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Some fixed point theorems on non-convex sets

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Abstract

In this paper, we prove that if K is a nonempty weakly compact set in a Banach space X, $T : K \to K$ is a nonexpansive map satisfying $\frac{x+Tx}{2} \in K$ for all $x \in K$ and if X is $3-$uniformly convex or X has the Opial property, then T has a fixed point in K.

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1. Introduction

Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

The following theorem was proved independently by Browder [2] and Göhde [8] in the setting of uniformly convex Banach spaces.

Theorem 1.1 ([2]). *Let K be a nonempty weakly compact convex subset of a uniformly convex Banach space X and $T : K \to K$ be a nonexpansive map. Then T has a fixed point in K.*

Using the notion of normal structure, Kirk [10] proved the following theorem which is more general than Theorem 1.1.

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Theorem 1.2 ([10]). *Let K be a nonempty weakly compact convex subset having normal structure in a Banach space X and $T : K \to K$ be a nonexpansive map. Then T has a fixed point in K.*

The convexity assumption cannot be dispense in the above theorems as can be seen from the following simple example.

Let $K = [-2, -1] \cup [1, 2] \subseteq \mathbb{R}$ and T is a self map on K defined by $Tx = -x$ for all $x \in K$. Then T is nonexpansive, but T has no fixed points in K. This implies that nonexpansive map on a non-convex set in a Banach space need not have a fixed point.

Motivated by Theorem 1.1 and Theorem 1.2, Veeramani [20] introduced the notion of $T-$regular set as follows:

Let T be a self map on a nonempty subset K of a Banach space X. Then K is said to be a $T-$regular set if $\frac{x+Tx}{2} \in K$ for all $x \in K$.

Clearly, if K is a convex set and $T : K \to K$, then K is $T-$regular. But a $T-$regular set need not be a convex set(see Example 3.2). Further, Veeramani [20] proved the following fixed point theorem.

Theorem 1.3 ([20]). *Let K be a nonempty weakly compact subset of a uniformly convex Banach space X and $T : K \to K$ be a nonexpansive map. Further, assume that K is $T-$regular. Then T has a fixed point in K.*

Khan and Hussain [9] used the notion of $T-$regular sets to prove the existence of fixed points for nonexpansive mappings in the setting of metrizable topological vector space. Also, Goebel and Schöneberg [6] proved the existence of fixed point for a nonexpansive map on certain nonconvex sets in a Hilbert space.

Sullivan [18] introduced the concept of $k-$uniform convexity, $k-$UC in short, where k is any positive integer and proved that every $k-$uniformly convex Banach space has normal structure. Note that for $k = 1$, it is uniformly convex.

Sullivan [18] observed that every $k-$UC Banach space is a $(k + 1)-$UC. But the converse is not true. For example, the Banach space $l^{p,1}(\mathbb{N})$ [1] for $1 < p < \infty$ is $2-$UC but not $1-$UC where $l^{p,1}(\mathbb{N})$ is the $l^p(\mathbb{N})$ space with suitable renorm.

Motivated by Theorem 1.2, Theorem 1.3 and the fact that $k-$UC Banach spaces have normal structure [18], we raise the following question:

Does a nonexpansive map T on a nonempty weakly compact set K in a $k-$UC Banach space have a fixed point if $\frac{x+Tx}{2} \in K$ for all $x \in K$?

In this paper, we give an affirmative answer to the above question, if X is a $3-$UC Banach space. For the proof of this result, Lemma 3.3 and Lemma 3.4 (the geometric inequality on $k-$UC Banach space) are crucial.

In another direction, Opial [16] introduced a class of spaces for which the asymptotic center of a weakly convergent sequence coincides with the weak limit point of the sequence. Gossez and Lami Dozo [7] have observed that all such spaces have normal structure. Hence, in view of Kirk's theorem, every nonempty weakly compact convex set in a Banach space which satisfy

Opial's condition has fixed point property for a nonexpansive mapping. Recently, Suzuki [19] introduced a new class of mappings which also includes nonexpansive maps and proved that every nonempty weakly compact convex set in a Banach space which satisfy Opial's condition also has fixed point property for all such maps.

In this paper, we prove that if K is a nonempty weakly compact set in a Banach space X having the Opial property, $T : K \to K$ is a nonexpansive map and if K is $T-$regular set, then T has a fixed point point in K. Moreover, the Krasnoseleskii's [12] iterated sequence $\{x_n\}$ where $x_{n+1} = \frac{x_n+Tx_n}{2}$ for all $n \in \mathbb{N}$ and $x_1 \in K$ weakly converges to a fixed point.

2. Preliminaries

Now, we give some basic definitions and results which are used in this paper. Let X be a Banach space. For a nonempty subset A of X, let

$$\mathrm{co}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \lambda_i \geq 0, \text{ for } i = 1, 2, \ldots, n \text{ and } \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N} \right\}$$

$$\mathrm{aff}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \lambda_i \in \mathbb{R}, \text{ for } i = 1, 2, \ldots, n \text{ and } \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N} \right\}$$

The sets $\mathrm{co}(A)$ and $\mathrm{aff}(A)$ are called the convex hull and the affine hull of A respectively.

A set A is affine if $A = \mathrm{aff}(A)$. Every affine set is a translation of a subspace and the subspace is uniquely defined by the affine set. The dimension of an affine set is the dimension of the corresponding subspace. Further, the dimension of a convex set A is defined as the dimension of the smallest affine set which contains A. This shows that the dimension of $\mathrm{co}(A)$ is the dimension of $\mathrm{aff}(A)$.

