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Research Article

On Second-Order Duality for Minimax Fractional Programming Problems with Generalized Convexity

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We focus our study on a discussion of duality relationships of a minimax fractional programming problem with its two types of second-order dual models under the second-order generalized convexity type assumptions. Results obtained in this paper naturally unify and extend some previously known results on minimax fractional programming in the literature.

1. Introduction

Fractional programming is an interesting subject applicable to many types of optimization problems such as portfolio selection, production, and information theory and numerous decision making problems in management science. More specifically, it can be used in engineering and economics to minimize a ratio of physical or economical functions, or both, such as cost/time, cost/volume, and cost/benefit, in order to measure the efficiency or productivity of the system (see Stancu-Minasian [1]).

Minimax type functions arise in the design of electronic circuits; however, minimax fractional problems appear in the formulation of discrete and continuous rational approximation problems with respect to the Chebyshev norm [2], continuous rational games [3], multiobjective programming [4, 5], and engineering design as well as some portfolio selection problems discussed by Bajona-Xandri and Martinez-Legaz [6].

In this paper, we consider the minimax fractional programming problem

$$\text{minimize} \quad \phi(x) = \sup_{y \in Y} \frac{f(x, y)}{h(x, y)},$$

$$\text{subject to} \quad g(x) \leq 0, \qquad x \in R^n, \tag{1.1}$$

where Y is a compact subset of R^l and $f(\cdot,\cdot) : R^n \times R^l \to R$, $h(\cdot,\cdot) : R^n \times R^l \to R$, and $g(\cdot) : R^n \to R^m$ are twice continuously differentiable functions on $R^n \times R^l$, $R^n \times R^l$, and R^n, respectively. It is assumed that, for each (x, y) in $R^n \times R^l$, $f(x, y) \geq 0$ and $h(x, y) > 0$.

For the case of convex differentiable minimax fractional programming, Yadav and Mukherjee [7] formulated two dual models for (1.1) and derived duality theorems. Chandra and Kumar [8] pointed out certain omissions and inconsistencies in the dual formulation of Yadav and Mukherjee [7]; they constructed two modified dual problems for (1.1) and proved appropriate duality results. Liu and Wu [9, 10] and Ahmad [11] obtained sufficient optimality conditions and duality theorems for (1.1) assuming the functions involved to be generalized convex.

Second-order duality provides tighter bounds for the value of the objective function when approximations are used. For more details, one can consult ([12, page 93]). One more advantage of second-order duality, when applicable, is that, if a feasible point in the primal is given and first-order duality does not apply, then we can use second order duality to provide a lower bound of the value of the primal (see [13]).

Mangasarian [14] first formulated the second-order dual for a nonlinear programming problem and established second-order duality results under certain inequalities. Mond [12] reproved second-order duality results assuming rather simple inequalities. Subsequently, Bector and Chandra [15] formulated a second-order dual for a fractional programming problem and obtained usual duality results under the assumptions [14] by naming these as convex/concave functions.

Based upon the ideas of Bector et al. [16] and Rueda et al. [17], Yang and Hou [18] proposed a new concept of generalized convexity and discussed sufficient optimality conditions for (1.1) and duality results for its corresponding dual. Recently, Husain et al. [19] formulated two types of second-order dual models to (1.1) and discussed appropriate duality results involving η-convexity/generalized η-convexity assumptions.

In this paper, we are inspired by Chandra and Kumar [8], Bector et al. [16], Liu [20], and Husain et al. [19] to discuss weak, strong, and strict converse duality theorems connecting (1.1) with its two types of second-order duals by using second-order generalized convexity type assumptions [21].

2. Notations and Preliminaries

Let $S = \{x \in R^n : g(x) \leq 0\}$ denote the set of all feasible solutions of (1.1). For each $(x, y) \in R^n \times R^l$, we define

$$J(x) = \{j \in M : g_j(x) = 0\}, \tag{2.1}$$

where $M = \{1, 2, \dots, m\}$,

$$Y(x) = \left\{ y \in Y : f(x, y) = \sup_{z \in Y} f(x, z) \right\},$$

$$K(x) = \left\{ (s,t,\tilde{y}) \in \mathbb{N} \times R_+^s \times R^{ls} : 1 \leq s \leq n+1,\ t = (t_1, t_2, \ldots, t_s) \in R_+^s \right.$$
$$\left. \text{with} \sum_{i=1}^{s} t_i = 1, \tilde{y} = (\overline{y}_1, \overline{y}_2, \ldots . \overline{y}_s) \text{ with } \overline{y}_i \in Y(x),\ i = 1, 2, \ldots, s \right\}. \tag{2.2}$$

Definition 2.1. A functional $\mathcal{F} : X \times X \times R^n \to R$, where $X \subseteq R^n$ is said to be sublinear in its third argument, if $\forall x, \overline{x} \in X$,

(i) $\mathcal{F}(x, \overline{x}; a_1 + a_2) \leq \mathcal{F}(x, \overline{x}; a_1) + \mathcal{F}(x, \overline{x}; a_2)\ \forall a_1, a_2 \in R^n$,

(ii) $\mathcal{F}(x, \overline{x}; \alpha a) = \alpha \mathcal{F}(x, \overline{x}; a)\ \forall \alpha \in R_+, a \in R^n$.

