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Research Article

Fractional Heat Conduction Models and Thermal Diffusivity Determination

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The contribution deals with the fractional heat conduction models and their use for determining thermal diffusivity. A brief historical overview of the authors who have dealt with the heat conduction equation is described in the introduction of the paper. The one-dimensional heat conduction models with using integer- and fractional-order derivatives are listed. Analytical and numerical methods of solution of the heat conduction models with using integer- and fractional-order derivatives are described. Individual methods have been implemented in MATLAB and the examples of simulations are listed. The proposal and experimental verification of the methods for determining thermal diffusivity using half-order derivative of temperature by time are listed at the conclusion of the paper.

1. Introduction

Unsteady heat conduction process, described by partial differential equation, was first formulated by Jean Baptiste Joseph Fourier (1768–1830). In 1807, he wrote an article "Partial differential equation for heat conduction in solids." The issue of heat conduction was addressed by other scientists as well, such as Adolf Fick (1829–1901) [1, 2], James Clerk Maxwell (1831–1879) [3–5], Albert Einstein (1879–1955) [6], Lorenzo Richards (1904–1993) [7], and Geoffrey Taylor (1886–1975) [8, 9].

The various analytical and numerical methods are used to solve the Fourier heat conduction equation (FHCE) [10, 11]. In the case of heat conduction in materials with nonstandard structure, such as polymers, granular and porous materials, and composite materials, a standard description is insufficient and required the creation of more adequate models with using derivatives of fractional-order [12–15]. The causes are mainly memory systems and ongoing processes [16–20], roughness, or porosity of the material [21–23] and also fractality and chaotic behavior of systems [24–28].

The more adequate models of processes subsequently require new methods to determine the parameters of these models. In the case of FHCE, the basic parameter of this equation is thermal diffusivity, which characterizes the dynamics of temperature changes in the substance. Measurement of thermal diffusivity can be realized by many ways. The latest methods for determining thermal diffusivity are mainly *laser flash method* [29, 30], *Kennedy transient heat flow method* [31–33], *single rectangular pulse heating method* [34], and *thermal wave method* [35, 36].

The issue of research and development methods and tools for processes modeling with using fractional-order derivatives is very actual, since it means a qualitatively new level of modeling. Important authors of the first articles were Fourier (1768–1830), Abel, Leibniz (1646–1716), Grünwald (1838–1920), and Letnikov (1837–1888). Mathematicians like Liouville (1809–1882) [37, 38] and Riemann (1826–1866) [39] made major contributions to the theory of fractional calculus. Nowadays the fractional calculus interests many scientists and engineers from different fields, such as mechanics, physics, chemistry, and control theory [40].

At the present time, there are a number of analytical [41–48] and numerical solutions of fractional heat conduction equation. In the case of numerical methods different methods are developed based on the random walk models [49–52], the finite difference method (FDM) [53–55], the finite element method [56–59], numerical quadrature [60–62], the method of Adomian decomposition [63, 64], Monte Carlo simulation [65, 66], matrix approach [12, 13, 67], or the matrix transform method [68, 69]. The finite difference method is an extended method where an explicit [53, 70, 71], an implicit [54, 72–74], and a Crank-Nicolson scheme [55, 75] are used. For the Crank-Nicolson scheme, the literature describes the use of Grünwald-Letnikov definition only for a spatial derivative [73, 76–78].

2. Models of Heat Conduction Processes

Heat conduction is a molecular transfer of thermal energy in solids, liquids, and gases due to the temperature difference. The process of heat conduction takes place between the particles of the substance to touch directly each other and has different temperature. Existing models of heat conduction processes are divided according to various criterions. We consider a division of models into two groups, namely models with using derivatives of integer- and fractional-order.

Models with using derivatives of integer order are the nonstationary and stationary models. Nonstationary models are described by Fourier heat conduction equation, where the temperature T (K) is a function of spatial coordinate x (m) and time τ (s). In the case of one-dimensional heat conduction, it has the following form [11, 79]:

$$\frac{\partial T(x,\tau)}{\partial \tau} = \left(\sqrt{a}\right)^2 \frac{\partial^2 T(x,\tau)}{\partial x^2} \quad \text{for } 0 < x < L,\ \tau > 0, \tag{1}$$

$$\begin{gathered} T(0,\tau) = T_1, \quad T(L,\tau) = T_2 \quad \text{for } \tau > 0, \\ T(x,0) = f(x) \quad \text{for } 0 \le x \le L, \end{gathered} \tag{2}$$

where $a = \lambda/(\rho c_p)$ is thermal diffusivity ($\mathrm{m^2 \cdot s^{-1}}$), ρ is density ($\mathrm{kg \cdot m^{-3}}$), c_p is specific heat capacity ($\mathrm{J \cdot kg^{-1} \cdot K^{-1}}$), and λ is thermal conductivity ($\mathrm{W \cdot m^{-1} \cdot K^{-1}}$).