Sliverman [17] introduced the notion of volume of $k+1$ vectors, denoted by $V(x_1, x_2, \ldots, x_{k+1})$, as follows:

Given $x_1, x_2, \ldots, x_{k+1} \in X$,

$$V(x_1, x_2, \ldots, x_{k+1}) = \frac{1}{k!} \sup \left\{ \begin{vmatrix} f_1(x_2 - x_1) & \ldots & f_1(x_{k+1} - x_1) \\ f_2(x_2 - x_1) & \ldots & f_2(x_{k+1} - x_1) \\ \vdots & \vdots & \vdots \\ f_k(x_2 - x_1) & \ldots & f_k(x_{k+1} - x_1) \end{vmatrix} : f_1, \ldots, f_k \in B_{X^*} \right\}$$

By the consequences of Hahn-Banach theorem, $V(x_1, x_2) = \|x_1 - x_2\|$ for any $x_1, x_2 \in X$. Note that $V(x_1, x_2, \ldots, x_{k+1}) = 0$ iff the dimension of the convex hull of $\{x_1, x_2, \ldots, x_{k+1}\}$ does not exceed $k - 1$.

Using the notion of volume of $k+1$ vectors, Sullivan [18] defined the concept of $k-$uniform convexity.

We put $\mu_X^{(k)} = \sup\{V(x_1, \ldots, x_{k+1}) : x_1, \ldots, x_{k+1} \in B_X\}$.

Definition 2.1 ([18]). The modulus of $k-$convexity is defined as

$$\delta_X^{(k)}(\epsilon) = \inf\left\{1 - \frac{1}{k+1}\left\|\sum_{i=1}^{k+1} x_i\right\| : x_1, \ldots, x_{k+1} \in B_X \text{ and } V(x_1, \ldots, x_{k+1}) \geq \epsilon\right\}$$

where $\epsilon \in [0, \mu_X^{(k)})$.

A Banach space X is said to be $k-$uniformly convex if $\delta_X^{(k)}(\epsilon) > 0$ for every $0 < \epsilon < \mu_X^{(k)}$.

Note that all Banach spaces of dimension less than $k+1$ are $k-$UC. For more information on $k-$UC, one can refer to [11, 14, 15].

Lim [13] proved the continuity of modulus $\delta_X^{(k)}$ of $k-$convexity using the following inequality.

Theorem 2.2 ([13]). *Let X be a Banach space and k be any positive integer. For every $0 < \epsilon_1 < c < \epsilon_2 < \mu_X^{(k)}$,*

$$\frac{\delta_X^{(k)}(c) - \delta_X^{(k)}(\epsilon_1)}{c - \epsilon_1} \leq \frac{1}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k})\epsilon_1^{1-1/k}}$$

Corollary 2.3 ([13]). *Let X be a Banach space. Then $\delta_X^{(k)}(\epsilon)$ is continuous on $[0, \mu_X^{(k)})$.*

Definition 2.4 ([16]). A Banach space X is said to have the Opial property if $\{x_n\}$ is a weakly convergent sequence in X with limit z, then

$$\liminf_{n\to\infty} \|x_n - z\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq z$.

It is known that [5] Hilbert spaces, finite dimensional Banach spaces and $l^p(\mathbb{N})$ $(1 < p < \infty)$ have the Opial property.

Edelstein [3] introduced the notion of asymptotic center as follows:

Definition 2.5 ([3]). Let K be a nonempty subset of a Banach space X and $\{x_n\}$ be a bounded sequence in X. For each $x \in X$, define $r(x) = \limsup_{n\to\infty} \|x - x_n\|$. The number $r = \inf_{x \in K} r(x)$ and the set $A(K, \{x_n\}) = \{x \in K : r(x) = r\}$ are called the asymptotic radius and asymptotic center of $\{x_n\}$ with respect to K respectively.

We use the next lemma in the sequel, which is proved by Goebel and Kirk [4].

Lemma 2.6 ([4]). *Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space X and let $\lambda \in (0,1)$. Suppose that $z_{n+1} = \lambda w_n + (1-\lambda) z_n$ and $\|w_{n+1} - w_n\| \leq \|z_{n+1} - z_n\|$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \|w_n - z_n\| = 0$.*

3. Main results

3.1. $3-$**UC Banach spaces.** In this section, we first give the convergence theorem for a nonexpansive map T defined on a compact $T-$regular set in a Banach space X. Also, we prove the existence of fixed points for a nonexpansive map T defined on a weakly compact $T-$regular set in a $3-$UC Banach space X.

Theorem 3.1. *Let K be a nonempty compact subset of a Banach space X and $T : K \to K$ be a nonexpansive map. Further, assume that K is $T-$regular. Define a sequence $\{x_n\}$ in K by $x_{n+1} = \frac{x_n+Tx_n}{2}$ for $n \in \mathbb{N}$ and $x_1 \in K$. Then T has a fixed point in K and $\{x_n\}$ strongly converges to a fixed point of T.*

Proof. Since $x_{n+1} = \frac{x_n+Tx_n}{2}$ for $n \in \mathbb{N}$, by Lemma 2.6, we have $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$.

Since K is compact and $\{x_n\} \subseteq K$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in K$ such that $\{x_{n_k}\}$ converges to z. Now, by the continuity of T, $\{Tx_{n_k}\}$ converges to Tz.

But, note that $\lim\limits_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$. Hence $\{x_{n_k}\}$ also converges to Tz. This implies that $Tz = z$.

Also, note that $\{\|x_n - z\|\}$ is a decreasing sequence. For,

$$\|x_{n+1} - z\| \leq \frac{1}{2}\|x_n - z\| + \frac{1}{2}\|Tx_n - z\| \leq \|x_n - z\|, \text{ for all } n \in \mathbb{N}$$

Therefore $\{x_n\}$ converges to z, as $\{x_{n_k}\}$ converges to z in norm. □

Example 3.2. Let $K = \{(x, 0, \frac{1}{2^n}), (0, y, \frac{1}{2^n}), (x, x, \frac{1}{2^n}), (x, 0, 0), (0, y, 0), (x, x, 0) : 0 \leq x, y \leq 1 \text{ and } n \in \mathbb{N}\}$ be a subset of $(\mathbb{R}^3, \|.\|_2)$. Define a map $T : K \to K$ by $T(x, y, z) = (y, x, 0)$ for all $(x, y, z) \in K$.