By (ii), it is clear that $\mathcal{F}(x, \overline{x}; 0\ a) = 0$.

Definition 2.2. A point $\overline{x} \in S$ is said to optimal solution of (1.1) if $\phi(x) \geq \phi(\overline{x})$ for each $x \in S$.

The following theorem [8] will be needed in the subsequent analysis.

Theorem 2.3 (necessary conditions). *Let x^* be a solution (local or global) of* (1.1)*, and let $\nabla g_j(x^*), j \in J(x^*)$ be linearly independent. Then there exist $(s^*, t^*, \overline{y}^*) \in K(x^*), \lambda^* \in R_+$, and $\mu^* \in R_+^m$ such that*

$$\begin{gathered}
\nabla \sum_{i=1}^{s^*} t_i^* \left(f(x^*, \overline{y}_i^*) - \lambda^* h(x^*, \overline{y}_i^*) \right) + \nabla \sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0, \\
f(x^*, \overline{y}_i^*) - \lambda^* h(x^*, \overline{y}_i^*) = 0, \quad i = 1, 2, \ldots, s^*, \\
\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0, \\
t_i^* \geq 0, \quad \sum_{i=1}^{s^*} t_i^* = 1, \quad \overline{y}_i^* \in Y(x^*), \quad i = 1, 2, \ldots, s^*.
\end{gathered} \tag{2.3}$$

Throughout the paper, we assume that $\mathcal{F}$ is a sublinear functional. For $\beta = 1, 2, \ldots, r$ let $b, b_0, b_\beta : X \times X \to R_+, \phi, \phi_0, \phi_\beta : R \to R, \rho, \rho_0, \rho_\beta$ be real numbers, and let $\theta : R^n \times R^n \to R$.

3. First Duality Model

In this section, we discuss usual duality results for the following dual [19]:

$$\max_{(s,t,\overline{y}) \in K(z)} \ \sup_{(z,\mu,\lambda,p) \in H_1(s,t,\overline{y})} \lambda, \tag{3.1}$$

where $H_1(s,t,\overline{y})$ denotes the set of all $(z,\mu,\lambda,p) \in R^n \times R_+^m \times R_+ \times R^n$ satisfying

$$\nabla\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\right)+\nabla^2\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\right)p + \nabla\sum_{j=1}^{m}\mu_j g_j(z)+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p=0, \tag{3.2}$$

$$\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\right)-\frac{1}{2}p^T\nabla^2\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\right)p\geq 0, \tag{3.3}$$

$$\sum_{j=1}^{m}\mu_j g_j(z)-\frac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\geq 0. \tag{3.4}$$

If, for a triplet $(s,t,\overline{y}) \in K(z)$, the set $H_1(s,t,\overline{y})=\emptyset$, then we define the supremum over it to be $-\infty$.

Remark 3.1. If $P=0$, then (3.1) becomes the dual considered in [9].

Theorem 3.2 (weak duality). *Let x and $(z,\mu,\lambda,s,t,\overline{y},p)$ be the feasible solutions of* (1.1) *and* (3.1), *respectively. Suppose that there exist $\digamma,\theta,\phi,b$ and ρ such that*

$$\begin{aligned} b(x,z)\phi\Bigg[&\sum_{i=1}^{s}t_i\left(f(x,\overline{y}_i)-\lambda h(x,\overline{y}_i)\right)-\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\right)-\sum_{j=1}^{m}\mu_j g_j(z)\\ &+\frac{1}{2}p^T\nabla^2\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\right)p+\frac{1}{2}p^T\sum_{j=1}^{m}\mu_j g_j(z)p\Bigg]<0\\ \Longrightarrow \digamma\Bigg(&x,z;\nabla\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\right)+\nabla\sum_{j=1}^{m}\mu_j g_j(z)\\ &+\nabla^2\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\right)p+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\Bigg)<-\rho\|\theta(x,z)\|^2. \end{aligned} \tag{3.5}$$

Further assume that

$$a<0\Longrightarrow\phi(a)<0, \tag{3.6}$$

$$b(x,z)>0, \tag{3.7}$$

$$\rho\geq 0. \tag{3.8}$$

Then

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)} \geq \lambda. \tag{3.9}$$

Proof. Suppose contrary to the result that

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)} < \lambda. \tag{3.10}$$

Thus, we have

$$f(x,\overline{y}_i) - \lambda h(x,\overline{y}_i) < 0, \quad \forall \overline{y}_i \in Y(x), \quad i = 1,2,\ldots,s. \tag{3.11}$$

