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Research Article

Multiparametric Quantum Algebras and the Cosmological Constant

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With a view towards applications for de Sitter, we construct the multiparametric q-deformation of the $\mathrm{so}(5,\mathbb{C})$ algebra using the Faddeev-Reshetikhin-Takhtadzhyan (FRT) formalism.

1. Introduction

There are reasons based on arguments of holography and finiteness of entropy to believe that the Hilbert space for quantum theory in a de Sitter background is finite-dimensional [1–4]. Since the isometry group of de Sitter, $SO(4,1)$, has to be represented on this Hilbert space, and since we expect quantum theory to be unitary, this gives rise to an immediate problem; that is, $SO(4,1)$ cannot have finite-dimensional, unitary representations, because it is a noncompact group. It is in this context that the possibility of considering a deformed de Sitter space with a q-deformed isometry group becomes interesting [5–8] (some of these references work in the context of dS/CFT). It is a well-known fact that for certain values of the deformation parameter, (noncompact) quantum groups have unitary, finite-dimensional representations [9–11].

But recently it was shown [12] that single-parameter quantum deformation can give rise to deformed de Sitter space only when the deformation parameter is real. This throws a spanner in the above program because finite-dimensional representations for one-parameter deformations exist only when the deformation parameter q is a root of unity. One obvious way to work around this problem is to consider multiparametric deformations of the de Sitter isometry group, and the aim of this paper is to take a first step in that direction by writing down the algebra for this case explicitly.

Another reason why multiparametric deformations are interesting is because in the coordinate system of a static observer in de Sitter, the full $SO(4,1)$ isometry group is not visible; the manifest isometries are $SO(3)$ and a time translation (see the appendix for an elementary demonstration of this fact). So one of the questions we need to answer when we quantize in de Sitter is to understand how the static observer and the full isometry group are related to each other. One hope behind the construction of multiparametric deformations of $SO(4,1)$ is finding representations of such an algebra that will shed some light on the states of the observer and their relation to the representations of the full isometry group. We will be working at the level of complexified algebras; so what we refer to as the algebra of $SO(4,1)$ or $SO(5)$ is in fact B_2 in the Cartan scheme.

The usual one-parameter q-deformation for a Lie algebra is the Drinfeld-Jimbo (DJ) algebra. We will be interested in a construction of this algebra starting with a dual description in terms of R-matrices, using the Faddeev-Reshetikhin-Takhtadzhyan [13] approach. What we will do in this paper is to take the DJ algebras to be defined by the FRT method, and then we will extend the definition by using a generalized, multiparametric R-matrix [14, 15]. We will do this explicitly for $SO(5,\mathbb{C})$ and the result will be a multiparametric generalization of the DJ algebra.

In the next section, we will provide an introduction to the DJ algebra and how it can be derived from a dual description. In Section 4, we will write down the explicit form of the multiparametric R-matrix for $SO(5)$ from [15], and use that in the dual description to construct the multiparametric algebra for $SO(5)$. We conclude with some speculations and possibilities for future research.

Finite-dimensionality of de Sitter Hilbert space has also been discussed in [16, 17], and q-deformation in the context of AdS/CFT has been considered in [18, 19].

2. One-parameter DJ algebra and its dual description

Drinfeld-Jimbo algebra is a deformation of the universal enveloping algebra of the Lie algebra of a classical group. A universal enveloping algebra is the algebra spanned by polynomials in the generators, modulo the commutation relations. When we deform it, we mod out by a set of deformed relations, instead of the usual commutation relations. These relations are what defines the DJ algebra. When the deformation parameter tends to the limit unity, the algebra reduces to the universal enveloping algebra of the usual Lie algebra.

We will write down the algebra relations on the so-called Chevalley-Cartan-Weyl basis. The rest of the generators of the Lie algebra can be generated through commutations between these. The Drinfeld-Jimbo algebra is constructed as a deformation of the relations between the Chevalley generators. So, without any further ado, let us write down the form of the DJ algebra [20] for a generic semisimple Lie algebra $\mathbf{g}$ of rank l and Cartan matrix (a_{ij}). In what follows, q is a fixed nonzero complex number (the deformation parameter) and $q_i = q^{d_i}$, with $d_i = (\alpha_i, \alpha_i)$, where α_i are the simple roots of the Lie algebra. The norm used in the definition of d_i is the norm defined in the dual space of the Cartan subalgebra, through the Killing form. These are all defined in the standard references [20, 21]. The indices run from 1 to l.