It is easy to see that K is $T-$regular. Also, note that T is nonexpansive. For, let $x = (x_1, y_1, z_1), y = (x_2, y_2, z_2) \in K$.

$$\begin{aligned} \text{Then } \|Tx - Ty\|_2 &= \|(y_1 - y_2, x_1 - x_2, 0)\|_2 \\ &\leq \|(x_1 - x_2, y_1 - y_2, z_1 - z_2)\|_2 = \|x - y\|_2 \end{aligned}$$

By Theorem 3.1, T has a fixed point in K, since K is compact and $T-$regular. Also, note that $\text{Fix}(T) = \{(x, x, 0) : 0 \leq x \leq 1\}$.

Lemma 3.3. *Let K be a nonempty weakly compact subset of a Banach space X and $T : K \to K$ be a nonexpansive map. Further, assume that K is $T-$regular. Define a sequence $\{x_n\}$ in K by $x_{n+1} = \frac{x_n+Tx_n}{2}$ for $n \in \mathbb{N}$ and $x_1 \in K$. Then the asymptotic center $A(K, \{x_n\})$ of $\{x_n\}$ with respect to K is also a nonempty weakly compact $T-$regular subset of K. Moreover, if K is a minimal weakly compact $T-$regular set, then $A(K, \{x_n\}) = K$.*

Proof. Since $r(x) = \limsup\limits_{n\to\infty} \|x - x_n\|$ is a weakly lower semicontinuous function on X and K is weakly compact, $A(K, \{x_n\}) = \{x \in K : r(x) = \inf\limits_{y\in K} r(y) = r\}$ is nonempty.

Also $\{x \in X : r(x) \leq \inf_{y \in K} r(y)\}$ is a weakly closed set, this implies that $A(K, \{x_n\}) = \{x \in X : r(x) \leq \inf_{y \in K} r(y)\} \cap K$ is a weakly closed set.

Moreover, since T is nonexpansive and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, $A(K, \{x_n\})$ is $T-$invariant.

Now, it is claimed that $A(K, \{x_n\})$ is a $T-$regular set.

Let $x \in A(K, \{x_n\})$. Then $Tx \in A(K, \{x_n\})$ and

$$\left\| \frac{x + Tx}{2} - x_n \right\| \leq \frac{1}{2}\|x - x_n\| + \frac{1}{2}\|Tx - x_n\|.$$

This implies that

$$\limsup_{n\to\infty} \left\| \frac{x + Tx}{2} - x_n \right\| = r.$$

Therefore $\frac{x+Tx}{2} \in A(K, \{x_n\})$. Hence $A(K, \{x_n\})$ is a nonempty weakly compact $T-$regular subset of K.

Suppose that K is a nonempty minimal weakly compact $T-$regular set. Then $A(K, \{x_n\}) = K$, as $A(K, \{x_n\}) \subseteq K$ is also a nonempty weakly compact $T-$regular set. □

Lemma 3.4. *Let X be a $k-$UC Banach space, for some $k \in \mathbb{N}$ and $x_1, x_2, \ldots, x_{k+1} \in B_X$ such that $V(x_1, x_2, \ldots, x_{k+1}) = \epsilon > 0$.*

Then $\|t_1x_1 + t_2x_2 + \cdots + t_{k+1}x_{k+1}\| \leq 1 - (k+1)\min\{t_1, t_2, \ldots, t_{k+1}\}\delta_X^{(k)}(\epsilon)$, where $\sum_{i=1}^{k+1} t_i = 1,\ t_i \geq 0$ for $i = 1, 2, \ldots, k+1$.

Proof. Without loss of generality, we can assume that $t_1 = \min\{t_1, t_2, \ldots, t_{k+1}\}$.

$$\begin{aligned}
\|t_1x_1 + t_2x_2 + \cdots + t_{k+1}x_{k+1}\| &= \|t_1(x_1 + \cdots + x_{k+1}) + (t_2 - t_1)x_2 + (t_3 - t_1)x_3 \\
&\quad + \cdots + (t_{k+1} - t_1)x_{k+1}\| \\
&\leq (k+1)t_1 \left\| \frac{x_1 + x_2 + \cdots + x_{k+1}}{k+1} \right\| + (t_2 - t_1)\|x_2\| \\
&\quad + (t_3 - t_1)\|x_3\| + \cdots + (t_{k+1} - t_1)\|x_{k+1}\| \\
&\leq (k+1)t_1(1 - \delta_X^{(k)}(\epsilon)) + t_2 + t_3 + \cdots + t_{k+1} - kt_1 \\
&= (k+1)t_1 - (k+1)t_1\delta_X^{(k)}(\epsilon) + 1 - (k+1)t_1 \\
&= 1 - (k+1)t_1\delta_X^{(k)}(\epsilon)
\end{aligned}$$

Hence $\|t_1x_1 + t_2x_2 + \cdots + t_{k+1}x_{k+1}\| \leq 1 - (k+1)\min\{t_1, t_2, \ldots, t_{k+1}\}\delta_X^{(k)}(\epsilon)$. □

Remark 3.5. Now from Lemma 3.4, we have:

(1) If $k = 2$ and $t_1 = t_2 = \frac{1}{4}$, then

$$\left\| \frac{x_1}{4} + \frac{x_2}{4} + \frac{x_3}{2} \right\| \leq 1 - \frac{3}{4}\delta_X^{(2)}(\epsilon).$$

(2) If $k=3$ and $t_1=t_2=\frac{1}{8}, t_3=\frac{1}{4}$ then

$$\left\|\frac{x_1}{8}+\frac{x_2}{8}+\frac{x_3}{4}+\frac{x_4}{2}\right\| \leq 1-\frac{1}{2}\delta_X^{(3)}(\epsilon).$$

(3) If $k=3$ and $t_1+t_2+t_3=\frac{1}{2}$, then

$$\left\|t_1x_1+t_2x_2+t_3x_3+\frac{1}{2}x_4\right\| \leq 1-4\min\{t_1,t_2,t_3\}\delta_X^{(3)}(\epsilon).$$

We obtain the intuitive and geometric idea for the proof of our main result Theorem 3.7 from the proof technique of the following theorem.