It follows from $t_i \geq 0$, $i = 1,2,\ldots,s$, that

$$t_i\left[f(x,\overline{y}_i) - \lambda h(x,\overline{y}_i)\right] \leq 0, \tag{3.12}$$

with at least one strict inequality since $t = (t_1, t_2, \ldots, t_s) \neq 0$. Taking summation over i, we have

$$\sum_{i=1}^{s} t_i\left(f(x,\overline{y}_i) - \lambda h(x,\overline{y}_i)\right) < 0, \tag{3.13}$$

which together with (3.3) gives

$$\begin{aligned}\sum_{i=1}^{s} t_i\left(f(x,\overline{y}_i) - \lambda h(x,\overline{y}_i)\right) &< 0\\ &\leq \sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i)\right)\\ &\quad - \frac{1}{2}p^T\nabla^2\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i)\right)p.\end{aligned} \tag{3.14}$$

The above inequality along with (3.4) implies

$$\begin{aligned}&\sum_{i=1}^{s} t_i\left(f(x,\overline{y}_i) - \lambda h(x,\overline{y}_i)\right) - \sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i)\right) - \sum_{j=1}^{m}\mu_j g_j(z)\\ &\quad + \frac{1}{2}p^T\nabla^2\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i)\right)p + \frac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p < 0.\end{aligned} \tag{3.15}$$

Using (3.6) and (3.7), it follows from (3.15) that

$$b(x,z)\phi\left[\sum_{i=1}^{s}t_i(f(x,\overline{y}_i)-\lambda h(x,\overline{y}_i))-\sum_{i=1}^{s}t_i(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i))-\sum_{j=1}^{m}\mu_j g_j(z)\right.$$
$$\left.+\frac{1}{2}p^T\nabla^2\sum_{i=1}^{s}t_i(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i))p+\frac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\right]<0, \tag{3.16}$$

which along with (3.5) and (3.8) yields

$$\mathcal{F}\left(x,z;\nabla\sum_{i=1}^{s}t_i(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i))+\nabla\sum_{j=1}^{m}\mu_j g_j(z)\right.$$
$$\left.+\nabla^2\sum_{i=1}^{s}t_i(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i))p+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\right)<0, \tag{3.17}$$

which contradicts (3.2) since $\mathcal{F}(x,z;0)=0$. □

Theorem 3.3 (strong duality). *Assume that x^* is an optimal solution of* (1.1) *and $\nabla g_j(x^*), j\in J(x^*)$ are linearly independent. Then there exist $(s^*,t^*,\overline{y}^*)\in K(x^*)$ and $(x^*,\mu^*,\lambda^*,p^*=0)\in H_1(s^*,t^*,\overline{y}^*)$ such that $(x^*,\mu^*,\lambda^*,s^*,t^*,\overline{y}^*,p^*=0)$ is a feasible solution of* (3.1) *and the two objectives have the same values. Further, if the assumptions of weak duality (Theorem 3.2) hold for all feasible solutions $(z,\mu,\lambda,s,t,\overline{y},p)$ of* (3.1)*, then $(x^*,\mu^*,\lambda^*,s^*,t^*,\overline{y}^*,p^*=0)$ is an optimal solution of* (3.1).

Proof. Since x^* is an optimal solution of (1.1) and $\nabla g_j(x^*), j\in J(x^*)$ are linearly independent, then, by Theorem 2.3, there exist $(s^*,t^*,\overline{y}^*)\in K(x^*)$ and $(x^*,\mu^*,\lambda^*,p^*=0)\in H_1(s^*,t^*,\overline{y}^*)$ such that $(x^*,\mu^*,\lambda^*,s^*,t^*,\overline{y}^*,p^*=0)$ is a feasible solution of (3.1) and the two objectives have the same values. Optimality of $(x^*,\mu^*,\lambda^*,s^*,t^*,\overline{y}^*,p^*=0)$ for (3.1) thus follows from weak duality (Theorem 3.2). □

Theorem 3.4 (Strict converse duality). *Let x^* and $(z^*,\mu^*,\lambda^*,s^*,t^*,\overline{y}^*,p^*)$ be the optimal solutions of* (1.1) *and* (3.1)*, respectively. Suppose that $\nabla g_j(x^*), j\in J(x^*)$ are linearly independent and there exist $\mathcal{F},\theta,\phi,b$ and ρ such that*

$$b(x^*,z^*)\phi\left[\sum_{i=1}^{s^*}t_i^*(f(x^*,\overline{y}_i^*)-\lambda^*h(x^*,\overline{y}_i^*))-\sum_{i=1}^{s^*}t_i^*(f(z^*,\overline{y}_i^*)-\lambda^*h(z^*,\overline{y}_i^*))-\sum_{j=1}^{m}\mu_j^*g_j(z^*)\right.$$
$$\left.+\frac{1}{2}p^{*T}\nabla^2\sum_{i=1}^{s^*}t_i^*(f(z^*,\overline{y}_i^*)-\lambda^*h(z^*,\overline{y}_i^*))p^*+\frac{1}{2}p^{*T}\nabla^2\sum_{j=1}^{m}\mu_j^*g_j(z^*)p^*\right]\leq 0$$

$$\Longrightarrow \mathcal{F}\left(x^*, z^*; \nabla \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \overline{y}_i^*\right) - \lambda^* h\left(z^*, \overline{y}_i^*\right)\right) + \nabla \sum_{j=1}^{m} \mu_j^* g_j(z^*) \right.$$
$$\left. + \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \overline{y}_i^*\right) - \lambda^* h\left(z^*, \overline{y}_i^*\right)\right) p^* + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^* \right) < -\rho \|\theta(x^*, z^*)\|^2. \tag{3.18}$$

