for all $x,y,z \in S$, we have

(1) $(x\cdot y)\cdot z = x\cdot(y\cdot z)$,

(2) $x \leq y$ implies $x\cdot z \leq y\cdot z$ and $z\cdot x \leq z\cdot y$,

(3) $x\cdot y \leq x$ and $x\cdot y \leq y$.

An n.p.o. semigroup $(S;\leq,\cdot)$ is said to be *implicative* if there is an additional binary operation $*: S\times S \to S$ such that for any elements x,y,z of S,

(4) $z \leq x*y$ if and only if $z\cdot x \leq y$.

The operation $*$ is called *implication*. From now on, an implicative n.p.o. semigroup is simply called an *implicative semigroup*.

An implicative semigroup $(S;\leq,\cdot,*)$ is said to be *commutative* if it satisfies

(5) $x\cdot y = y\cdot x$ for all $x,y \in S$, that is, $(S,\cdot)$ is a commutative semigroup.

In any implicative semigroup $(S;\leq,\cdot,*)$, $x*x = y*y$ for every $x,y \in S$ and this element is the greatest element, written 1, of $(S,\leq)$.

PROPOSITION 2.1 (see [3, Theorem 1.4]). *Let S be an implicative semigroup. Then for every $x,y,z \in S$, the following hold:*

(6) $x \leq 1$, $x*x = 1$, $x = 1*x$,

(7) $x \leq y*(x\cdot y)$,

(8) $x \le x * x^2$,
(9) $x \le y * x$,
(10) *if* $x \le y$ *then* $x * z \ge y * z$ *and* $z * x \le z * y$,
(11) $x \le y$ *if and only if* $x * y = 1$,
(12) $x * (y * z) = (x \cdot y) * z$,
(13) *if* S *is commutative then* $x * y \le (s \cdot x) * (s \cdot y)$ *for all* s *in* S.

Now we note important elementary properties of a commutative implicative semigroup, which follows from (5), (6), and (12).

Observation 2.2. If S is a commutative implicative semigroup, then for any $x, y, z \in S$,
(14) $x * (y * z) = y * (x * z)$,
(15) $y * z \le (x * y) * (x * z)$,
(16) $x \le (x * y) * y$.

3. Ideals of implicative semigroups. In what follows let S denote an implicative semigroup unless otherwise specified. We begin by defining the notion of ideals of S.

Definition 3.1. A subset I of S is called an *ideal* of S if
(I1) $x \in S$ and $a \in I$ imply $x * a \in I$,
(I2) $x \in S$ and $a, b \in I$ imply $(a * (b * x)) * x \in I$.

Example 3.2. Consider an implicative semigroup $S := \{1, a, b, c, d, 0\}$ with Cayley tables (Tables 3.1 and 3.2) and Hasse diagram (Figure 3.1) as follows:

Table 3.1

$\cdot$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

Table 3.2

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

We know that $\{1, a, b\}$ is an ideal of S, but $\{1, a\}$ is not an ideal of S, since $(a * (a * b)) * b = b \notin \{1, a\}$.

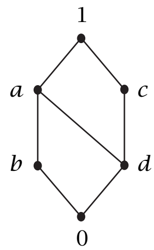

FIGURE 3.1

LEMMA 3.3. *Every ideal of S contains* 1.

PROOF. The proof follows from (6) and (I1). □

LEMMA 3.4. *If I is an ideal of S, then $(a*x)*x \in I$ for all $a \in I$ and $x \in S$.*

PROOF. The proof follows by taking $b = a$ and $a = 1$ in (I2). □

COROLLARY 3.5. *Let I be an ideal of S. If $a \in I$ and $a \leq x$, then $x \in I$.*

PROOF. Let $a \in I$ and $x \in S$ be such that $a \leq x$. Using (6) and Lemma 3.4, we have $x = 1*x = (a*x)*x \in I$. This completes the proof. □

LEMMA 3.6. *Let I be a subset of S such that*
(I3) $1 \in I$,
(I4) $x*(y*z) \in I$ *and* $y \in I$ *imply* $x*z \in I$ *for all* $x,y,z \in S$. *If* $a \in I$ *and* $a \leq x$, *then* $x \in I$.