Theorem 3.6. *Let K be a nonempty weakly compact subset of a $2-$uniformly convex Banach space X and $T: K \to K$ be a nonexpansive map. Further, assume that K is $T-$regular. Then T has a fixed point in K.*

Proof. Define $\mathcal{F}=\{F\subseteq K : F \text{ is nonempty weakly compact } T-\text{regular set}\}$.

It is easy to see that the set inclusion $\subseteq$, defines a partial order relation on $\mathcal{F}$. By Zorn's lemma, we get a minimal element in $\mathcal{F}$.

Without loss of generality, we can assume that K is minimal in $\mathcal{F}$.

Let $x_1 \in K$ and define $x_{k+1}=\frac{x_k+Tx_k}{2} \in K$, for $k\in\mathbb{N}$.

By Lemma 3.3, we have $A(K,\{x_k\})=K$ i.e., $r(x)=\limsup\limits_{k\to\infty}\|x-x_k\|=r$, for all $x\in K$.

Note that $r=0$ if and only if K is singleton.

For, if $r=0$, then $\limsup\limits_{k\to\infty}\|x-x_k\|=0$, for all $x\in K$. This gives $\{x_k\}$ converges to every point in K. Hence K is singleton.

Conversely, suppose that K is singleton. Then it is easy to see that $r=0$, as $\{x_k\}\subseteq K$.

We claim that $r=0$. Suppose that $r>0$. This implies that $x\neq Tx$, for all $x\in K$.

It is claimed that $Tx_n \in \operatorname{aff}\{x_1,Tx_1\}$ for all $n\in\mathbb{N}$.

Suppose that there exists $n\in\mathbb{N}$ such that $Tx_n\notin \operatorname{aff}\{x_1,Tx_1\}$.

Without loss of generality, we can assume that $Tx_2\notin\operatorname{aff}\{x_1,Tx_1\}$.

This gives $\{x_1,Tx_1,Tx_2\}$ is affinely independent and $dim(\operatorname{co}\{x_1,Tx_1,Tx_2\}=2$. Hence $V(x_1,Tx_1,Tx_2)=\epsilon$ for some $\epsilon>0$.

Since X is $2-$UC and $\delta_X^{(2)}$ is continuous, we have

$$\lim_{\rho\to 0}(r+\rho)\left(1-\frac{3}{4}\delta_X^{(2)}\left(\frac{\epsilon}{(r+\rho)^2}\right)\right)=r\left(1-\frac{3}{4}\delta_X^{(2)}\left(\frac{\epsilon}{r^2}\right)\right)<r$$

This implies that there is a $\rho_0>0$ such that

$$(r+\rho_0)\left(1-\frac{3}{4}\delta_X^{(2)}\left(\frac{\epsilon}{(r+\rho_0)^2}\right)\right)<r.$$

Since $A(K, \{x_k\}) = K$ and for this $\rho_0 > 0$, there exists $N \in \mathbb{N}$ such that for $k \geq N$, we have

$$\begin{aligned} \|x_1 - x_k\| &\leq r + \rho_0 \\ \|Tx_1 - x_k\| &\leq r + \rho_0 \\ \|Tx_2 - x_k\| &\leq r + \rho_0 \end{aligned}$$

As X is $2-$UC, we have

$$\left\|\frac{x_1 + Tx_1 + Tx_2}{3} - x_k\right\| \leq (r+\rho_0)\left(1 - \delta_X^{(2)}\left(\frac{\epsilon}{(r+\rho_0)^2}\right)\right), \text{ for } k \geq N.$$

Note that $x_3 = \frac{x_1}{4} + \frac{Tx_1}{4} + \frac{Tx_2}{2} \in \text{co}\{x_1, Tx_1, Tx_2\}$ and by Lemma 3.4, we get

$$\begin{aligned} \|x_3 - x_k\| &= \left\|\frac{x_1}{4} + \frac{Tx_1}{4} + \frac{Tx_2}{2} - x_k\right\| \\ &\leq (r+\rho_0)\left(1 - \frac{3}{4}\delta_X^{(2)}\left(\frac{\epsilon}{(r+\rho_0)^2}\right)\right), \text{ for } k \geq N. \end{aligned}$$

This implies that

$$\begin{aligned} r(x_3) &= \limsup_{k\to\infty} \|x_3 - x_k\| \\ &\leq (r+\rho_0)\left(1 - \frac{3}{4}\delta_X^{(2)}\left(\frac{\epsilon}{(r+\rho_0)^2}\right)\right) < r. \end{aligned}$$

This gives a contradiction to $A(K, \{x_k\}) = K$.

Therefore $Tx_n \in \text{aff}\{x_1, Tx_1\}$, for all $n \in \mathbb{N}$. This implies that $\{x_n\} \subseteq \text{aff}\{x_1, Tx_1\}$.

Since $\{x_n\}$ is a bounded sequence and $dim(\text{aff}\{x_1, Tx_1\}) = 1$, so it has a convergent subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ and $z \in K$ such that $x_{n_j} \to z$ as $j \to \infty$. Since $\lim\limits_{j\to\infty} \|x_{n_j} - Tx_{n_j}\| = 0$ and T is nonexpansive, $Tz = z$. Hence $r = 0$.

This implies that K is singleton and T has a fixed point in K. □

Next we prove the main result of this paper.