Further Assume

$$a < 0 \Longrightarrow \phi(a) \leq 0, \tag{3.19}$$

$$b(x^*, z^*) > 0, \tag{3.20}$$

$$\rho \geq 0. \tag{3.21}$$

Then $z^* = x^*$, that is, z^* is an optimal solution of (1.1).

Proof. Suppose contrary to the result that $z^* \neq x^*$. Since x^* and $(z^*, \mu^*, \lambda^*, s^*, t^*, \overline{y}^*, p^*)$ are optimal solutions of (1.1) and (3.1), respectively, and $\nabla g_j(x^*), j \in J(x^*)$ are linearly independent, therefore, from strong duality (Theorem 3.3), we reach

$$\sup_{y^* \in Y} \frac{f(x^*, y^*)}{h(x^*, y^*)} = \lambda^*. \tag{3.22}$$

Thus, we have

$$f\left(x^*, \overline{y}_i^*\right) - \lambda^* h\left(x^*, \overline{y}_i^*\right) \leq 0, \quad \forall \overline{y}_i^* \in Y(x^*),\ i = 1, 2, \ldots, s^*. \tag{3.23}$$

Now, proceeding as in Theorem 3.2, we get

$$\sum_{i=1}^{s^*} t_i^* \left(f\left(x^*, \overline{y}_i^*\right) - \lambda^* h\left(x^*, \overline{y}_i^*\right)\right) - \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \overline{y}_i^*\right) - \lambda^* h\left(z^*, \overline{y}_i^*\right)\right) - \sum_{j=1}^{m} \mu_j^* g_j(z^*)$$
$$+ \frac{1}{2} p^{*T} \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \overline{y}_i^*\right) - \lambda^* h\left(z^*, \overline{y}_i^*\right)\right) p^* + \frac{1}{2} p^{*T} \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^* < 0. \tag{3.24}$$

Using (3.19) and (3.20), it follows from (3.24) that

$$b(x^*,z^*)\phi\left[\sum_{i=1}^{s^*}t_i^*\left(f\left(x^*,\overline{y}_i^*\right)-\lambda^*h\left(x^*,\overline{y}_i^*\right)\right)-\sum_{i=1}^{s^*}t_i^*\left(f\left(z^*,\overline{y}_i^*\right)-\lambda^*h\left(z^*,\overline{y}_i^*\right)\right)-\sum_{j=1}^{m}\mu_j^*g_j(z^*)\right.$$
$$\left.+\frac{1}{2}p^{*T}\nabla^2\sum_{i=1}^{s^*}t_i^*\left(f\left(z^*,\overline{y}_i^*\right)-\lambda^*h\left(z^*,\overline{y}_i^*\right)\right)p^*+\frac{1}{2}p^{*T}\nabla^2\sum_{j=1}^{m}\mu_j^*g_j(z^*)p^*\right]\leq 0, \tag{3.25}$$

which along with (3.18) and (3.21) implies

$$\mathcal{F}\left(x^*,z^*;\nabla\sum_{i=1}^{s^*}t_i^*\left(f\left(z^*,\overline{y}_i^*\right)-\lambda^*h\left(z^*,\overline{y}_i^*\right)\right)+\nabla\sum_{j=1}^{m}\mu_j^*g_j(z^*)\right.$$
$$\left.+\nabla^2\sum_{i=1}^{s^*}t_i^*\left(f\left(z^*,\overline{y}_i^*\right)-\lambda^*h\left(z^*,\overline{y}_i^*\right)\right)p^*+\nabla^2\sum_{j=1}^{m}\mu_j^*g_j(z^*)p^*\right)<0, \tag{3.26}$$

which contradicts (3.2) since $\mathcal{F}(x^*,z^*;0)=0$. □

4. Second Duality Model

This section deals with duality theorems for the following second-order dual to (1.1):

$$\max_{(s,t,\overline{y})\in K(z)}\ \sup_{(z,\mu,\lambda,p)\in H_2(s,t,\overline{y})}\lambda, \tag{4.1}$$

where $H_2(s,t,\overline{y})$ denotes the set of all $(z,\mu,\lambda,p)\in R^n\times R_+^m\times R_+\times R^n$ satisfying

$$\nabla\sum_{i=1}^{s}t_i\left(f\left(z,\overline{y}_i\right)-\lambda h\left(z,\overline{y}_i\right)\right)+\nabla^2\sum_{i=1}^{s}t_i\left(f\left(z,\overline{y}_i\right)-\lambda h\left(z,\overline{y}_i\right)\right)p$$
$$+\nabla\sum_{j=1}^{m}\mu_jg_j(z)+\nabla^2\sum_{j=1}^{m}\mu_jg_j(z)p=0, \tag{4.2}$$