PROOF. Let $a \in I$ and $x \in S$ be such that $a \leq x$. Then $x*(a*1) = x*1 = 1 \in I$ by (6) and (I3), and so $x = x*1 \in I$ by (I4). This completes the proof. □

The following is a characterization of ideals.

THEOREM 3.7. *Let S be a commutative implicative semigroup. A subset I of S is an ideal of S if and only if it satisfies conditions (I3) and (I4).*

PROOF. Let I be an ideal of S. Then $1 \in I$ by Lemma 3.3. Let $x,y,z \in S$ be such that $x*(y*z) \in I$ and $y \in I$. Using Lemma 3.4, we get $(y*z)*z \in I$. It follows from (6), (15), and (I2) that

$$x*z = 1*(x*z) = \big(\big((y*z)*z\big)*\big((x*(y*z))*(x*z)\big)\big)*(x*z) \in I. \tag{3.1}$$

Conversely, assume that I satisfies conditions (I3) and (I4). Let $x \in S$ and $a \in I$. Since $x*(a*a) = x*1 = 1 \in I$ by (I3), it follows from (I4) that $x*a \in I$, that is, (I1) holds. Since $(a*x)*(a*x) = 1 \in I$, we have $(a*x)*x \in I$ by (I4). Note from (15) that

$$((a*x)*x)*((b*(a*x))*(b*x)) = 1, \tag{3.2}$$

that is,

$$(a*x)*x \leq (b*(a*x))*(b*x) \tag{3.3}$$

for all $b \in I$. Thus, by Lemma 3.6, we have $(b*(a*x))*(b*x) \in I$. Using (I4), we conclude that $(b*(a*x))*x \in I$ which proves (I2). Hence I is an ideal of S. □

For any $u, v \in S$, consider a set

$$S(u,v) = \{z \in S \mid u * (v * z) = 1\}. \tag{3.4}$$

In Example 3.2, the set $S(1,a) = \{1,a\}$ is not an ideal of S. Hence we know that $S(u,v)$ may not be an ideal of S in general.

THEOREM 3.8. *Let S satisfy the left self-distributive law under $*$, that is, $x*(y*z) = (x*y)*(x*z)$ for all $x,y,z \in S$. For any $u,v \in S$, the set $S(u,v)$ is an ideal of S.*

PROOF. Let $x \in S$ and $a,b \in S(u,v)$. Then

$$\begin{aligned} u*(v*(x*a)) &= (u*(v*x))*(u*(v*a)) = (u*(v*x))*1 = 1, \\ u*(v*((a*(b*x))*x)) &= (u*(v*(a*(b*x))))*(u*(v*x)) \\ &= ((u*(v*a))*(u*(v*(b*x))))*(u*(v*x)) \\ &= (1*((u*(v*b))*(u*(v*x))))*(u*(v*x)) \\ &= (u*(v*x))*(u*(v*x)) = 1. \end{aligned} \tag{3.5}$$

Hence $x*a \in S(u,v)$ and $(a*(b*x))*x \in S(u,v)$, which shows that $S(u,v)$ is an ideal of S. □

LEMMA 3.9. *Let S be an implicative semigroup. If $y \in S$ satisfies $y*z = 1$ for all $z \in S$, then $S(x,y) = S = S(y,x)$ for all $x \in S$.*

PROOF. The proof is straightforward. □

EXAMPLE 3.10. Let $S := \{1,a,b,c,d\}$ be an implicative semigroup with Cayley tables (Tables 3.3 and 3.4) and Hasse diagram (Figure 3.2) as follows:

TABLE 3.3

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	d	c	d
b	b	d	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