Theorem 3.7. *Let K be a nonempty weakly compact subset of a $3-$uniformly convex Banach space X and $T : K \to K$ be a nonexpansive map. Further, assume that K is $T-$regular. Then T has a fixed point in K.*

Proof. Note that by using Zorn's lemma, we get a nonempty minimal weakly compact $T-$regular subset of K.

Without loss of generality, we can assume that K is a nonempty minimal weakly compact $T-$regular set.

Let $x_1 \in K$ and define $x_{k+1} = \frac{x_k + Tx_k}{2} \in K$, for $k \in \mathbb{N}$.

By Lemma 3.3, we have $A(K, \{x_k\}) = K$ i.e., $r(x) = \limsup\limits_{k\to\infty} \|x - x_k\| = r$, for all $x \in K$.

We claim that $r = 0$. Suppose that $r > 0$. This implies that $x \neq Tx$, for all $x \in K$.

Suppose that for every $n \in \mathbb{N}$, $Tx_n \in \text{aff}\{x_1, Tx_1\}$. Then $\{x_n\}$ is a bounded sequence in $\text{aff}\{x_1, Tx_1\}$, as K is bounded.

Hence $\{x_n\}$ has a convergent subsequence. This implies that T has a fixed point in K.

Suppose that there exists $n \in \mathbb{N}$ such that $Tx_n \notin \text{aff}\{x_1, Tx_1\}$.

Without loss of generality, we can assume that $Tx_2 \notin \text{aff}\{x_1, Tx_1\}$.

It is claimed that $Tx_n \in \text{aff}\{x_1, Tx_1, Tx_2\}$, for all $n \in \mathbb{N}$.

We use mathematical induction to prove our claim.

Case 1. It is claimed that $Tx_3 \in \text{aff}\{x_1, Tx_1, Tx_2\}$. Suppose that $Tx_3 \notin \text{aff}\{x_1, Tx_1, Tx_2\}$.

This gives $\{x_1, Tx_1, Tx_2, Tx_3\}$ is affinely independent and $dim(\text{co}\{x_1, Tx_1, Tx_2, Tx_3\}) = 3$. Hence $V(x_1, Tx_1, Tx_2, Tx_3) = \epsilon$, for some $\epsilon > 0$.

Since X is $3-$UC and $\delta_X^{(3)}$ is continuous, there is a $\rho_0 > 0$ such that

$$(r+\rho_0)\left(1 - \frac{1}{2}\delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right) < r.$$

Since $A(K, \{x_k\}) = K$, there exists $N \in \mathbb{N}$ such that for $k \geq N$, we have

$$\begin{aligned} \|x_1 - x_k\| &\leq r + \rho_0 \\ \|Tx_1 - x_k\| &\leq r + \rho_0 \\ \|Tx_2 - x_k\| &\leq r + \rho_0 \\ \|Tx_3 - x_k\| &\leq r + \rho_0 \end{aligned}$$

As X is $3-$UC, we have for $k \geq N$

$$\left\|\frac{x_1 + Tx_1 + Tx_2 + Tx_3}{4} - x_k\right\| \leq (r+\rho_0)\left(1 - \delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right).$$

Note that $x_4 = \frac{x_3 + Tx_3}{2} = \frac{x_2 + Tx_2}{4} + \frac{Tx_3}{2} = \frac{x_1}{8} + \frac{Tx_1}{8} + \frac{Tx_2}{4} + \frac{Tx_3}{2} \in \text{co}\{x_1, Tx_1, Tx_2, Tx_3\}$. Now, by Lemma 3.4, we get

$$\begin{aligned} \|x_4 - x_k\| &= \left\|\frac{x_1}{8} + \frac{Tx_1}{8} + \frac{Tx_2}{4} + \frac{Tx_3}{2} - x_k\right\| \\ &\leq (r+\rho_0)\left(1 - \frac{1}{2}\delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right), \text{for } k \geq N. \end{aligned}$$

This implies that

$$\begin{aligned} r(x_4) &= \limsup_{k\to\infty} \|x_4 - x_k\| \\ &\leq (r+\rho_0)\left(1 - \frac{1}{2}\delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right) < r. \end{aligned}$$

This gives a contradiction to $A(K, \{x_k\}) = K$. Hence $Tx_3 \in \text{aff}\{x_1, Tx_1, Tx_2\}$.

Case 2. It is claimed that $Tx_4 \in \text{aff}\{x_1, Tx_1, Tx_2\}$. Suppose that $Tx_4 \notin \text{aff}\{x_1, Tx_1, Tx_2\}$.

This gives $\{x_1, Tx_1, Tx_2, Tx_4\}$ is affinely independent and $dim(\text{co}\{x_1, Tx_1, Tx_2, Tx_4\}) = 3$.

Since $Tx_3 \in \text{aff}\{x_1, Tx_1, Tx_2\}$, we have the following cases:

(a). $Tx_3 \in \text{aff}\{x_2, Tx_2\}$
(b). $Tx_3 \notin \text{aff}\{x_2, Tx_2\}$.

Subcase 2(a). Suppose that $Tx_3 \in \text{aff}\{x_2, Tx_2\}$. Then $Tx_3 = (1-\mu_3)x_2 + \mu_3 Tx_2$, for some $\mu_3 \in \mathbb{R}$. By the nonexpansiveness of T, we have
$\frac{1}{2}\|Tx_2 - x_2\| = \|x_3 - x_2\| \geq \|Tx_3 - Tx_2\| = |1-\mu_3|\|Tx_2 - x_2\|$.