With these at hand, we can define the Drinfeld-Jimbo algebra $U_q(\mathbf{g})$ as the algebra generated by E_i, F_i, K_i, K_i^{-1}, $1 \leq i \leq l$, and the defining relations

$$\begin{gathered} K_iK_j = K_jK_i, \qquad K_iK_i^{-1} = K_i^{-1}K_i, \\ K_iE_j = q_i^{a_{ij}}E_jK_i, \qquad K_iF_j = q_i^{-a_{ij}}F_jK_i, \\ E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \left[\left[1 - a_{ij}; r\right]\right]_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0, \quad i \neq j, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \left[\left[1 - a_{ij}; r\right]\right]_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r = 0, \quad i \neq j, \end{gathered} \tag{2.1}$$

with

$$\begin{gathered} [[n;r]]_q = \frac{[n]_q!}{[r]_q![n-r]_q!}, \\ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [1]_q[2]_q \cdots [n]_q, \quad [0]_q \equiv 1. \end{gathered} \tag{2.2}$$

The relations containing only the Es or the Fs are called Serre relations and they should be thought of as the price that we have to pay in order to write the algebra relations entirely in terms of the Chevalley generators. Sometimes, it is useful to write K_i as $q_i^{H_i}$. In the limit of $q \to 1$, the DJ algebra relations reduce to the Lie algebra relations written in the Chevalley basis, with H_i's being the generators in the Cartan subalgebra and E_i's and F_i's being the raising and lowering operators.

We will be interested in the specific case of SO(5) (Cartan's B_2), and we will rewrite the DJ algebra $U_{q^{1/2}}(\mathbf{so(5)})$ for that case in a slightly different form for later convenience:

$$\begin{gathered} k_1k_2 = k_2k_1, \qquad k_1^{-1} = q^{H_1+H_2/2}, \qquad k_2^{-1} = q^{H_2/2}, \\ k_1E_1 = q^{-1}E_1k_1, \qquad k_2E_1 = qE_1k_2, \\ k_1E_2 = E_2k_1, \qquad k_2E_2 = q^{-1}E_2k_2, \\ k_1F_1 = qF_1k_1, \qquad k_2F_1 = q^{-1}F_1k_2, \\ k_1F_2 = F_2k_1, \qquad k_2F_2 = qF_2k_2, \\ \left[E_1, F_1\right] = \frac{k_2k_1^{-1} - k_2^{-1}k_1}{q - q^{-1}}, \quad \left[E_2, F_2\right] = \frac{k_2^{-1} - k_2}{q^{1/2} - q^{-1/2}}. \end{gathered} \tag{2.3}$$

The Serre relations take the form

$$\begin{gathered} E_1^2E_2 - \left(q + q^{-1}\right)E_1E_2E_1 + E_2E_1^2 = 0, \\ E_1E_2^3 - \left(q + q^{-1} + 1\right)E_2E_1E_2^2 + \left(q + q^{-1} + 1\right)E_2^2E_1E_2 - E_2^3E_1 = 0 \end{gathered} \tag{2.4}$$

with analogous expressions for the Fs.

Drinfeld-Jimbo algebra is one way to describe a "quantum group." Another way to do this is to work with the groups directly and deform the group structure using the so-called R-matrices, rather than to deform the universal envelope of the Lie algebra. It turns out that both of these approaches are dual to each other, and one can obtain the DJ algebra by starting with R-matrices. Faddeev, Reshetikhin, and Takhtadzhyan have constructed a formalism for working with the R-matrices and constructing the DJ algebra starting from the dual approach. So, a natural place to look for, when trying to generalize the DJ algebra of SO(5), is this dual construction and try to see whether it admits any generalizations.

In the rest of this section, we will review the construction of the DJ algebra starting with the R-matrices. In the next section, we will start with a multiparametric generalization of the R-matrix for SO(5) and follow an analogous procedure to obtain the multiparametric SO(5) DJ algebra.

As already mentioned, the deformation of the group structure is done in the dual picture through the introduction of the R-matrix. The duality between the two approaches is manifested through the so-called L-functionals [20]. If one defines the L-functionals as certain matrices constructed from the DJ algebra generators, then the R-matrix and the L-functionals would together satisfy certain relations (which we will call the duality relations), as a consequence of the fact that the generators satisfy the DJ algebra. Conversely, we could start with L-functionals thought of as matrices with previously unconstrained matrix elements, and then the duality relations would be the statement that the matrix elements should satisfy the DJ algebra. Thus the L-functionals, together with the duality relations, are equivalent to the DJ algebra.

