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Research Article

Green's-Like Relations on Algebras and Varieties

K. Denecke[1] and S. L. Wismath[2]

[1] *Institut für Mathematik, Universität Potsdam, D-14415 Potsdam, PF 601553 , Germany*
[2] *Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, AB., Canada T1K-3M4*

Correspondence should be addressed to S. L. Wismath, wismaths@uleth.ca

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There are five equivalence relations known as Green's relations definable on any semigroup or monoid, that is, on any algebra with a binary operation which is associative. In this paper, we examine whether Green's relations can be defined on algebras of any type τ. Some sort of (super-)associativity is needed for such definitions to work, and we consider algebras which are clones of terms of type τ, where the clone axioms including superassociativity hold. This allows us to define for any variety V of type τ two Green's-like relations $\mathcal{L}_V$ and $\mathcal{R}_V$ on the term clone of type τ. We prove a number of properties of these two relations, and describe their behaviour when V is a variety of semigroups.

1. Introduction

A semigroup is an algebra of type (2) for which the single binary operation satisfies the associativity identity. A monoid is a semigroup with an additional nullary operation which acts as an identity element for the binary operation. On any semigroup or monoid, the five equivalence relations known as Green's relations provide information about the structure of the semigroup.

To define Green's relations on a semigroup $\mathcal{A}$, we follow the convention of denoting the binary operation of the semigroup by juxtaposition. For any elements a and b of A, we say that $a\mathcal{L}^A b$ if and only if $a = b$ or there exist some c and d in A such that $ca = b$ and $db = a$. When the semigroup $\mathcal{A}$ is clear from the context, we usually omit the superscript A on the name of the relation $\mathcal{L}^A$ and just write $a\mathcal{L}b$. Dual to this "left" relation is the "right" relation $\mathcal{R}$ defined by $a\mathcal{R}b$ if and only if $a = b$ or there exist c and d in A such that $ac = b$ and $bd = a$. Both $\mathcal{L}$ and $\mathcal{R}$ are equivalence relations on any semigroup $\mathcal{A}$. The remaining Green's relations are $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$, $\mathfrak{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$, and $\mathcal{J}$, defined by $a\mathcal{J}b$ if and only if $a = b$ or there exist elements c, d, p and q in A such that $a = cbd$ and $b = paq$. For more information about Green's relations in general, we refer the reader to [1].

In this paper, we consider how one might extend the definitions of the five Green's relations to algebras of any arbitrary type. In Section 2, we propose some definitions for $\mathcal{L}$ and $\mathcal{R}$, and show what properties are needed to make our relations into equivalence relations. Then we consider a variation which extends our definition of two relations $\mathcal{L}$ and $\mathcal{R}$ to relations $\mathcal{L}_V$ and $\mathcal{R}_V$ on the term clone of any variety V. In Section 3, we deduce a number of properties of these two relations, and then in Section 4 we examine their behaviour when V is a variety of semigroups.

2. Green's relations for any type

We begin with some notation. Throughout this paper, we will assume a type $\tau = (n_i)_{i\in I}$, with an n_i-ary operation symbol f_i for each index i in some set I. For each $n \geq 1$, we let $X_n = \{x_1, \dots, x_n\}$ be an n-element alphabet of variables, and let $W_\tau(X_n)$ be the set of all n-ary terms of type τ. Then we set $X = \{x_1, x_2, x_3, \dots\}$, and let $W_\tau(X)$ denote the set of all (finitary) terms of type τ. Terms can be represented by tree diagrams called semantic trees. We will use the well-known Galois connection Id-Mod between classes of algebras and sets of identities. For any class K of algebras of type τ and any set Σ of identities of type τ, Mod Σ is the class of all algebras $\mathcal{A}$ of type τ which satisfy all the identities in Σ, while IdK is the set of all identities $s \approx t$ of type τ which are satisfied by all algebras in K.

As a preliminary step in defining Green's relations on any algebra of arbitrary type, let us consider first the case of type $\tau = (n)$, where we have a single operation symbol f of arity $n \geq 1$. In analogy with the two left and right Green's relations $\mathcal{L}$ and $\mathcal{R}$ for type (2), we can define n different Green's-like relations here. Let $\mathcal{A}$ be an algebra of type (n) and let a and b be elements of A. For each $1 \leq j \leq n$, set $a\,\mathcal{G}_j\,b$ if and only if $a = b$ or there exist elements $b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_n$ and $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n$ in A such that

$$a = f^A(b_1, \dots, b_{j-1}, b, b_{j+1}, \dots, b_n), \qquad b = f^A(a_1, \dots, a_{j-1}, a, a_{j+1}, \dots, a_n). \tag{2.1}$$

Each $\mathcal{G}_j$ for $1 \leq j \leq n$ is clearly a reflexive and symmetric relation on A, but as we will see is not necessarily transitive for $n \geq 2$. Of particular interest are the two relations $\mathcal{G}_1$ and $\mathcal{G}_n$, which we will denote by $\mathcal{R}$ and $\mathcal{L}$, respectively.