Heat conduction model with using derivatives of fractional-order for various one-dimensional geometric cases was expressed by the following Oldham-Spanier equation [22]:

$$\frac{\partial T(x,\tau)}{\partial x} + \frac{1}{\sqrt{a}} \frac{\partial^{1/2}\left[T(x,\tau) - T_0\right]}{\partial \tau^{1/2}} + \frac{g\left[T(x,\tau) - T_0\right]}{x + R} = 0, \tag{3}$$

where g is a geometric factor and R is a radius of curvature.

In the case of one-dimensional heat conduction planar wall ($g = 0$), (3) will take the following form:

$$\frac{\partial^{1/2}\left[T(x,\tau) - T_0\right]}{\partial \tau^{1/2}} = -\sqrt{a} \frac{\partial T(x,\tau)}{\partial x}. \tag{4}$$

A more general formulation of the task for modeling not only one-dimensional heat conduction is based on the model in which, on the left-hand side of (1) instead of the first derivative with respect to time, the derivative of order α occurs; that is, we can find it in the form [42, 79]

$$\frac{\partial^\alpha u}{\partial \tau^\alpha} = (b)^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < L,\ \tau > 0, \tag{5}$$

$$\begin{gathered} u(0,\tau) = U_1, \quad u(L,\tau) = U_2 \quad \text{for } \tau > 0, \\ u(x,0) = f(x) \quad \text{for } 0 \le x \le L, \end{gathered} \tag{6}$$

where b represents a constant coefficient with the unit $\mathrm{m \cdot s^{-\alpha/2}}$.

Fractional-order models can also be described by the following equation, where α and β are of arbitrary order [12, 13, 67]:

$$\frac{\partial^\alpha T(x,\tau)}{\partial \tau^\alpha} = \left(\sqrt{a}\right)^2 \frac{\partial^\beta T(x,\tau)}{\partial |x|^\beta}. \tag{7}$$

3. Solutions

One-dimensional heat conduction models using integer and fractional derivatives can be solved by analytical and numerical methods.

3.1. Analytical Methods of Solution. Analytical methods can be used for solving problems in a bounded, semibounded, or unbounded interval.

Analytical solution of heat conduction model (1) for a bounded interval $\langle 0, L\rangle$ has the following shape [11, 42]:

$$T(x,\tau) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi\sqrt{a}}{L}\right)^2 \tau\right) c_k. \tag{8}$$

Analytical solution for a fractional diffusion-wave equation (5) has the form

$$T(x,\tau) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) E_\alpha\left(-\left(\frac{n\pi b}{L}\right)^2 \tau^\alpha\right) c_k. \tag{9}$$

For models (1) and (5),

$$c_k = \frac{2}{L}\int_0^L \left[f(\xi) - \frac{1}{L}(T_2 - T_1)\xi - T_1\right] \sin\frac{k\pi\xi}{L} d\xi. \tag{10}$$

We developed and derived the coefficient c_k for the form of the function $f(\xi) = a_0 + a_1\xi + a_2\xi^2$, in order to implement simulations for different initial conditions (constant, straight line, and parabola). Coefficient c_k has this final shape [79]:

$$\begin{aligned} c_k = \frac{2}{nL}\Bigg[&\left[T_2 - a_0 - a_1 L + a_2\left(\frac{2}{n^2} - L^2\right)\right](-1)^k \\ &- T_1 + a_0 - a_2\frac{2}{n^2}, \end{aligned} \tag{11}$$

where $n = k\pi/L$.

3.2. Numerical Methods of Solution. The best known numerical methods include finite element method, finite difference method, and boundary element methods.

Finite difference methods according to the type of differential expression can be divided into explicit, implicit, and Crank-Nicolson scheme.

Explicit Scheme. Explicit scheme for solving the heat conduction model defined by (1) in the case of homogeneous material has the form

$$T_{m,p} = MT_{m-1,p-1} + T_{m,p-1} - 2MT_{m,p-1} + MT_{m+1,p-1}, \quad (12)$$

where module M is determined by the relation

$$M = \left(\frac{\sqrt{a}}{\Delta x}\right)^2 \quad \Delta\tau \leq 0,5, \quad (13)$$

and in the case of nonhomogeneous material, it has the following form:

$$\begin{aligned} T_{m,p} &= M_{m-1}T_{m-1,p-1} + T_{m,p-1} \\ &\quad - (M_{m-1} + M_m) T_{m,p-1} + M_m T_{m+1,p-1}, \end{aligned} \quad (14)$$

where module M_m is

$$M_m = \left(\frac{\sqrt{a_m}}{\Delta x}\right)^2 \quad \Delta\tau \leq 0,5. \quad (15)$$