For any R-matrix,[1] we can define an algebra $A(R)$, with $N(N+1)$ generators l_{ij}^{+}, l_{ij}^{-}, $i \leq j$, $j = 1, 2, \ldots, N$, and the defining relations

$$\begin{gathered} L_1^{\pm} L_2^{\pm} R = R L_2^{\pm} L_1^{\pm}, \qquad L_1^{-} L_2^{+} R = R L_2^{+} L_1^{-}, \\ l_{ii}^{+} l_{ii}^{-} = l_{ii}^{-} l_{ii}^{+} = 1, \quad i = 1, 2, \ldots, N, \end{gathered} \tag{2.5}$$

where the matrices $L^{\pm} \equiv (l_{ij}^{\pm})$ and $l_{ij}^{+} = 0 = l_{ji}^{-}$, for $i > j$ (i.e., they are upper or lower triangular). The subscripts 1 and 2 have the following meanings. L_1^{+} stands for L^{+} tensored with the $N \times N$ identity matrix, and L_2^{+} stands for the $N \times N$ identity matrix tensored with L^{+}. So, the matrix multiplication with R is well defined because the R-matrix is an $N^2 \times N^2$ matrix. The above relations will be referred to as the duality relations. It turns out that this algebra has a Hopf algebra structure with

$$\begin{aligned} \text{comultiplication:} \quad & \Delta\left(l_{ij}^{\pm}\right) = \sum_k l_{ik}^{\pm} \otimes l_{kj}^{\pm}, \\ \text{counit:} \quad & \epsilon\left(l_{ij}^{\pm}\right) = \delta_{ij}, \\ \text{antipode:} \quad & S\left(L^{\pm}\right) = \left(L^{\pm}\right)^{-1}. \end{aligned} \tag{2.6}$$

[1] It is useful here to keep in mind that R-matrices are $N^2 \times N^2$ matrices.

Now, let us choose the R-matrix in the above case to be the one-parameter R-matrix for $\mathrm{SO}(N)$, with $N = 2n+1$:

$$\begin{aligned} R = {} & q \sum_{i \neq i'}^{2n} E_{ii} \otimes E_{ii} + q^{-1} \sum_{i \neq i'}^{2n} E_{ii} \otimes E_{i'i'} + E_{n+1,n+1} \otimes E_{n+1,n+1} \\ & + \sum_{i \neq j, j'}^{2n} E_{ii} \otimes E_{jj} + \left(q - q^{-1}\right) \left[\sum_{i>j}^{2n} E_{ij} \otimes E_{ji} - \sum_{i>j}^{2n} q^{\rho_i - \rho_j} E_{ij} \otimes E_{i'j'} \right]. \end{aligned} \tag{2.7}$$

Here E_{ij} is the $2n \times 2n$ matrix with 1 in the (i,j)-position and 0 everywhere else, and the symbol $\otimes$ stands for the tensoring of two matrices. $i' = 2n + 2 - i$, similarly for j'. The deformation parameter is q. Finally $(\rho_1, \rho_2, \ldots, \rho_{2n}) = (n - 1/2, n - 3/2, \ldots, 1/2, 0, -1/2, \ldots, -n + 1/2)$.

Let $I(\mathrm{so}(N))$ be the two-sided ideal in $A(R)$ generated by

$$L^{\pm} C^t \left(L^{\pm}\right)^t \left(C^{-1}\right)^t = I = C^t \left(L^{\pm}\right)^t \left(C^{-1}\right)^t L^{\pm}, \tag{2.8}$$

where I is the identity matrix, and the metric C defines a length in the vector space where the quantum matrices are acting. C provides the constraint arising from the fact that the underlying classical group is an orthogonal group: $TC^{-1}T^tC = I = C^{-1}T^tCT$ for quantum matrices T (see [15]). For $\mathrm{SO}(N)$,

$$C = \left(C^i_j\right), \quad C^i_j = \delta_{ij'} q^{-\rho_i} \tag{2.9}$$

and j' and ρ_i are as defined above.