$$\sum_{i=1}^{s}t_i\left(f\left(z,\overline{y}_i\right)-\lambda h\left(z,\overline{y}_i\right)\right)+\sum_{j\in J_\circ}\mu_jg_j(z)$$
$$-\frac{1}{2}p^T\nabla^2\left[\sum_{i=1}^{s}t_i\left(f\left(z,\overline{y}_i\right)-\lambda h\left(z,\overline{y}_i\right)\right)+\sum_{j\in J_\circ}\mu_jg_j(z)\right]p\geq 0, \tag{4.3}$$

$$\sum_{j\in J_\alpha}\mu_jg_j(z)-\frac{1}{2}p^T\nabla^2\sum_{j\in J_\alpha}\mu_jg_j(z)p\geq 0,\quad \alpha=1,2,\ldots,r, \tag{4.4}$$

where $J_\alpha\subseteq M$, $\alpha=0,1,2,\ldots,r$, with $\bigcup_{\alpha=0}^{r}J_\alpha=M$ and $J_\alpha\cap J_\beta=\emptyset$, if $\alpha\neq\beta$.

If, for a triplet $(s,t,\overline{y})\in K(z)$, the set $H_2(s,t,\overline{y})=\emptyset$, then we define the supremum over it to be $-\infty$.

Theorem 4.1 (weak duality). *Let x and $(z,\mu,\lambda,s,t,\overline{y},p)$ be the feasible solutions of* (1.1) *and* (4.1)*, respectively. Suppose that there exist $\mathcal{F},\theta,\phi_0,b_0,\rho_0$ and $\phi_\beta,b_\beta,\rho_\beta,\beta=1,2,\ldots,r$ such that*

$$\begin{aligned}
&b_0(x,z)\phi_0\Bigg[\sum_{i=1}^{s}t_i\big(f(x,\overline{y}_i)-\lambda h(x,\overline{y}_i)\big)-\sum_{i=1}^{s}t_i\big(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\big)-\sum_{j\in J_0}\mu_j g_j(z)\\
&\qquad+\frac{1}{2}p^T\nabla^2\Bigg(\sum_{i=1}^{s}t_i\big(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\big)+\sum_{j\in J_0}\mu_j g_j(z)\Bigg)p\Bigg]<0\\
&\Longrightarrow \mathcal{F}\Bigg(x,z;\nabla\sum_{i=1}^{s}t_i\big(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\big)+\nabla^2\sum_{i=1}^{s}t_i\big(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\big)p\\
&\qquad+\nabla\sum_{j\in J_0}\mu_j g_j(z)+\nabla^2\sum_{j\in J_0}\mu_j g_j(z)p\Bigg)<-\rho_0\|\theta(x,z)\|^2,
\end{aligned}\tag{4.5}$$

$$\begin{aligned}
&-b_\alpha(x,z)\phi_\alpha\Bigg[\sum_{j\in J_\alpha}\mu_j g_j(z)-\frac{1}{2}p^T\nabla^2\sum_{j\in J_\alpha}\mu_j g_j(z)p\Bigg]\leq 0\\
&\Longrightarrow \mathcal{F}\Bigg(x,z;\nabla\sum_{j\in J_\alpha}\mu_j g_j(z)+\nabla^2\sum_{j\in J_\alpha}\mu_j g_j(z)p\Bigg)\leq-\rho_\alpha\|\theta(x,z)\|^2,\quad \alpha=1,2,\ldots,r.
\end{aligned}\tag{4.6}$$

Further assume that

$$a\geq 0\Longrightarrow \phi_\alpha(a)\geq 0,\quad \alpha=1,2,\ldots,r,\tag{4.7}$$

$$a<0\Longrightarrow \phi_0(a)<0,\tag{4.8}$$

$$b_0(x,z)>0,\quad b_\alpha(x,z)\geq 0,\quad \alpha=1,2,\ldots,r,\tag{4.9}$$

$$\rho_0+\sum_{\alpha=1}^{r}\rho_\alpha\geq 0.s\tag{4.10}$$

Then

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}\geq\lambda.\tag{4.11}$$

Proof. Suppose contrary to the result that

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}<\lambda.\tag{4.12}$$

Thus, we have

$$f(x,\overline{y}_i) - \lambda h(x,\overline{y}_i) < 0, \quad \forall \overline{y}_i \in Y(x), \quad i = 1,2,\ldots,s. \tag{4.13}$$