This gives $\frac{1}{2} \leq \mu_3 \leq \dfrac{3}{2}$. Note that $\mu_3 \neq \frac{1}{2}$. For, if $\mu_3 = \frac{1}{2}$, then $Tx_3 = x_3$.

$$
\begin{aligned}
\text{Now } x_4 = \frac{x_3 + Tx_3}{2} &= \frac{1}{2}\left(\frac{x_2 + Tx_2}{2} + Tx_3\right) \\
&= \frac{x_2}{4} + \frac{1}{4}\left(\frac{Tx_3 - (1-\mu_3)x_2}{\mu_3}\right) + \frac{Tx_3}{2} \\
&= \left(\frac{2\mu_3 - 1}{4\mu_3}\right)x_2 + \left(\frac{2\mu_3 + 1}{4\mu_3}\right)Tx_3 \\
&= \left(\frac{2\mu_3 - 1}{8\mu_3}\right)x_1 + \left(\frac{2\mu_3 - 1}{8\mu_3}\right)Tx_1 + \left(\frac{2\mu_3 + 1}{4\mu_3}\right)Tx_3 \\
&= t_1 x_1 + t_1 Tx_1 + (1-2t_1)Tx_3 \text{ where } t_1 = \frac{2\mu_3 - 1}{8\mu_3}.
\end{aligned}
$$

Since $\mu_3 > \frac{1}{2}$, we have $t_1 > 0$ and $1 - 2t_1 > 0$. This gives x_4 lies in the interior of $\text{co}\{x_1, Tx_1, Tx_3\}$.

Since $\{x_1, Tx_1, Tx_2, Tx_4\}$ is affinely independent and $Tx_3 \in \text{aff}\{x_2, Tx_2\}$, we have $\{x_1, Tx_1, Tx_3, Tx_4\}$ is affinely independent and $dim(\text{co}\{x_1, Tx_1, Tx_3, Tx_4\}) = 3$. Hence $V(x_1, Tx_1, Tx_3, Tx_4) = \epsilon$ for some $\epsilon > 0$.

Since $\delta_X^{(3)}$ is continuous and X is $3-$UC, there is a $\rho_0 > 0$ such that

$$(r+\rho_0)\left(1 - 2\min\{t_1, 1-2t_1\}\delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right) < r$$

As $A(K, \{x_k\}) = K$, there exist $N \in \mathbb{N}$ such that for $k \geq N$, we have

$$\left\|\frac{x_1 + Tx_1 + Tx_3 + Tx_4}{4} - x_k\right\| \leq (r+\rho_0)\left(1 - \delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right).$$

Note that $x_5 = \frac{x_4 + Tx_4}{2} = \frac{1}{2}\left(t_1 x_1 + t_1 Tx_1 + (1-2t_1)Tx_3 + Tx_4\right)$.

This implies that x_5 lies in the interior of $\text{co}\{x_1, Tx_1, Tx_3, Tx_4\}$. Now, by Lemma 3.4, for $k \geq N$ we have

$$
\begin{aligned}
\|x_5 - x_k\| &= \left\|\frac{1}{2}(t_1 x_1 + t_1 Tx_1 + (1-2t_1)Tx_3 + Tx_4) - x_k\right\| \\
&\leq (r+\rho_0)\left(1 - 2\min\{t_1, 1-2t_1\}\delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
r(x_5) &= \limsup_{k\to\infty}\|x_5 - x_k\| \\
&\leq (r+\rho_0)\left(1 - 2\min\{t_1, 1-2t_1\}\delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right) < r.
\end{aligned}
$$

This gives a contradiction to $A(K,\{x_k\}) = K$. Hence $Tx_4 \in \mathrm{aff}\{x_1, Tx_1, Tx_2\}$.
Subcase 2(b). Suppose that $Tx_3 \notin \mathrm{aff}\{x_2, Tx_2\}$. Then $\{x_2, Tx_2, Tx_3\}$ is affinely independent and $dim(\mathrm{co}\{x_2, Tx_2, Tx_3\}) = 2$.

Since $Tx_3 \in \mathrm{aff}\{x_1, Tx_1, Tx_2\}$ and $Tx_3 \notin \mathrm{aff}\{x_2, Tx_2\}$, we have $Tx_3 = ax_1 + bTx_1 + (1-(a+b))Tx_2$, for $a, b \in \mathbb{R}$ with $a \neq b$.

Since $\{x_1, Tx_1, Tx_2, Tx_4\}$ is affinely independent and $Tx_3 = ax_1 + bTx_1 + (1-(a+b))Tx_2$, we have $\{x_2, Tx_2, Tx_3, Tx_4\}$ is affinely independent and $dim(\mathrm{co}\{x_2, Tx_2, Tx_3, Tx_4\}) = 3$. This implies that $V(x_2, Tx_2, Tx_3, Tx_4) = \epsilon$, for some $\epsilon > 0$.

Therefore by case 1, we get $r(x_5) < r$.

This gives a contradiction to $A(K,\{x_k\}) = K$. Hence $Tx_4 \in \mathrm{aff}\{x_1, Tx_1, Tx_2\}$.
Case 3. Now, we assume that $Tx_n \in \mathrm{aff}\{x_1, Tx_1, Tx_2\}$, for $1 \leq n \leq m-1$.

To prove that $Tx_m \in \mathrm{aff}\{x_1, Tx_1, Tx_2\}$.

Suppose not. Then $\{x_1, Tx_1, Tx_2, Tx_m\}$ is affinely independent.

Since $Tx_k \in \mathrm{aff}\{x_1, Tx_1, Tx_2\}$ for $3 \leq k \leq m-1$, we have the following cases:

(a). $Tx_k \in \mathrm{aff}\{x_2, Tx_2\}$ for $k = 3, 4, \ldots, m-1$
(b). $Tx_k \notin \mathrm{aff}\{x_2, Tx_2\}$ for some $k \in \{3, 4, \ldots, m-1\}$.

Subcase 3(a). Suppose that $Tx_k \in \mathrm{aff}\{x_2, Tx_2\}$ for $3 \leq k \leq m-1$. Then $x_k \in \mathrm{aff}\{x_2, Tx_2\}$ for $3 \leq k \leq m$, as $x_k = \frac{x_{k-1}+Tx_{k-1}}{2}$.