Example 2.1. Let $\tau = (1)$ be a type with one unary operation symbol f. In this case $\mathcal{L} = \mathcal{R}$, and we see that for any algebra $\mathcal{A} = (A; f^A)$ and any elements $a, b \in A$, we have $a\mathcal{L}b$ if and only if $a = b$, or $a = f^A(b)$ and $b = f^A(a)$. Thus two distinct elements are related if and only if there is a cycle between them in the algebra $\mathcal{A}$. The relation $\mathcal{L}$ is transitive and hence an equivalence relation: if $a\mathcal{L}b$ and $b\mathcal{L}c$, and $a \neq b$ and $b \neq c$, then we have $a = f^A(b)$, $b = f^A(a)$, $b = f^A(c)$, and $c = f^A(b)$. This forces $a = c = f^A(b)$, and so $a\mathcal{L}c$. This also tells us that each element $b \in A$ can be $\mathcal{L}$-related to at most one element other than itself.

If the type (1) algebra $\mathcal{A}$ has no cycles in it, we get simply $\mathcal{L} = \Delta_A$, the diagonal relation on A. If $A = \{a, b\}$ with $f^A(a) = b$ and $f^A(b) = a$, then $\mathcal{L} = A \times A$. An algebra $\mathcal{A}$ in which there are some cycles but not every element that has a cycle will result in $\mathcal{L}$ strictly between Δ_A and $A \times A$.

Now consider an algebra $\mathcal{A}$ of an arbitrary type τ. Since there can be different operation symbols of different arities in our type, we cannot define our relations $\mathcal{G}_j$ using the jth position as before. But we can use the first and last position entries to define left and right relations. This motivates the following definition.

Definition 2.2. Let $\mathcal{A}$ be any algebra of type τ. We define relations $\mathcal{R}$ and $\mathcal{L}$ on A as follows. For any $a, b \in A$, we set

(i) $a\mathcal{R}b$ if and only if $a = b$ or $a = f_i^A(b, b_2, \ldots, b_{n_i})$ and $b = f_k^A(a, a_2, \ldots, a_{n_k})$, for some $i, k \in I$ and some elements $b_2, \ldots, b_{n_i}$ and $a_2, \ldots, a_{n_k}$ in A.

(ii) $a\mathcal{L}b$ if and only if $a = b$ or $a = f_i^A(b_1, \ldots, b_{n_i-1}, b)$ and $b = f_k^A(a_1, \ldots, a_{n_k-1}, a)$, for some $i, k \in I$ and some elements $b_1, \ldots, b_{n_i-1}$ and $a_1, \ldots, a_{n_k-1}$ in A.

Again these two relations are clearly seen to be reflexive and symmetric on the base set A of any algebra $\mathcal{A}$. It is the requirement of transitivity that causes problems, and forces us to impose some restrictions on our algebra. For transitivity of $\mathcal{R}$ on an algebra $\mathcal{A}$, suppose that a, b, and c are in A, $a\mathcal{R}b$, and $b\mathcal{R}c$. In the special cases that $a = b$ or $b = c$, we certainly have $a\mathcal{R}c$, so let us assume that $a \neq b$ and $b \neq c$. Then we have $a = f_i^A(b, b_2, \ldots, b_{n_i})$ and $b = f_k^A(a, a_2, \ldots, a_{n_k})$, and also $b = f_p^A(c, c_2, \ldots, c_{n_p})$ and $c = f_q^A(b, d_1, \ldots, d_{n_q})$, for some operation symbols f_i, f_k, f_p, and f_q of our type and some elements $b_2, \ldots, b_{n_i}$, $a_2, \ldots, a_{n_k}$, $c_2, \ldots c_{n_p}$, $d_2, \ldots, d_{n_q}$ of set A. By substitution, we get

$$a = f_i^A\left(f_p^A\left(c, c_2, \ldots, c_{n_p}\right), b_2, \ldots, b_{n_i}\right), \qquad c = f_q^A\left(f_k^A\left(a, a_2, \ldots, a_{n_k}\right), d_1, \ldots, d_{n_q}\right). \tag{2.2}$$