Implicit Scheme. In the case of the implicit scheme, the temperature at a given point is calculated for a homogeneous body according to the following formula:

$$-MT_{m-1,p} + (1 + 2M) T_{m,p} - MT_{m+1,p} = T_{m,p-1}, \quad (16)$$

and for a nonhomogeneous body, it has the following formula:

$$\begin{aligned} &- M_{m-1}T_{m-1,p} + (1 + M_{m-1} + M_m) T_{m,p} \\ &- M_m T_{m+1,p} = T_{m,p-1}. \end{aligned} \quad (17)$$

Crank-Nicolson Scheme. For a homogeneous body, it has the form

$$\begin{aligned} T_{m,p} &= \frac{M}{2}\left(T_{m-1,p} - 2T_{m,p} + T_{m+1,p}\right) \\ &\quad + \frac{M}{2}\left(T_{m-1,p-1} - 2T_{m,p-1} + T_{m+1,p-1}\right) + T_{m,p-1} \end{aligned} \quad (18)$$

and for a nonhomogeneous body it has the form

$$\begin{aligned} T_{m,p} &= \frac{1}{2}\left(M_{m-1}T_{m-1,p} - (M_{m-1} + M_m) T_{m,p} \right. \\ &\qquad \left. + M_m T_{m+1,p}\right) \\ &\quad + \frac{1}{2}\left(M_{m-1}T_{m-1,p-1} - (M_{m-1} + M_m) T_{m,p-1} \right. \\ &\qquad \left. + M_m T_{m+1,p-1}\right) \\ &\quad + T_{m,p-1}. \end{aligned} \quad (19)$$

Numerical Methods of Fractional-Order. For solving numerical methods of fractional-order, we use Grünwald-Letnikov definition with using the principle of "short memory" [16, 80]:

$$\frac{\partial^\alpha T(x,\tau)}{\partial\tau^\alpha} = \frac{\sum_{j=0}^{N_f} bc_j T(x, \tau - j\Delta\tau)}{\Delta\tau^\alpha}, \quad (20)$$

where L is the "length memory," τ is the time step, and the value of $N(f)$ will be determined by the following relation:

$$\begin{gathered} N(f) = \min\left\{\left[\frac{\tau}{\Delta\tau}\right], \left[\frac{L}{\Delta\tau}\right]\right\}, \\ bc_0 = 1, \qquad bc_j = \left(1 - \frac{1+\alpha}{j}\right) \cdot bc_{j-1}, \\ \text{where } j \geq 1. \end{gathered} \quad (21)$$

Explicit Scheme. Explicit scheme for the heat conduction model using derivative of *fractional-order* (5) for a homogeneous material has the form

$$\begin{aligned} T_{m,p} &= MT_{m-1,p-1} - \sum_{j=1}^{N_f} bc_j T_{m,p-j} \\ &\quad - 2MT_{m,p-1} + MT_{m+1,p-1} \end{aligned} \quad (22)$$

and for a nonhomogeneous material it has the form

$$\begin{aligned} T_{m,p} &= M_{m-1}T_{m-1,p-1} - \sum_{j=1}^{N_f} bc_j T_{m,p-j} \\ &\quad - (M_{m-1} + M_m) T_{m,p-1} + M_m T_{m+1,p-1}. \end{aligned} \quad (23)$$

Implicit Scheme. Fractional shape for a homogeneous body is given by the following relation:

$$\begin{aligned} &- MT_{m-1,p} + (1 + 2M) T_{m,p} - MT_{m+1,p} \\ &= -\sum_{j=1}^{N_f} bc_j T_{m,p-j} \end{aligned} \quad (24)$$

and for a nonhomogeneous body, it has the following relation:

$$\begin{aligned} &- M_{m-1}T_{m-1,p} + (1 + M_{m-1} + M_m) T_{m,p} \\ &- M_m T_{m+1,p} = -\sum_{j=1}^{N_f} bc_j T_{m,p-j}. \end{aligned} \quad (25)$$

Crank-Nicolson Scheme. The fractional shape for a homogeneous body has the form

$$\begin{aligned} T_{m,p} &= \frac{M}{2}\left(T_{m-1,p} - 2T_{m,p} + T_{m+1,p}\right) \\ &\quad + \frac{M}{2}\left(T_{m-1,p-1} - 2T_{m,p-1} + T_{m+1,p-1}\right) \\ &\quad - \sum_{j=1}^{N_f} bc_j T_{m,p-j} \end{aligned} \quad (26)$$