Let $x_k = (1-\lambda_k)x_2 + \lambda_k Tx_2$ for some $\lambda_k \in \mathbb{R}$, $2 \leq k \leq m$ and $Tx_k = (1-\mu_k)x_2 + \mu_k Tx_2$ for some $\mu_k \in \mathbb{R}$, $2 \leq k \leq m-1$. Note that $\lambda_{k+1} = \frac{\lambda_k+\mu_k}{2}$, for $2 \leq k \leq m-1$, as $x_{k+1} = \frac{x_k+Tx_k}{2}$. Hence $\lambda_3 = \frac{1}{2}$, as $\lambda_2 = 0$, $\mu_2 = 1$.

Since we work with the $\mathrm{aff}\{x_2, Tx_2\}$, we can identify the $\mathrm{aff}\{x_2, Tx_2\}$ with the real line $\mathbb{R}$ by assuming $x_2 = 0$ and $Tx_2 = 1$. In this way, we get that $x_k = \lambda_k$ and $Tx_k = \mu_k$ for $2 \leq k \leq m-1$.

As $Tx_k \neq x_k$, we have $\lambda_k \neq \mu_k$ and $\lambda_k \neq \lambda_{k+1}$ for $2 \leq k \leq m-1$.

Note that, from case 2(a), we have $\lambda_3 < \mu_3$. This implies that $\lambda_3 < \lambda_4 < \mu_3$, as $\lambda_{k+1} = \frac{\lambda_k+\mu_k}{2}$.

It is claimed that $\lambda_k < \lambda_{k+1}$ and $\lambda_k < \mu_k$, for $4 \leq k \leq m-1$.

Since T is nonexpansive, we have

$|\mu_4 - \mu_3|\|x_2 - Tx_2\| = \|Tx_3 - Tx_4\| \leq \|x_3 - x_4\| = (\lambda_4 - \lambda_3)\|x_2 - Tx_2\|$.

This implies that $-\lambda_4 + \lambda_3 \leq \mu_4 - \mu_3 \leq \lambda_4 - \lambda_3$. Now, since $\lambda_4 = \frac{\lambda_3+\mu_3}{2}$, we have $\lambda_4 < \mu_4$. This gives $\lambda_4 < \lambda_5 < \mu_4$.

Continuing in this way, we get $\lambda_k < \lambda_{k+1} < \mu_k$ for $3 \leq k \leq m-1$.

Hence $0 = \lambda_2 < \lambda_3 < \lambda_4 < \cdots < \lambda_{m-1} < \lambda_m < \mu_{m-1}$.

This implies that λ_k lies in the interior of $\mathrm{co}\{\lambda_2, \mu_{m-1}\}$ for $3 \leq k \leq m$.

Hence x_k lies in the interior of $\mathrm{co}\{x_2, Tx_{m-1}\}$ for $3 \leq k \leq m$.

This implies that x_m lies in the interior of $\mathrm{co}\{x_1, Tx_1, Tx_{m-1}\}$, as $x_2 = \frac{x_1+Tx_1}{2}$.

Now, since $\mathrm{aff}\{x_1, Tx_1, Tx_2\} = \mathrm{aff}\{x_1, Tx_1, Tx_{m-1}\}$ and $Tx_m \notin \mathrm{aff}\{x_1, Tx_1, Tx_2\}$, we have $\{x_1, Tx_1, Tx_{m-1}, Tx_m\}$ is affinely independent and $dim(\mathrm{co}\{x_1, Tx_1, Tx_{m-1}, Tx_m\}) = 3$.

Hence x_{m+1} lies in the interior of $\mathrm{co}\{x_1, Tx_1, Tx_{m-1}, Tx_m\}$, as $x_{m+1} = \frac{x_m+Tx_m}{2}$.

Now, by using the arguments as in case 2(a), it is easy to see that $r(x_{m+1}) = \limsup_{k\to\infty} \|x_{m+1} - x_k\| < r$.

This gives a contradiction to $A(K, \{x_k\}) = K$. Hence $Tx_m \in \mathrm{aff}\{x_1, Tx_1, Tx_2\}$.

Subcase 3(b). Suppose that there exists $k \in \mathbb{N}$ such that $3 \leq k \leq m-1$ and $Tx_k \notin \mathrm{aff}\{x_2, Tx_2\}$.

Let k_0 be the least integer satisfying $Tx_{k_0} \notin \mathrm{aff}\{x_2, Tx_2\}$. This implies $Tx_3, Tx_4, \ldots, Tx_{k_0-1} \in \mathrm{aff}\{x_2, Tx_2\}$.

Then $\{x_{k_0-1}, Tx_{k_0-1}, Tx_{k_0}\}$ is affinely independent and $\mathrm{aff}\{x_{k_0-1}, Tx_{k_0-1}, Tx_{k_0}\} = \mathrm{aff}\{x_1, Tx_1, Tx_2\}$.

Now, we consider the set $\{x_{k_0-1}, Tx_{k_0-1}, Tx_{k_0}\}$.

Suppose that $Tx_k \in \mathrm{aff}\{x_{k_0}, Tx_{k_0}\}$ for $k_0 + 1 \leq k \leq m-1$.

Then using the arguments as in case 3(a), it is easy to see that x_{m+1} lies in the interior of $\mathrm{co}\{x_{k_0-1}, Tx_{k_0-1}, Tx_{m-1}, Tx_m\}$ and $\{x_{k_0-1}, Tx_{k_0-1}, Tx_{m-1}, Tx_m\}$ is affinely independent. Now, it is apparent that $r(x_{m+1}) < r$, as X is $3-$UC.