It follows from $t_i \geq 0,\ i = 1,2,\ldots,s$, that

$$t_i\left[f(x,\overline{y}_i) - \lambda h(x,\overline{y}_i)\right] \leq 0, \tag{4.14}$$

with at least one strict inequality since $t = (t_1,t_2,\ldots,t_s) \neq 0$. Taking summation over i, we have

$$\sum_{i=1}^{s} t_i\left(f(x,\overline{y}_i) - \lambda h(x,\overline{y}_i)\right) < 0, \tag{4.15}$$

which together with (4.3) implies

$$\begin{aligned}
\sum_{i=1}^{s} t_i\left(f(x,\overline{y}_i) - \lambda h(x,\overline{y}_i)\right) &< 0 \\
&\leq \sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i)\right) + \sum_{j\in J_0} \mu_j g_j(z) \\
&\quad - \frac{1}{2} p^T \nabla^2 \left[\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i)\right) + \sum_{j\in J_0} \mu_j g_j(z)\right] p.
\end{aligned} \tag{4.16}$$

Using (4.8) and (4.9), it follows from (4.16) that

$$\begin{aligned}
b_0(x,z)\phi_0 \Bigg[&\sum_{i=1}^{s} t_i\left(f(x,\overline{y}_i) - \lambda h(x,\overline{y}_i)\right) - \sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i)\right) - \sum_{j\in J_0} \mu_j g_j(z) \\
&+ \frac{1}{2} p^T \nabla^2 \left[\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i)\right) + \sum_{j\in J_0} \mu_j g_j(z)\right] p \Bigg] < 0,
\end{aligned} \tag{4.17}$$

which by (4.5) implies

$$\begin{aligned}
\mathcal{F}\Bigg(&x, z; \nabla \sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i)\right) + \nabla^2 \sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i)\right) p \\
&+ \nabla \sum_{j\in J_0} \mu_j g_j(z) + \nabla^2 \sum_{j\in J_0} \mu_j g_j(z) p \Bigg) < -\rho_0 \|\theta(x,z)\|^2.
\end{aligned} \tag{4.18}$$

Also, inequality (4.4) along with (4.7) and (4.9) yields

$$-b_{\alpha}(x,z)\phi_{\alpha}\left[\sum_{j\in J_{\alpha}}\mu_j g_j(z)-\frac{1}{2}p^T\nabla^2\sum_{j\in J_{\alpha}}\mu_j g_j(z)p\right]\leq 0,\quad \alpha=1,2,\ldots,r. \tag{4.19}$$

From (4.6) and the above inequality, we have

$$\mathcal{F}\left(x,z;\nabla\sum_{j\in J_{\alpha}}\mu_j g_j(z)+\nabla^2\sum_{j\in J_{\alpha}}\mu_j g_j(z)p\right)\leq -\rho_{\alpha}\|\theta(x,z)\|^2,\quad \alpha=1,2,\ldots,r. \tag{4.20}$$

On adding (4.18) and (4.20) and making use of the sublinearity of $\mathcal{F}$ with (4.10), we obtain

$$\begin{aligned}\mathcal{F}\Bigg(x,z;\nabla\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\right)+\nabla^2\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i)\right)p\\ +\nabla\sum_{j=1}^{m}\mu_j g_j(z)+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\Bigg)<0,\end{aligned} \tag{4.21}$$

which contradicts (4.2) since $\mathcal{F}(x,z;0)=0$. □

The proof of the following theorem is similar to that of Theorem 3.3 and, hence, is omitted.

Theorem 4.2 (strong duality). *Assume that x^* is an optimal solution of* (1.1) *and $\nabla g_j(x^*)$, $j\in J(x^*)$, are linearly independent. Then there exist $(s^*,t^*,\overline{y}^*)\in K(x^*)$ and $(x^*,\mu^*,\lambda^*,p^*=0)\in H_2(s^*,t^*,\overline{y}^*)$ such that $(x^*,\mu^*,\lambda^*,s^*,t^*,\overline{y}^*,p^*=0)$ is a feasible solution of* (4.1) *and the two objectives have the same values. Further, if the assumptions of weak duality (Theorem 4.1) hold for all feasible solutions $(z,\mu,\lambda,s,t,\overline{y},p)$ of* (4.1)*, then $(x^*,\mu^*,\lambda^*,s^*,t^*,\overline{y}^*,p^*=0)$ is an optimal solution of* (4.1).

Theorem 4.3 (strict converse duality). *Let x^* and $(z^*,\mu^*,\lambda^*,s^*,t^*,\overline{y}^*,p^*)$ be the optimal solutions of* (1.1) *and* (4.1)*, respectively. Suppose that $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent and there exist $\mathcal{F},\theta,\phi_0,b_0,\rho_0$ and $\phi_\beta,b_\beta,\rho_\beta$, $\beta=1,2,\ldots,r$ such that*

$$\begin{aligned}b_0(x^*,z^*)\phi_0\Bigg[&\sum_{i=1}^{s^*}t_i^*\left(f(x^*,\overline{y}_i^*)-\lambda^*h(x^*,\overline{y}_i^*)\right)-\sum_{i=1}^{s^*}t_i^*\left(f(z^*,\overline{y}_i^*)-\lambda^*h(z^*,\overline{y}_i^*)\right)\\ &-\sum_{j\in J_0}\mu_j^*g_j(z^*)+\frac{1}{2}p^{*T}\nabla^2\left(\sum_{i=1}^{s^*}t_i^*\left(f(z^*,\overline{y}_i^*)-\lambda^*h(z^*,\overline{y}_i^*)\right)+\sum_{j\in J_0}\mu_j^*g_j(z^*)\right)p^*\Bigg]\leq 0\end{aligned}$$