TABLE 3.4

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

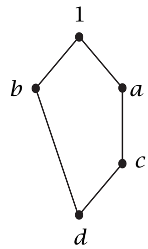

FIGURE 3.2

It is easy to check that S satisfies the left self-distributive law under $*$, that is, $x*(y*z) = (x*y)*(x*z)$ for all $x,y,z \in S$. By Lemma 3.9 we have $S(x,d) = S(d,x) = S$ for all $x \in S$. Furthermore we know that $S(1,1) = \{1\}$, $S(1,a) = S(a,1) = S(a,a) = S(a,b) = \{1,a\}$, $S(1,b) = S(b,1) = S(b,b) = \{1,b\}$, $S(1,c) = S(a,c) = S(c,1) = S(c,a) = S(c,c) = \{1,a,c\}$, $S(b,a) = \{1,a,b\}$, and $S(c,b) = S$ are ideals of S.

Using the set $S(u,v)$, we describe a characterization of ideals.

THEOREM 3.11. *Let S be a commutative implicative semigroup and let I be a nonempty subset of S. Then I is an ideal of S if and only if $S(u,v) \subseteq I$ for all $u,v \in I$.*

PROOF. Assume that I is an ideal of S and let $u,v \in I$. If $z \in S(u,v)$, then $u*(v*z) = 1 \in I$ and so $z = 1*z = (u*(v*z))*z \in I$ by (I2). Hence $S(u,v) \subseteq I$.

Conversely, suppose that $S(u,v) \subseteq I$ for all $u,v \in I$. Note that $1 \in S(u,v) \subseteq I$. Let $x,y,z \in S$ be such that $x*(y*z) \in I$ and $y \in I$. Since

$$(x*(y*z))*(y*(x*z)) = (y*(x*z))*(y*(x*z)) = 1, \tag{3.6}$$

we have $x*z \in S(x*(y*z),y) \subseteq I$. Applying Theorem 3.7, we conclude that I is an ideal of S. □

THEOREM 3.12. *Let S be a commutative implicative semigroup. If I is an ideal of S, then*

$$I = \cup_{u,v\in I} S(u,v). \tag{3.7}$$

PROOF. Let I be an ideal of S and let $x \in I$. Obviously, $x \in S(x,1)$ and so

$$I \subseteq \cup_{x\in I} S(x,1) \subseteq \cup_{u,v\in I} S(u,v). \tag{3.8}$$

Now let $y \in \cup_{u,v\in I} S(u,v)$. Then there exist $a,b \in I$ such that $y \in S(a,b)$. It follows from Theorem 3.11 that $y \in I$. Hence $\cup_{u,v\in I} S(u,v) \subseteq I$. This completes the proof. □

COROLLARY 3.13. *If I is an ideal of a commutative implicative semigroup S, then*

$$I = \cup_{w\in I} S(w,1). \tag{3.9}$$

References

[1] G. Birkhoff, *Lattice Theory*, 3rd ed., American Mathematical Society Colloquium Publications, vol. 25, American Mathematical Society, Rhode Island, 1967. MR 37#2638. Zbl 153.02501.

[2] T. S. Blyth, *Pseudo-residuals in semigroups*, J. London Math. Soc. **40** (1965), 441–454. MR 31#1211. Zbl 136.26903.

[3] M. W. Chan and K. P. Shum, *Homomorphisms of implicative semigroups*, Semigroup Forum **46** (1993), no. 1, 7–15. MR 93g:20127. Zbl 776.06012.

[4] H. B. Curry, *Foundations of Mathematical Logic*, McGraw-Hill Book, New York, 1963. MR 26#6036. Zbl 163.24209.

[5] Y. B. Jun, J. Meng, and X. L. Xin, *On ordered filters of implicative semigroups*, Semigroup Forum **54** (1997), no. 1, 75–82. MR 98a:06022. Zbl 862.06005.

[6] W. C. Nemitz, *Implicative semi-lattices*, Trans. Amer. Math. Soc. **117** (1965), 128–142. MR 31#1212. Zbl 128.24804.

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