Now, $I(\mathrm{so}(N))$ is a Hopf ideal of $A(R)$ [20], so the quotient $A(R)/I(\mathrm{so}(N))$ is also a Hopf algebra which we will call $U_q^L(\mathbf{so}(\mathbf{N}))$. Now, there is a theorem (see, e.g., [20] or [13] for a proof) which says that $U_q^L(\mathbf{so}(\mathbf{N}))$ is isomorphic to $U_{q^{1/2}}(\mathbf{so}(\mathbf{2n+1}))$, which is the DJ algebra for $\mathrm{SO}(2n+1)$ with deformation parameter $q^{1/2}$. Explicitly, this isomorphism can be written down as

$$\begin{aligned} l^+_{ii} &= q^{-H'_i}, \quad l^+_{i'i'} = q^{H'_i}, \\ l^+_{n+1,n+1} &= l^-_{n+1,n+1} = 1, \\ l^+_{k,k+1} &= \left(q - q^{-1}\right) q^{-H'_k} E_k, \\ l^+_{2n-k+1,2n-k+2} &= -\left(q - q^{-1}\right) q^{H'_{k+1}} E_k, \\ l^-_{k+1,k} &= -\left(q - q^{-1}\right) F_k q^{H'_k}, \\ l^-_{2n-k+2,2n-k+1} &= \left(q - q^{-1}\right) F_k q^{-H'_{k+1}}, \\ l^+_{n,n+1} &= \left(q^{1/2} + q^{-1/2}\right)^{1/2} \left(q^{1/2} - q^{-1/2}\right) q^{-H'_n} E_n, \\ l^+_{n+1,n+2} &= -q^{-1/2} \left(q^{1/2} + q^{-1/2}\right)^{1/2} \left(q^{1/2} - q^{-1/2}\right) E_n, \\ l^-_{n+1,n} &= -\left(q^{1/2} + q^{-1/2}\right)^{1/2} \left(q^{1/2} - q^{-1/2}\right) F_n q^{H'_n}, \\ l^-_{n+2,n+1} &= q^{1/2} \left(q^{1/2} + q^{-1/2}\right)^{1/2} \left(q^{1/2} - q^{-1/2}\right) F_n. \end{aligned} \tag{2.10}$$

Here, $i = 1, 2, \ldots, n$ as always, and $1 \leq k \leq n-1$. $H'_i = H_i + H_{i+1} + \cdots + H_{n-1} + H_n/2$. The above relations (which we will call the isomorphism relations) define the relations between elements of the L-matrices and the Chevalley-Cartan-Weyl generators. Sometimes, it will be convenient to call $q^{-H'_i}$ as k_i because it makes comparison with SO(5) DJ algebra (written earlier) more directly.

3. The multiparametric algebra

Our procedure for constructing the multiparametric algebra is straightforward. Instead of using the usual one-parametric R-matrices in the duality relations, we use the multiparametric R-matrices that Schirrmacher has written down [15]. We keep the isomorphism relations the same as above and use the duality relations to define the new multiparametric algebra.

In principle, this procedure could be done for all the multiparametric R-matrices of all the different Cartan groups using their associated isomorphism relations. We have endeavored to do this procedure for only the case of SO(5), but at least for the smaller Cartan groups, the exact same procedure can be performed on a computer using the appropriate R-matrices. Writing down the form of the multiparametric DJ algebra for a generic semisimple Lie algebra is an interesting problem which we have not attempted to tackle here.

The multiparametric R-matrix for SO$(2n+1)$ (which for our purposes is the same thing as B_n) looks like

$$\begin{aligned} R = {} & r \sum_{i \neq i'}^{2n} E_{ii} \otimes E_{ii} + r^{-1} \sum_{i \neq i'}^{2n} E_{ii} \otimes E_{i'i'} + E_{n+1,n+1} \otimes E_{n+1,n+1} \\ & + \sum_{i<j,\, i \neq j'}^{2n} \frac{r}{q_{ij}} E_{ii} \otimes E_{jj} + \sum_{i>j,\, i \neq j'}^{2n} \frac{q_{ij}}{r} E_{ii} \otimes E_{jj} \\ & + \left(r - r^{-1}\right) \left[\sum_{i>j}^{2n} E_{ij} \otimes E_{ji} - \sum_{i>j}^{2n} r^{\rho_i - \rho_j} E_{ij} \otimes E_{i'j'} \right]. \end{aligned} \tag{3.1}$$

The deformation parameters are r and q_{ij} and they are not all independent: $q_{ii} = 1$, $q_{ji} = r^2/q_{ij}$, and $q_{ij} = r^2/q_{ij'} = r^2/q_{i'j} = q_{i'j'}$. These relations basically imply that q_{ij} with $i < j \leq n$ determine all the deformation parameters. It should be noted that when all the independent deformation parameters are set equal to each other ($= q$), then the R-matrix reduces to the usual one parametric version. In the case of SO(5), the multiparametric R-matrix has only two independent parameters, which we will call r and q.