$$\Longrightarrow \mathcal{F}\left(x^*, z^*; \nabla \sum_{i=1}^{s^*} t_i^*\left(f\left(z^*, \overline{y}_i^*\right) - \lambda^* h\left(z^*, \overline{y}_i^*\right)\right) + \nabla^2 \sum_{i=1}^{s^*} t_i^*\left(f\left(z^*, \overline{y}_i^*\right) - \lambda^* h\left(z^*, \overline{y}_i^*\right)\right) p^* \right.$$
$$\left. + \nabla \sum_{j\in J_0} \mu_j^* g_j(z^*) + \nabla^2 \sum_{j\in J_0} \mu_j^* g_j(z^*) p^* \right) < -\rho_0 \|\theta(x^*, z^*)\|^2 \tag{4.22}$$

$$- b_\alpha(x^*, z^*)\phi_\alpha\left[\sum_{j\in J_\alpha} \mu_j^* g_j(z^*) - \frac{1}{2} p^{*T} \nabla^2 \sum_{j\in J_\alpha} \mu_j^* g_j(z^*) p^*\right] \le 0$$
$$\Longrightarrow \mathcal{F}\left(x^*, z^*; \nabla \sum_{j\in J_\alpha} \mu_j^* g_j(z^*) + \nabla^2 \sum_{j\in J_\alpha} \mu_j^* g_j(z^*) p^*\right) \le -\rho_\alpha \|\theta(x^*, z^*)\|^2, \quad \alpha = 1, 2, \ldots, r. \tag{4.23}$$

Further assume that

$$a \ge 0 \Longrightarrow \phi_\alpha(a) \ge 0, \quad \alpha = 1, 2, \ldots, r, \tag{4.24}$$

$$a < 0 \Longrightarrow \phi_0(a) \le 0, \tag{4.25}$$

$$b_0(x^*, z^*) > 0, \quad b_\alpha(x^*, z^*) \ge 0, \quad \alpha = 1, 2, \ldots, r, \tag{4.26}$$

$$\rho_0 + \sum_{\alpha=1}^{r} \rho_\alpha \ge 0. \tag{4.27}$$

Then $z^* = x^*$, that is, z^* is an optimal solution of (1.1).

Proof. Suppose contrary to the result that $z^* \neq x^*$. Since x^* and $(z^*, \mu^*, \lambda^*, s^*, t^*, \overline{y}^*, p^*)$ are optimal solutions of (1.1) and (4.1), respectively, and $\nabla g_j(x^*), j \in J(x^*)$ are linearly independent, therefore, from strong duality (Theorem 4.2), we reach

$$\sup_{y^*\in Y} \frac{f(x^*, y^*)}{h(x^*, y^*)} = \lambda^*. \tag{4.28}$$

Thus, we have

$$f\left(x^*, \overline{y}_i^*\right) - \lambda^* h\left(x^*, \overline{y}_i^*\right) \le 0, \quad \forall \overline{y}_i^* \in Y(x^*), \quad i = 1, 2, \ldots, s^*. \tag{4.29}$$

Now, proceeding as in Theorem 4.1, we get

$$\sum_{i=1}^{s^*} t_i^*\left(f\left(x^*, \overline{y}_i^*\right) - \lambda^* h\left(x^*, \overline{y}_i^*\right)\right) - \sum_{i=1}^{s^*} t_i^*\left(f\left(z^*, \overline{y}_i^*\right) - \lambda^* h\left(z^*, \overline{y}_i^*\right)\right)$$
$$- \sum_{j\in J_0} \mu_j^* g_j(z^*) + \frac{1}{2} p^{*T} \nabla^2 \left(\sum_{i=1}^{s^*} t_i^*\left(f\left(z^*, \overline{y}_i^*\right) - \lambda^* h\left(z^*, \overline{y}_i^*\right)\right) + \sum_{j\in J_0} \mu_j^* g_j(z^*)\right) p^* < 0. \tag{4.30}$$

Using (4.25) and (4.26), it follows from (4.30) that

$$b_0(x^*,z^*)\phi_0\left[\sum_{i=1}^{s^*}t_i^*\left(f\left(x^*,\overline{y}_i^*\right)-\lambda^*h\left(x^*,\overline{y}_i^*\right)\right)-\sum_{i=1}^{s^*}t_i^*\left(f\left(z^*,\overline{y}_i^*\right)-\lambda^*h\left(z^*,\overline{y}_i^*\right)\right)-\sum_{j\in J_0}\mu_j^*g_j(z^*)\right.$$
$$\left.+\frac{1}{2}p^{*T}\nabla^2\left(\sum_{i=1}^{s^*}t_i^*\left(f\left(z^*,\overline{y}_i^*\right)-\lambda^*h\left(z^*,\overline{y}_i^*\right)\right)+\sum_{j\in J_0}\mu_j^*g_j(z^*)\right)p^*\right]\leq 0, \tag{4.31}$$