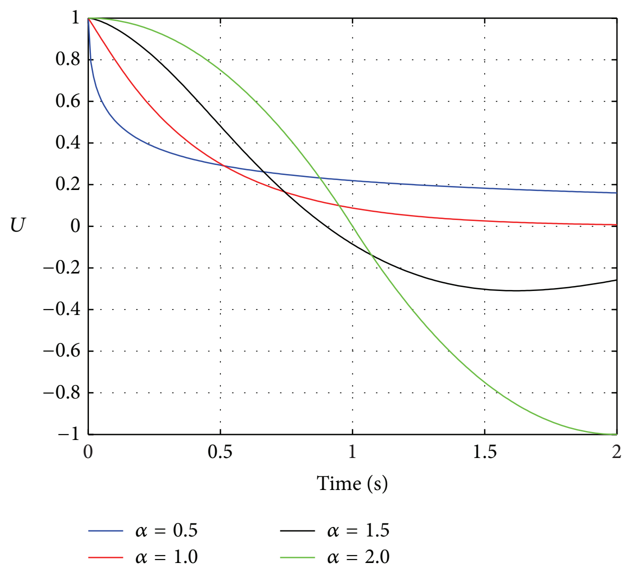

FIGURE 1: Comparison of analytical method for four different derivatives of temperature according to the time.

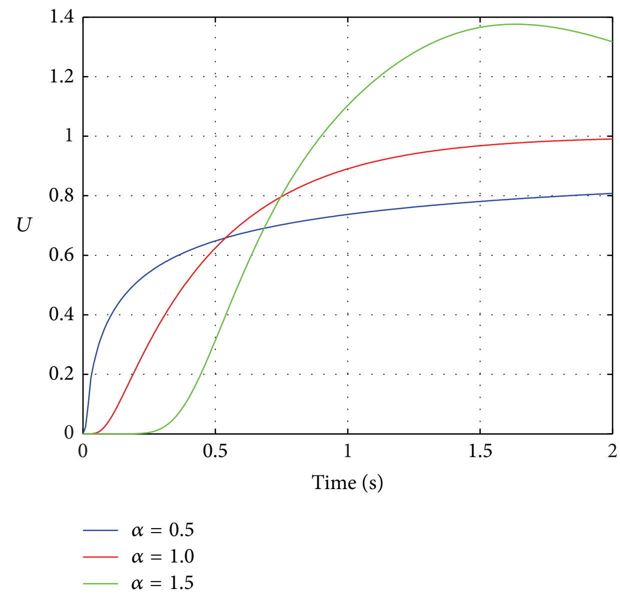

FIGURE 2: Crank-Nicolson scheme for derivatives of 0.5, 1, and 1.5.

and for a nonhomogeneous body it has the form

$$\begin{aligned} T_{m,p} = {} & \frac{1}{2}\left(M_{m-1}T_{m-1,p} - \left(M_{m-1}+M_m\right)T_{m,p} + M_m T_{m+1,p}\right) \\ & + \frac{1}{2}\left(M_{m-1}T_{m-1,p-1} - \left(M_{m-1}+M_m\right)T_{m,p-1} + M_m T_{m+1,p-1}\right) \\ & - \sum_{j=1}^{N_f} bc_j T_{m,p-j}. \end{aligned} \tag{27}$$

4. Simulations

Implementation of the one-dimensional heat conduction model was realized in the programming environment MATLAB. Two toolboxes for the one-dimensional heat conduction model with using integer- and fractional-order derivatives have been created. All implemented functions are published at Mathworks, Inc., MATLAB Central File Exchange as Heat Conduction Toolbox and Fractional Heat Conduction Toolbox [81, 82].

Simulations of heat conduction model for analytical solution have been implemented for four different derivatives temperatures according to time, namely, for the derivative order of 0.5, 1, 1.5, and 2 (Figure 1). The model input parameters were set as follows: initial temperature in the shape of parabolic function $f(x) = 2x - x^2$, boundary condition of the 1st kind for $U_1 = U_2 = 0$, total time simulation 2 s, time step 0.01 s, number of items' sum 100, distance 2 m, number of points 21, and coefficient for material properties $1\,\mathrm{m{\cdot}s}^{-\alpha/2}$.

Simulation with a heat conduction model for explicit, implicit, and Crank-Nicolson scheme was performed with the time step 0.01 s and order of the derivative of 1.5. Input parameters of the model were chosen as follows: initial value $U(x) = 0$, boundary condition of the 1st kind for $U_1 = U_2 = 1$, total time simulation 2 s, time step 0.01 s, number of items' sum 100, distance 2 m, number of points 21, and coefficient for material properties $1\,\mathrm{m{\cdot}s}^{-3/4}$.

From the numerical methods, we have chosen Crank-Nicolson scheme, in which we can see what effect a different order of the derivative has on the temperatures course (Figure 2).

In Figure 3, we see the comparison of courses of individual numerical methods and analytical solution.