We extensively used a Mathematica package called NCALGEBRA (version 3.7)[22] to do the computations since the matrix elements (being generators of an algebra) are not commuting objects. The first task is to obtain the duality relations between the matrix elements explicitly. The L-matrices are chosen to be upper and lower triangular. The task is straightforward but tedious because the duality relations are 25×25 matrix relations for the case of SO(5). So, one has to scan through the resulting output to filter out the relations that are dual to the relations between the Chevalley-Cartan-Weyl generators. Doing the calculation for the single-parameter case will give a hint about which relations are relevant in writing down the algebra.

The first line of the isomorphism relations (for the specific case of SO(5)) implies that we can use k_1, k_2, 1, k_2^{-1}, and k_1^{-1} instead of l_{11}, l_{22}, l_{33}, l_{44}, and l_{55}, respectively. With this caveat, the algebra looks like what follows in terms of the relevant L-matrix elements:

$$
\begin{gathered}
k_1k_2 = k_2k_1, \\
k_1 l_{12}^{+} = \frac{r}{q^2} l_{12}^{+} k_1, \qquad k_2 l_{12}^{+} = \frac{q^2}{r} l_{12}^{+} k_2, \\
k_1 l_{23}^{+} = \frac{q}{r} l_{23}^{+} k_1, \qquad k_2 l_{23}^{+} = q^{-1} l_{23}^{+} k_2, \\
k_1 l_{21}^{-} = r l_{21}^{-} k_1, \qquad k_2 l_{21}^{-} = \frac{r}{q^2} l_{21}^{-} k_2, \\
k_1 l_{32}^{-} = \frac{q}{r} l_{32}^{-} k_1, \qquad k_2 l_{32}^{-} = q l_{32}^{-} k_2, \\
\left[l_{45}^{+}, l_{21}^{-} \right] = \left(q - q^{-1}\right)\left(k_1^{-2} - k_2^{-2}\right), \\
\left[l_{23}^{+}, l_{32}^{-} \right] = \left(q - q^{-1}\right)\left(k_2 - k_2^{-1}\right), \\
{l_{12}^{+}}^{2} l_{23}^{+} - \left(\frac{q}{r} + \frac{r}{q^3}\right) l_{12}^{+} l_{23}^{+} l_{12}^{+} + \frac{1}{q^2} l_{23}^{+} {l_{12}^{+}}^{2} = 0, \\
\frac{q^2}{r^2} {l_{23}^{+}}^{3} - \left(\frac{q^2}{r} + \frac{q^5}{r^3} + \frac{q}{r}\right) {l_{23}^{+}}^{2} l_{12}^{+} l_{23}^{+} + \left(q + \frac{q^4}{r^2} + \frac{q^5}{r^2}\right) l_{23}^{+} l_{12}^{+} {l_{23}^{+}}^{2} - \frac{q^4}{r} l_{12}^{+} {l_{23}^{+}}^{3} = 0.
\end{gathered}
\tag{3.2}
$$

The last two equations correspond to the Serre relations (we write them down only for the L^{+} matrix elements). As an example of the general procedure for obtaining these algebra relations from the duality relations (i.e., the Mathematica output), we will demonstrate the derivation of the first Serre relation. The relevant expressions that one gets from Mathematica are

$$
\begin{gathered}
l_{12}^{+} l_{23}^{+} - \frac{q}{r} l_{23}^{+} l_{12}^{+} = -\left(q - q^{-1}\right) l_{13}^{+} k_2, \\
l_{12}^{+} l_{13}^{+} = \frac{1}{q} l_{13}^{+} l_{12}^{+}.
\end{gathered}
\tag{3.3}
$$

Solving for l_{13}^{+} from the first equation by multiplying by k_2^{-1} on the right, plugging it back into the second equation, and using the commutation rules for k_2, we get our Serre relation. This kind of manipulation is fairly typical in the derivation of the above algebra.