which by (4.22) implies

$$\mathcal{F}\left(x^*,z^*;\nabla\sum_{i=1}^{s^*}t_i^*\left(f\left(z^*,\overline{y}_i^*\right)-\lambda^*h\left(z^*,\overline{y}_i^*\right)\right)+\nabla^2\sum_{i=1}^{s^*}t_i^*\left(f\left(z^*,\overline{y}_i^*\right)-\lambda^*h\left(z^*,\overline{y}_i^*\right)\right)p^*\right.$$
$$\left.+\nabla\sum_{j\in J_0}\mu_j^*g_j(z^*)+\nabla^2\sum_{j\in J_0}\mu_j^*g_j(z^*)p^*\right)<-\rho_0\|\theta(x^*,z^*)\|^2. \tag{4.32}$$

Also, inequality (4.4) along with (4.24) and (4.26) yields

$$-b_\alpha(x^*,z^*)\phi_\alpha\left[\sum_{j\in J_\alpha}\mu_j^*g_j(z^*)-\frac{1}{2}p^{*T}\nabla^2\sum_{j\in J_\alpha}\mu_j^*g_j(z^*)p^*\right]\leq 0,\quad \alpha=1,2,\ldots,r. \tag{4.33}$$

From (4.23) and the above inequality, we have

$$\mathcal{F}\left(x^*,z^*;\nabla\sum_{j\in J_\alpha}\mu_j^*g_j(z^*)+\nabla^2\sum_{j\in J_\alpha}\mu_j^*g_j(z^*)p^*\right)\leq-\rho_\alpha\|\theta(x^*,z^*)\|^2,\quad \alpha=1,2,\ldots,r. \tag{4.34}$$

On adding (4.32) and (4.34) and making use of the sublinearity of $\mathcal{F}$ with (4.27), we obtain

$$\mathcal{F}\left(x^*,z^*;\nabla\sum_{i=1}^{s^*}t_i^*\left(f\left(z^*,\overline{y}_i^*\right)-\lambda^*h\left(z^*,\overline{y}_i^*\right)\right)+\nabla^2\sum_{i=1}^{s^*}t_i^*\left(f\left(z^*,\overline{y}_i^*\right)-\lambda^*h\left(z^*,\overline{y}_i^*\right)\right)p^*\right.$$
$$\left.+\nabla\sum_{j=1}^{m}\mu_j^*g_j(z^*)+\nabla^2\sum_{j=1}^{m}\mu_j^*g_j(z^*)p^*\right)<0, \tag{4.35}$$

which contradicts (4.2) since $\mathcal{F}(x^*,z^*;0)=0$. □

5. Conclusion and Further Developments

In this paper, we have established weak, strong, and strict converse duality theorems for a class of minimax fractional programming problems in the frame work of second-order generalized convexity. The second-order duality results developed in this paper can be further extended for the following nondifferentiable minimax fractional programming problem [22, 23]:

$$\begin{aligned} &\text{minimize} \quad \psi(x) = \sup_{y \in Y} \frac{f(x,y) + \left(x^T B x\right)^{1/2}}{h(x,y) - \left(x^T D x\right)^{1/2}}, \\ &\text{subject to} \quad g(x) \leq 0, \quad x \in R^n, \end{aligned} \tag{5.1}$$

where Y is a compact subset of R^l, B and D are $n \times n$ positive semidefinite symmetric matrices, and $f(\cdot,\cdot) : R^n \times R^l \to R$, $h(\cdot,\cdot) : R^n \times R^l \to R$, and $g(\cdot) : R^n \to R^m$ are twice continuously differentiable functions on $R^n \times R^l$, $R^n \times R^l$, and R^n, respectively.

The question arises as to whether the second-order fractional duality results developed in this paper hold for the following complex nondifferentiable minimax fractional problem:

$$\begin{aligned} &\text{minimize} \quad \Psi(\xi) = \sup_{\nu \in W} \frac{\operatorname{Re}\left[f(\xi,\nu) + \left(z^T B z\right)^{1/2}\right]}{\operatorname{Re}\left[h(\xi,\nu) - \left(z^T D z\right)^{1/2}\right]}, \\ &\text{subject to} \quad -g(\xi) \in S, \quad \xi \in C^{2n}, \end{aligned} \tag{5.2}$$

where $\xi = (z, \overline{z})$, $\nu = (\omega, \omega)$ for $z \in C^n$, $\omega \in C^l$, $f(\cdot,\cdot) : C^{2n} \times C^{2l} \to C$ and $h(\cdot,\cdot) : C^{2n} \times C^{2l} \to C$ are analytic with respect to ω, W ia a specified compact subset in C^{2l}, S is a polyhedral cone in C^m, and $g : C^{2n} \to C^m$ is analytic. Also $B, D \in C^{n\times n}$ are positive semidefinite Hermitian matrices.

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