5. Proposal Method for Thermal Diffusivity Determination

The method is based on the method of calculation of heat flows:

$$i_Q = \sqrt{c_p \rho \lambda}\cdot{}_0 D_\tau^{1/2} g(\tau), \quad g(\tau) = T_w(\tau) - T_0. \tag{28}$$

Determination of the heat flow i_Q is possible in two ways: namely,

(i) from the gradient of the two measured temperatures (T_1, T_2),

$$i_Q = -\lambda \frac{d}{dx} T_1(\tau), \tag{29}$$

(ii) from the half-order derivative of one measured temperature (T_1),

$$i_Q = \frac{\lambda}{\sqrt{a}} \frac{d^{1/2}}{d\tau^{1/2}} \left[T_1(\tau) - T_0\right]. \tag{30}$$

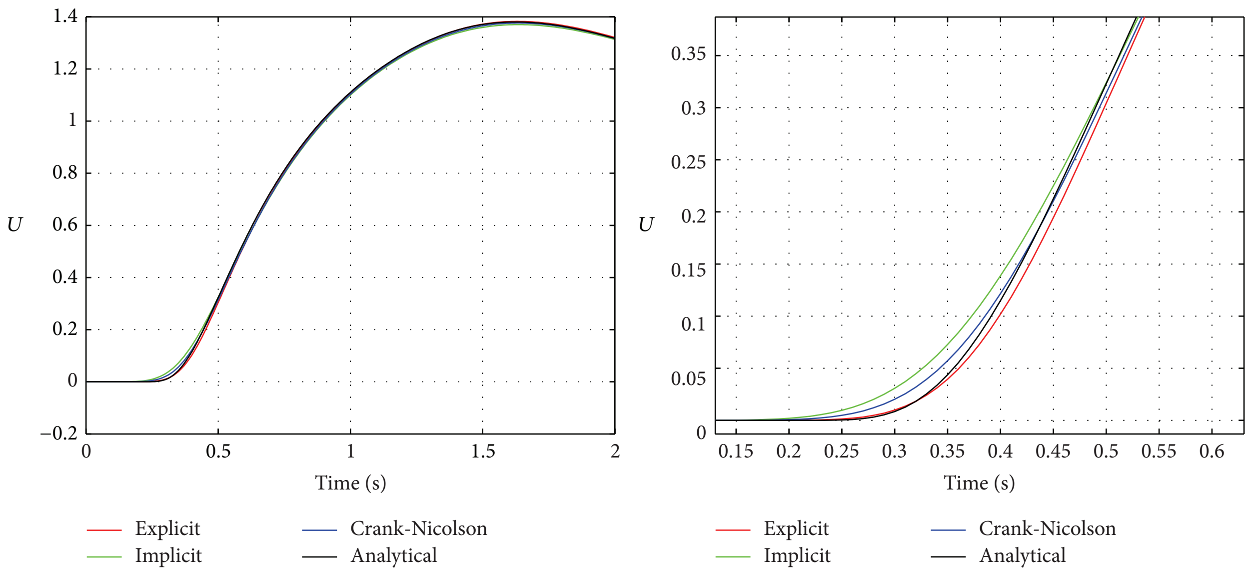

Figure 3: Comparison of analytical solution and numerical methods for the derivative of 1.5.

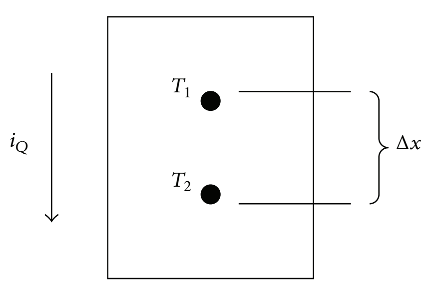

Figure 4: Measured temperatures.

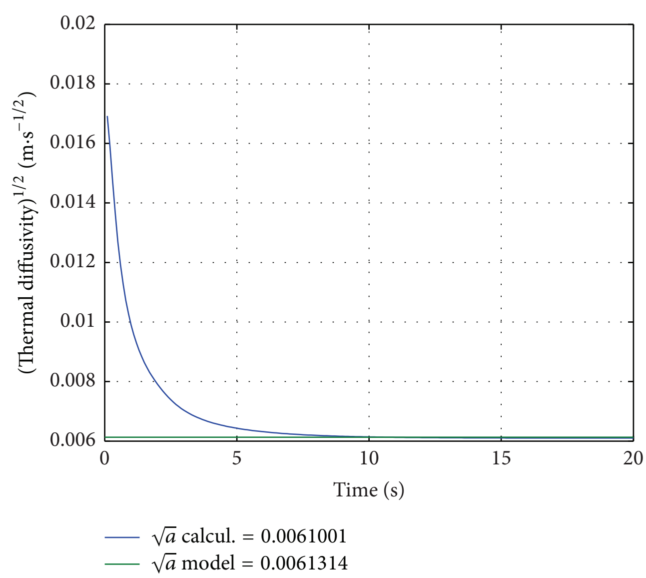

Figure 5: Rate of half-order temperature derivative to the temperature gradient.