But we need to be able to express a as $f_m^A(c, e_2, \ldots, e_{n_m})$ for some operation symbol f_m and some elements $e_2, \ldots, e_{n_m}$. For type (2), this is dealt with by the requirement that $f^A(f^A(c, c_2), b_2)$ can be changed to $f^A(c, f^A(c_2, b_2))$, that is, we have associativity in our algebra $\mathcal{A}$. For arbitrary types, it would suffice here to have a superassociative algebra, satisfying the superassociative law:

$$f_i\left(f_j\left(x_1, x_2, \ldots, x_{n_j}\right), y_2, \ldots, y_{n_i}\right) \approx f_j\left(x_1, f_i\left(x_2, y_2, \ldots, y_{n_i}\right), \ldots, f_i\left(x_{n_j}, y_2, \ldots, y_{n_i}\right)\right). \tag{2.3}$$

This identity would allow us to express a as an element with c in the left-most position and similarly to express c in terms of a. Another way to handle this would be to define $a\mathcal{R}b$ when $a = b$ or $a = t_1^A(b, b_2, \ldots, b_n)$ and $b = t_2^A(a, a_2, \ldots, a_m)$ for some term operations t_1^A, t_2^A on $\mathcal{A}$ and some elements $a_2, \ldots, a_m, b_2, \ldots, b_n$ of A. In either approach, we are led to consider clones of terms.

A *clone* is an important kind of algebra which satisfies a superassociative law that we need here. Although clones may be defined more generally (see [2]) we define here only the *term clone* of type τ. This term clone is a heterogeneous or multi-based algebra, having as universes or base sets the sets $W_\tau(X_n)$ of n-ary terms of type τ, for $n \geq 1$. For each $n \geq 1$, the n variable terms $x_1, \ldots, x_n$ are selected as nullary operations $e_1^n, \ldots, e_n^n$. And for each pair n, m of natural numbers, there is a superposition operation S_m^n, from $W_\tau(X_n) \times (W_\tau(X_m))^n$ to $W_\tau(X_m)$, defined by $S_m^n(s, t_1, \ldots, t_n) = s(t_1, \ldots, t_n)$.

This gives us the algebra

$$\text{clone}\,\tau := \left(W_\tau(X_n);\, S_m^n, e_i^n\right)_{n,m\geq 1, 1\leq i\leq n} \tag{2.4}$$

called the term clone of type τ. It satisfies the following three axioms called the *clone axioms*:

(C1) $S_m^p(z, S_m^n(y_1, x_1, \ldots, x_n), \ldots, S_m^n(y_p, x_1, \ldots, x_n)) \approx S_m^n(S_n^p(z, y_1, \ldots, y_p), x_1, \ldots, x_n)$, for m, $n, p \geq 1$;

(C2) $S_m^n(e_i^n(x_1, \ldots, x_n)) \approx x_i$, for $m, n \geq 1$ and $1 \leq i \leq n$;

(C3) $S_n^n(y, e_1^n, \ldots, e_n^n) \approx y$, for $n \geq 1$.

Definition 2.3. Let $\tau = (n_i)_{i\in I}$be any type, and let $(S^n_m)_{n,m\geq 1}$be the superposition operations on the term clone, clone τ. One defines two relations $\mathcal{R}$ and $\mathcal{L}$ on clone τ as follows. For any terms s and t in clone τ, of arities m and n, respectively,

(i) $s\mathcal{R}t$ if and only if $s = t$, or $s = S^m_n(t, t_1, \ldots, t_m)$ and $t = S^n_m(s, s_1, \ldots, s_n)$ for some terms $t_1, \ldots, t_m$ and $s_1, \ldots, s_n$ in clone τ;

(ii) $s\mathcal{L}t$ if and only if $s = t$, or $m = n$ and $s = S^m_m(t_1, \ldots, t_m, t)$ and $t = S^m_m(s_1, \ldots, s_m, s)$ for some terms $t_1, \ldots t_m$ and $s_1, \ldots s_m$ in clone τ.

Lemma 2.4. *For any type τ, the relation $\mathcal{R}$ defined on* clone τ *is an equivalence relation on* clone τ.