This gives a contradiction to $A(K, \{x_k\}) = K$. Hence $Tx_m \in \mathrm{aff}\{x_1, Tx_1, Tx_2\}$.

Suppose that there exists $k \in \mathbb{N}$ such that $k_0 + 1 \leq k \leq m-1$ and $Tx_k \notin \mathrm{aff}\{x_{k_0}, Tx_{k_0}\}$.

Let k_1 be the least integer satisfying $Tx_{k_1} \notin \mathrm{aff}\{x_{k_0}, Tx_{k_0}\}$. This implies that $Tx_{k_0+1}, Tx_{k_0+2}, \ldots, Tx_{k_1-1} \in \mathrm{aff}\{x_{k_0}, Tx_{k_0}\}$.

Then $\{x_{k_1-1}, Tx_{k_1-1}, Tx_{k_1}\}$ is affinely independent and $\mathrm{aff}\{x_{k_1-1}, Tx_{k_1-1}, Tx_{k_1}\} = \mathrm{aff}\{x_{k_0-1}, Tx_{k_0-1}, Tx_{k_0}\}$.

Now, we consider the set $\{x_{k_1-1}, Tx_{k_1-1}, Tx_{k_1}\}$.

Continuing in this way, we can find n_0 is the largest integer such that $k_1 \leq n_0 \leq m-1$ and $Tx_{n_0} \notin \mathrm{aff}\{x_{n_0-1}, Tx_{n_0-1}\}$. This implies that $Tx_n \in \mathrm{aff}\{x_{n_0}, Tx_{n_0}\}$ for $n_0 \leq n \leq m-1$.

Then using the arguments as in case 3(a), it is easy to see that x_{m+1} lies in the interior of $\mathrm{co}\{x_{n_0-1}, Tx_{n_0-1}, Tx_{m-1}, Tx_m\}$ and $\{x_{n_0-1}, Tx_{n_0-1}, Tx_{m-1}, Tx_m\}$ is affinely independent. Now, it is apparent that $r(x_{m+1}) < r$, as X is $3-$UC.

This gives a contradiction to $A(K, \{x_k\}) = K$. Hence $Tx_m \in \mathrm{aff}\{x_1, Tx_1, Tx_2\}$.

Hence, by mathematical indution $Tx_n \in \mathrm{aff}\{x_1, Tx_1, Tx_2\}$, for all $n \in \mathbb{N}$. This implies that $\{x_n\} \subseteq \mathrm{aff}\{x_1, Tx_1, Tx_2\}$.

Since $\{x_n\}$ is a bounded sequence and $dim(\mathrm{aff}\{x_1, Tx_1, Tx_2\}) = 2$, so it has a convergent subsequence i.e., there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $z \in K$ such that $x_{n_j} \to z$ as $j \to \infty$.

Since $\lim_{j\to\infty} \|x_{n_j} - Tx_{n_j}\| = 0$ and T is nonexpansive, we have $Tz = z$. Hence $r = 0$. This implies that K is singleton and T has a fixed point in K. □

Remark 3.8. In the light of Theorem 3.6 and Theorem 3.7, it is natural to expect that if K is a nonempty weakly compact subset of a $k-$UC Banach space X, for $k > 3$ and if $T : K \to K$ is a nonexpansive map satisfying $\frac{x+Tx}{2} \in K$ for all $x \in K$, then T has a fixed point in K.

3.2. **Banach space with Opial property.**

Theorem 3.9. *Let K be a nonempty weakly compact subset of a Banach space X having the Opial property and $T : K \to K$ be a nonexpansive map. Further, assume that K is $T-$regular. Define a sequence $\{x_n\}$ in K by $x_{n+1} = \frac{x_n + Tx_n}{2}$ for $n \in \mathbb{N}$ and $x_1 \in K$. Then T has a fixed point in K and $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. By Lemma 2.6, we have $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$. Since K is weakly compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in K$ such that $\{x_{n_k}\}$ converges weakly to z. Also, we have

$$\|x_{n_k} - Tz\| \leq \|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tz\|, \text{ for all } k \in \mathbb{N}.$$

Hence

$$\liminf_{k\to\infty} \|x_{n_k} - Tz\| \leq \liminf_{k\to\infty} \|x_{n_k} - z\|.$$

Since X has the Opial property, we obtain $Tz = z$. Also note that, $\{\|x_n - z\|\}$ is a decreasing sequence.

It is claimed that $\{x_n\}$ converges weakly to z. Suppose that $\{x_n\}$ does not converge weakly to z.

Then there exists a subsequence $\{x_{\widehat{n}_j}\}$ of $\{x_n\}$ which does not converge weakly to z. Since K is weakly compact and $\{x_{\widehat{n}_j}\} \subseteq K$, there exists a subsequence of $\{x_{\widehat{n}_j}\}$ whose weak limit is $w \in K$ and $z \neq w$.

Without loss of generality, we can assume that $\{x_{\widehat{n}_j}\}$ converges weakly to w. It is easy to see that $Tw = w$, as $\lim\limits_{j\to\infty} \|x_{\widehat{n}_j} - Tx_{\widehat{n}_j}\| = 0$. Also, it is apparent that $\{\|x_n - w\|\}$ is a decreasing sequence, as $Tw = w$.

Since X has the Opial property, $\{x_{\widehat{n}_j}\}$ converges weakly to w and $\{x_{n_k}\}$ converges weakly to z, we have

$$\begin{aligned} \lim_{n\to\infty} \|x_n - z\| &= \lim_{k\to\infty} \|x_{n_k} - z\| < \lim_{k\to\infty} \|x_{n_k} - w\| = \lim_{n\to\infty} \|x_n - w\| \\ &= \lim_{j\to\infty} \|x_{\widehat{n}_j} - w\| < \lim_{j\to\infty} \|x_{\widehat{n}_j} - z\| = \lim_{n\to\infty} \|x_n - z\|. \end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ weakly converges to z. □

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