Share of half-order derivative and gradient of temperature is proportional to the square root of the thermal diffusivity:

$$\sqrt{a} = \frac{\left(d^{1/2}/d\tau^{1/2}\right)\left[T_1(\tau) - T_0\right]}{-\left(d/dx\right)T_1(\tau)}. \tag{31}$$

The differential form of (31) is shown in the shape

$$\sqrt{a} = \frac{\Delta\tau^{-1/2}\sum_{j=0}^{N(f)} bc_j \left[T_{1,p-j} - T_0\right]}{\Delta x^{-1}\left[T_{2,p} - T_{1,p}\right]}. \tag{32}$$

For the numerical calculation of the first derivative of temperature according to the coordinate, respectively, temperature gradient (31) is sufficient to measure two temperatures (Figure 4).

The calculation of thermal diffusivity is based on the ratio half-order derivative of temperature according to the time to the temperature gradient (Figure 5) which is observed based on the values of two neighbouring temperatures in space obtained from simulations.

More previous values of temperature in time are used for the calculation of the half-order derivative, as in the case of the first derivative, which uses only one previous value [83].

The method was tested on the model using Crank-Nicolson scheme on a brass sample. The value of thermal diffusivity for a brass is $3.7594 \times 10^{-5}\,\mathrm{m^2 \cdot s^{-1}}$. The initial temperature of simulation was determined on 20°C, boundary condition of the 1st kind for 20°C and 100°C, with a time step of the simulation 0.01 s. Input parameters of the brass: density 8,400 $\mathrm{kg \cdot m^{-3}}$, specific heat capacity 380 $\mathrm{J \cdot kg^{-1} \cdot K^{-1}}$, and thermal conductivity 120 $\mathrm{W \cdot m^{-1} \cdot K^{-1}}$.

In Figure 6, we can see the effect of time step to calculate the square root of thermal diffusivity.

The calculation accuracy of determining the value of the square root of thermal diffusivity depends on the number of

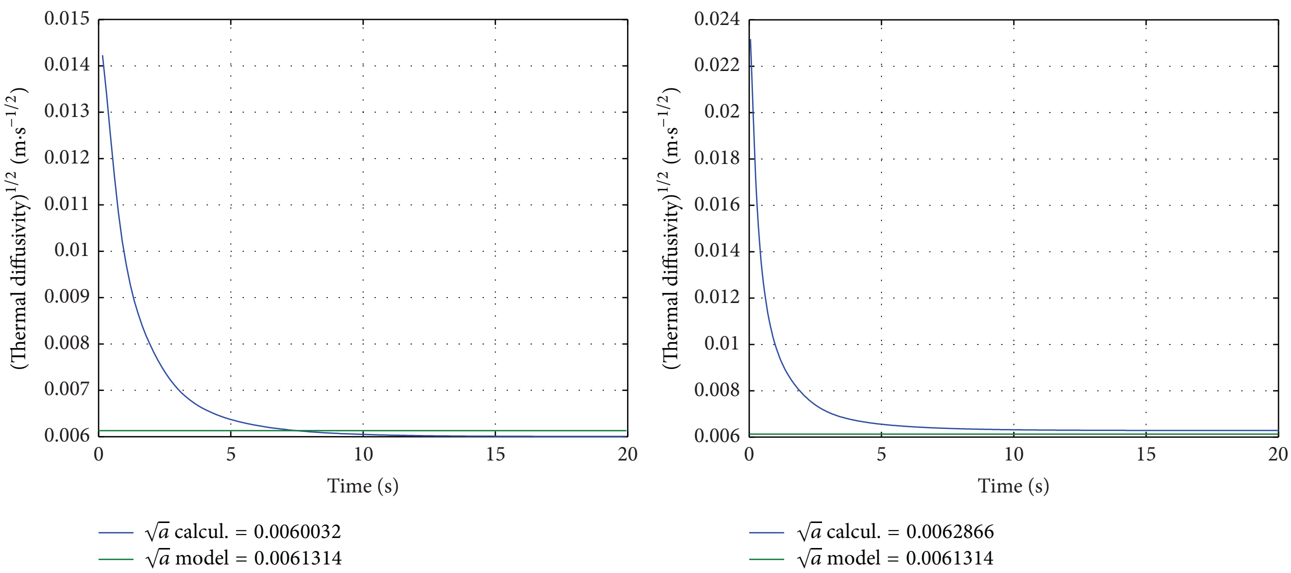

Figure 6: Crank-Nicolson scheme for a time step of 0.15 s and 0.05 s.