Proof. As noted above, both relations $\mathcal{R}$ and $\mathcal{L}$ are reflexive and symmetric by definition. Transitivity for $\mathcal{R}$ follows from the clone axiom (C1) as above. □

Transitivity of $\mathcal{L}$ does not follow directly from the clone axioms. We will show later that this relation is transitive, once we have deduced more information about it.

A similar definition of a Green's-like relation $\mathcal{R}$ was defined by Denecke and Jampachon in [3], but in the restricted special case of a Menger algebra of rank n. These are algebras of type $(n, 0, \ldots, 0)$, having one n-ary operation and n-nullary ones. Menger algebras can be formed using terms as the following: the base set $W_\tau(X_n)$ of all n-ary terms of type τ, along with the superposition operation S^n_n and the n-variable terms $x_1, \ldots, x_n$, form a Menger algebra of rank n called the n-clone of type τ. Such algebras also satisfy the clone axioms (C1), (C2), and (C3) (restricted to S^n_n). Denecke and Jampachon also defined a left Green's-like relation as well, again on the Menger algebra of rank n. Their left relation is a subset of our relation $\mathcal{L}$, and we will use the name $\overline{\mathcal{L}}$ in the next definition for the analogous relation in the term clone case.

Now, we extend our definition of Green's relations $\mathcal{L}$ and $\mathcal{R}$ on clone τ, to relations with respect to varieties of type τ.

Definition 2.5. Let V be any variety of type τ. One defines relations $\mathcal{R}_V$, $\mathcal{L}_V$, and $\overline{\mathcal{L}_V}$ on clone τ as follows. Let s and t be terms of type τ, of arities m and n, respectively. Then

(i) $s\mathcal{R}_V t$ if and only if $s = t$, or

$$s \approx S^m_n(t, t_1, \ldots, t_n) \in \mathrm{Id}V, \qquad t \approx S^n_m(s, s_1, \ldots s_m) \in \mathrm{Id}V \tag{2.5}$$

for some terms $t_1, \ldots, t_n$ and $s_1, \ldots, s_m$ in clone τ;

(ii) $s\mathcal{L}_V t$ if and only if $n = m$, and $s = t$ or

$$s \approx S^m_m(t_1, t_2, \ldots, t_m, t) \in \mathrm{Id}V, \qquad t \approx S^m_m(s_1, s_2, \ldots s_m, s) \in \mathrm{Id}V \tag{2.6}$$

for some terms $t_1, \ldots, t_m$ and $s_1, \ldots, s_m$ in clone τ;

(iii) $s\overline{\mathcal{L}_V}t$ if and only if $n = m$, and $s = t$ or

$$s \approx S^m_m(t_1, t, \ldots, t) \in \mathrm{Id}V, \qquad t \approx S^m_m(s_1, s, \ldots, s) \in \mathrm{Id}V \tag{2.7}$$

for some terms t_1 and s_1 in clone τ.

This definition actually includes Definition 2.3 as a special case: when V equals the variety $\mathrm{Alg}\,(\tau)$ of all algebras of type τ, the relation $\mathrm{Id}V$ is simply equality on $\mathrm{clone}\,\tau$ and we obtain the relations of Definition 2.3. We remark that similar definitions could be made for $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{L}_{\mathcal{A}}$ for any algebra $\mathcal{A}$, using identities of $\mathcal{A}$. Another possible variation is to restrict the existence of the terms $t_1, \dots, t_n$ and $s_1, \dots, s_m$ to terms from some subclone C of $\mathrm{clone}\,\tau$; in this case we could define subrelations $\mathcal{R}_V^C$ and $\mathcal{L}_V^C$.

The proof of the following lemma is similar to that of Lemma 2.4.

Lemma 2.6. *For any type τ and any variety V of type τ, the relation $\mathcal{R}_V$ defined on* $\mathrm{clone}\,\tau$ *is an equivalence relation on* $\mathrm{clone}\,\tau$.

3. The relations $\mathcal{R}_V$ and $\mathcal{L}_V$

In this section, we describe some properties of the relations $\mathcal{R}_V$, $\mathcal{L}_V$, and $\overline{\mathcal{L}_V}$, for any variety V. We begin with the relation $\mathcal{L}_V$.