As a next step, we use the isomorphism relations defined at the end of the last section to rewrite the above algebra in terms of the Chevalley-Cartan-Weyl-type generators. The result is

$$
\begin{gathered}
k_1k_2 = k_2k_1, \\
k_1 E_1 = \frac{r}{q^2} E_1 k_1, \qquad k_2 E_1 = \frac{q^2}{r} E_1 k_2, \\
k_1 E_2 = \frac{q}{r} E_2 k_1, \qquad k_2 E_2 = \frac{1}{q} E_2 k_2,
\end{gathered}
$$

$$\begin{aligned}
&k_1F_1 = rF_1k_1, \quad k_2F_1 = \frac{r}{q^2}F_1k_2,\\
&k_1F_2 = \frac{q}{r}F_2k_1, \quad k_2F_2 = qF_2k_2,\\
&\frac{q}{r}E_1F_1 - \frac{r}{q}F_1E_1 = \frac{k_2k_1^{-1} - k_2^{-1}k_1}{q-q^{-1}},\\
&[E_2,F_2] = \frac{k_2^{-1} - k_2}{q^{1/2} - q^{-1/2}},\\
&E_1^2E_2 - \left(\frac{q^2}{r} + \frac{r}{q^2}\right)E_1E_2E_1 + E_2E_1^2 = 0,\\
&E_2^3E_1 - \left(\frac{r}{q} + \frac{q^2}{r} + \frac{r}{q^2}\right)E_2^2E_1E_2 + \left(\frac{r^2}{q^3} + 1 + q\right)E_2E_2E_2^2 - \frac{r}{q}E_1E_2^3 = 0.
\end{aligned} \tag{3.4}$$

This is our final form for the multiparametric version of SO(5) Drinfeld-Jimbo algebra. Together with the Hopf algebra relations from (2.6), these relations complete our definition of the multiparametric algebra. Notice that they reduce to the one-parameter DJ algebra of SO(5) in the limit of $r \to q$.

4. Results and outlook

We have constructed the multiparametric version of the Drinfeld-Jimbo algebra for the case of SO(5) with the intention of investigating possible applications in de Sitter quantum mechanics and quantum gravity. As physicists, we are more interested in working with the algebra directly than working with the groups and the R-matrix because, presumably, finding representations of the algebra would be more direct (even though still nontrivial). Finding representations is interesting because that could be a first step in embedding the Hilbert space of the static patch of an observer, in the Hilbert space of the full de Sitter space. It might be the case that embedding the $SO(3)_q$ of the observer is easier to accomplish, in the added luxury of two parameters. Also, if it turns out that this embedding is possible only when there is a relationship between the parameters, it could translate into a statement about the surprising smallness of the cosmological statement in terms of scales which are more readily accessible to the observer. Of course, at this stage, this is pure speculation. The bottom line is that it seems like there is the exciting possibility of addressing the problem of the smallness of the positive cosmological constant using the multiparametric deformation.[2] Some of these issues are currently being investigated.

It is also interesting as a pure mathematical problem to write down the multiparametric DJ algebra for a generic Lie algebra. To the best of our knowledge, this is still an open problem.

[2] We thank Willy Fischler for suggesting this to us.

Appendix

In this appendix, we want to give an elementary demonstration that the boosts in SO(4, 1) correspond to time translations for the static observer. The metric for the static patch is

$$ds^2 = -(1 - r^2)dt^2 + \frac{dr^2}{1 - r^2} + r^2 d\Omega_2^2. \tag{A.1}$$

We take $\Lambda/3 = 1$, where Λ is the cosmological constant.

The easiest way to think about the de Sitter isometry group (SO(4, 1)) is to think of it as being embedded in a 5D Minkowski space. In terms of these Minkowski coordinates, the static patch can be written as

$$\begin{aligned} X^0 &= \sqrt{1 - r^2} \sinh t, \\ X^3 &= r \cos\theta, \\ X^1 &= r \sin\theta \cos\phi, \\ X^2 &= r \sin\theta \sin\phi, \\ X^4 &= \sqrt{1 - r^2} \cosh t. \end{aligned} \tag{A.2}$$

It is easy to check that $-(X^0)^2 + (X^i)^2 = 1$ and that $-dX^{0^2} + dX^{i^2}$ is equal to the metric on the static patch. Boosts in SO(4, 1) look like

$$\begin{pmatrix} X^{0'} \\ X^{4'} \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} X^0 \\ X^4 \end{pmatrix}. \tag{A.3}$$

Plugging into the expressions for X^0 and X^4 in terms of r and t, multiplying out the matrices, and simplifying them, we end up with

$$\begin{pmatrix} X^{0'} \\ X^{4'} \end{pmatrix} = \begin{pmatrix} \sqrt{1 - r^2} \sinh(t + \beta) \\ \sqrt{1 - r^2} \cosh(t + \beta) \end{pmatrix}, \tag{A.4}$$

which is just the time-translated version of the original expressions.

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