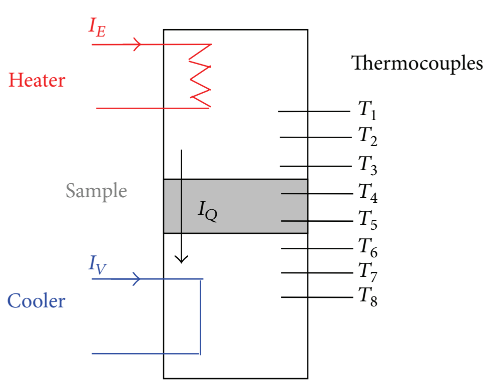

Figure 7: Scheme of HT11C.

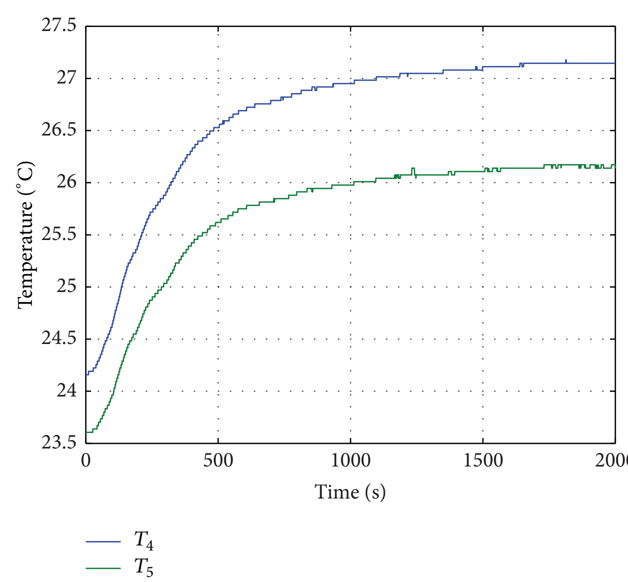

Figure 8: Experimental measurements of temperatures T_4 and T_5 for the brass.

previous values of temperatures in time and also from the selected time step.

Reducing the number of previous values of temperatures leads to higher inaccuracy of calculation.

6. Experimental Verification

The method has been verified on the experimental measurements. Measurements were carried out on the devices HT10XC and HT11C. Module HT11C is a physical model of one-dimensional heat conduction [84]. It consists of a heating and cooling section between which is inserted the sample of material (Figure 7).

Brass sample was used in the form of a cylinder with a diameter of 25 mm and a height of 30 mm. Contact areas of the sample were coated with a thin layer of thermal paste to minimize the transient thermal resistance. Module HT11C uses the thermocouples of type K in the temperature range from 0 to 133°C and the distance among them is 15 mm. The device HT10XC with HT11C module is connected via USB to a PC. The software that comes with the device allows setting conditions of the experiment and the measurement data saving to a file.

Experimental measurements which are referred to in this paper were carried out under the following conditions: namely, heater power 1.3 W, the water flow in the cooler 0.5 L/min, and the time step for recording of measured data 1.0 s. A unit jump in the heater power from 1.3 to 3.3 W was realized after stabilizing the temperatures. The transition from one steady state to another is shown in Figure 8.

On Figure 9 is determined the square root of thermal diffusivity from the measured values of the device HT11C

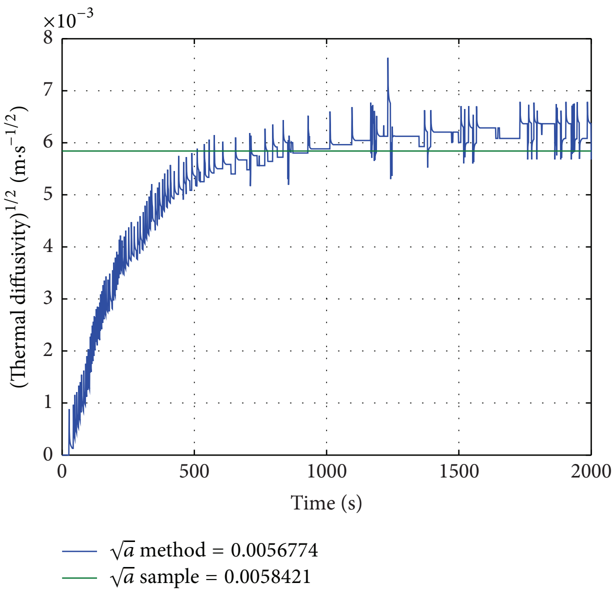

FIGURE 9: Thermal diffusivity.

(Figure 8), that is, from the ratio of half-order derivative the temperature according to the time to the temperature gradient.