Proposition 3.1. *Let V be any variety of type τ. Then*

(i) *two terms of type τ of arity, at least two, are $\mathcal{L}_V$-related if and only if they have the same arity;*
(ii) *the relation $\mathcal{L}_V$ is an equivalence relation on the set $W_\tau(X)$ of all terms of type τ.*

Proof. (i) It follows from the definition of superposition of terms that the term $S_m^n(t_1, t_2, \dots, t_m, t)$ has the same arity as t. Thus it is built into the definition of $\mathcal{L}_V$ that any two terms which are $\mathcal{L}_V$-related must have the same arity. Conversely, let both s and t be terms of arity $n \geq 2$. Then we can write $s = S_n^n(x_1, s, \dots, s, t)$ and $t = S_n^n(x_1, t, \dots, t, s)$, making $s \approx S_n^n(x_1, s, \dots, s, t) \in \mathrm{Id}V$ and $t \approx S_n^n(x_1, t, \dots, t, s) \in \mathrm{Id}V$ for any variety V, and so $s \,\mathcal{L}_V\, t$.

(ii) For any variety V, $\mathcal{L}_V$ is by definition reflexive and symmetric, and we need only verify transitivity. Since only elements of the same arity can be related, we see that $\mathcal{L}_V$ makes a partition of $W_\tau(X)$ in which all elements of $W_\tau(X_n)$ are related to each other for $n \geq 2$. This means that it suffices to verify transitivity for unary terms only. Let s, t, and u be unary terms with $s \,\mathcal{L}_V\, t$ and $t \,\mathcal{L}_V\, u$. Then there exist unary terms a, b, c, and d such that $s \approx S_1^1(a, t)$, $t \approx S_1^1(b, s)$, $t \approx S_1^1(c, u)$, and $u \approx S_1^1(d, t)$ all hold in $\mathrm{Id}V$. Then by substitution and the clone axiom (C1), we have $s \approx S_1^1(a, S_1^1(c, u)) \approx S_1^1(S_1^1(a, c), u)$ in $\mathrm{Id}V$, and similarly $u \approx S_1^1(S_1^1(d, b), s)$ in $\mathrm{Id}V$. This makes $s \,\mathcal{L}_V\, u$ as required. □

We have shown that any two terms of the same arity $n \geq 2$ are $\mathcal{L}_V$-related, for any variety V. Which unary terms are related, however, depends on the variety V. For instance, if the operation f_i is idempotent in V, we can express the unary terms x_1 and $f_i(x_1, \dots, x_1)$ in terms of each other:

$$x_1 \approx S_1^1(x_1, f_i(x_1, \dots, x_1)) \in \mathrm{Id}V, \qquad f_i(x_1, \dots, x_1) \approx S_1^1(f(x_1, \dots, x_1), x_1) \in \mathrm{Id}V. \tag{3.1}$$

Thus x_1 and $f_i(x_1, \dots, x_1)$ are $\mathcal{L}_V$-related when f_i is idempotent; but these terms need not be related if f_i is not idempotent. This question will be investigated in more detail in Section 4.

Proposition 3.2. *Let* $\mathrm{Alg}\,(\tau)$ *be the class of all algebras of type τ. The relation $\overline{\mathcal{L}_{\mathrm{Alg}\,(\tau)}}$ is equal to the identity relation $\Delta_{W_\tau(X)}$ on $W_\tau(X)$.*

Proof. This was proved in [3] for the analogous relation $\overline{\mathcal{L}_{\mathrm{Alg}\,(\tau)}}$ defined on the rank n Menger algebra, the n-clone of type τ. Since terms are $\overline{\mathcal{L}_V}$-related only if they have the same arity, the same proof covers the general term-clone case as well. □

Example 3.3. Let V be an idempotent variety of type τ. Then it is easy to show that for any terms s and t of the same arity n, we have $S_n^n(s,t,\ldots,t) \approx t \in \mathrm{Id}V$. It follows from this that $s \approx S_n^n(p,t,\ldots,t) \in \mathrm{Id}V$ for some term p if and only if $s \approx t \in \mathrm{Id}V$. This means that for any terms s and t, we have $s\overline{\mathcal{L}_V}t$ if and only if s and t have the same arity and $s \approx t \in \mathrm{Id}V$. In particular, any two unary terms of type τ are $\overline{\mathcal{L}_V}$-related in this case. Combining this with Proposition 3.1 and the fact that $\overline{\mathcal{L}_V} \subseteq \mathcal{L}_V$ shows that when V is idempotent, two terms are $\mathcal{L}_V$-related if and only if they have the same arity. We see also that $\overline{\mathcal{L}_V}$ is a proper subset of $\mathcal{L}_V$ when V is an idempotent variety.