The value of thermal diffusivity of the used brass sample for equipment HT11C was $3.2233 \times 10^{-5}\,\mathrm{m}^2{\cdot}\mathrm{s}^{-1}$ and it corresponds to the square root of thermal diffusivity $0.0056774\,\mathrm{m}{\cdot}\mathrm{s}^{-1/2}$. Brass sample was also measured on the device LFA [85] and the value of thermal diffusivity was $3.4130 \times 10^{-5}\,\mathrm{m}^2{\cdot}\mathrm{s}^{-1}$, which corresponds to the square root of thermal diffusivity $0.0058421\,\mathrm{m}{\cdot}\mathrm{s}^{-1/2}$. Calculated relative error between the measured values of the thermal diffusivity of the brass sample on HT11C and LFA is 5.5591% [79].

7. Conclusion

Benefits of this work are mainly the developed analytical and numerical methods for solving one-dimensional heat conduction using integer and fractional derivatives, which are implemented in the form of libraries functions in MATLAB. Another benefit is the designed, implemented, and verified method of determining thermal diffusivity using the half-order derivative of temperature according to the time on the experimental equipment HT10XC with module HT11C.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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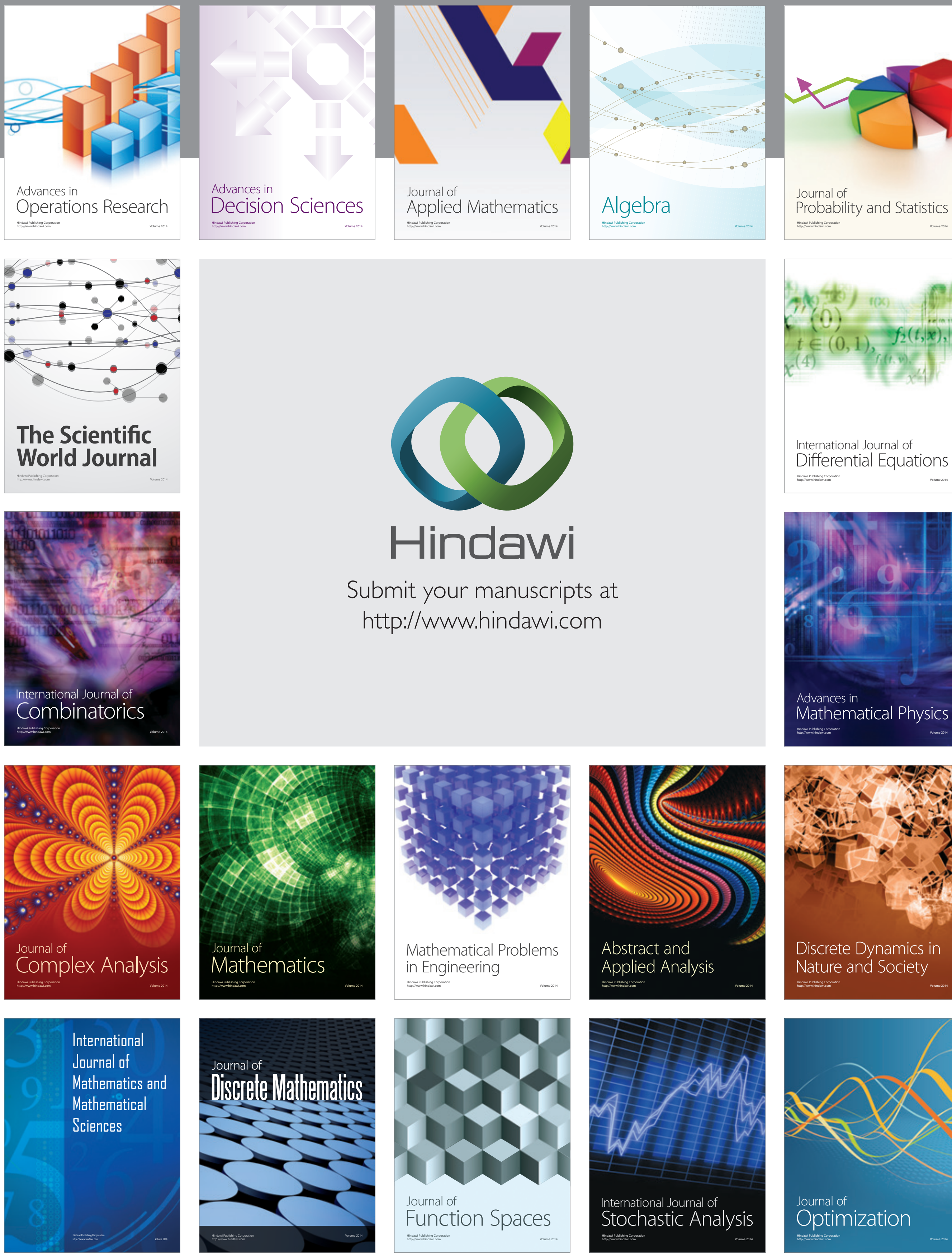
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