Next we consider the right relation $\mathcal{R}_V$. Denoting by $\mathcal{L}(\tau)$ the lattice of all varieties of type τ, ordered by inclusion, we show first that $\mathcal{R}_V$ is order-reversing as an operator on $\mathcal{L}(\tau)$.

Lemma 3.4. (i) *For any varieties $U, W \in \mathcal{L}(\tau)$, if $U \subseteq W$, then $\mathcal{R}_W \subseteq \mathcal{R}_U$.*

(ii) *If $\mathcal{R}_V$ is equal to $W_\tau(X)^2$ for some variety V, then $\mathcal{R}_W = W_\tau(X)^2$ for all varieties $W \subseteq V$.*

Proof. (i) follows immediately from the fact that $\mathrm{Id}W \subseteq \mathrm{Id}U$ when $U \subseteq W$, and (ii) follows immediately from (i). □

Now we want to prove some facts about which pairs of terms can be $\mathcal{R}_V$-related. Recall that $X = \{x_1, x_2, x_3, \ldots\}$ is the set of all variables used in forming terms. Our first observation is that for any two variables x_j and x_k of arities n and m, respectively, we can write $x_j = S_m^n(x_k, x_j, \ldots, x_j)$. This shows that any two variables are $\mathcal{R}_V$-related, for any variety V; we write this as $X \times X \subseteq \mathcal{R}_V$. Next suppose that $s \approx t$ is an identity of V, with s of arity n and t of arity m. Then $s \approx S_n^m(t, x_1, \ldots, x_m) \in \mathrm{Id}V$ and $t \approx S_m^n(s, x_1, \ldots, x_n) \in \mathrm{Id}V$, making $s\,\mathcal{R}_V t$. Identifying the set $\mathrm{Id}V$ of all identities of V with the subset $\{(s,t) \mid s \approx t \in \mathrm{Id}V\}$ of $W_\tau(X)^2$, we see that $\mathrm{Id}V \subseteq \mathcal{R}_V$.

Example 3.5. Let V be the trivial variety TR_τ of type τ, defined by the identity $x_1 \approx x_2$. Then $\mathrm{Id}V = W_\tau(X)^2$, since any identity is satisfied in V. From this and the previous comments, it follows that $\mathcal{R}_V$ also equals $W_\tau(X)^2$ for this choice of V.

To further describe $\mathcal{R}_V$, we need more notation. For any $m \geq 1$, let Sym_m be the symmetric group of permutations of the set $\{1, 2, \ldots, m\}$. Let $s = s(x_1, \ldots, x_n)$ be an n-ary term. For any $m \geq n$ and any permutation $\pi \in \mathrm{Sym}_m$, we will denote by $\pi(s)$ the m-ary term $S_m^n(s, x_{\pi(1)}, \ldots, x_{\pi(n)})$. That is, $\pi(s)$ is the term formed from s by relabelling the variables in s according to the permutation π.

Proposition 3.6. *Let V be any variety of type τ. For any term s of type τ of arity n, and any permutation $\pi \in \mathrm{Sym}_m$, where $m \geq n$, one gets $s\,\mathcal{R}_V\,\pi(s)$.*

Proof. By definition $\pi(s) = S_m^n(s, x_{\pi(1)}, \ldots, x_{\pi(n)})$, so that $\pi(s) \approx S_m^n(s, x_{\pi(1)}, \ldots, x_{\pi(n)}) \in \mathrm{Id}V$. For the other direction, to express s using $\pi(s)$, we use the inverse permutation $\pi^{-1} \in \mathrm{Sym}_m$:

$$\begin{aligned} &S_n^m\left(\pi(s), x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(m)}\right) \\ &\quad = S_n^m\left(S_m^n\left(s, x_{\pi(1)}, \ldots, x_{\pi(n)}\right), x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(m)}\right) \\ &\quad = S_n^n\left(s, S_n^m\left(x_{\pi(1)}, x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(m)}\right), \ldots, S_n^m\left(x_{\pi(n)}, x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(m)}\right)\right) \quad \text{by (C1)} \\ &\quad = S_n^n\left(s, x_1, \ldots, x_n\right) = s. \end{aligned} \tag{3.2}$$

This gives an identity in $\mathrm{Id}V$ and shows that $\pi(s)\,\mathcal{R}_V\,s$. □

Definition 3.7. Let Σ be any set of identities. For any identity $s \approx t$ in Σ, with s of arity n and t of arity m, let $\pi \in \mathrm{Sym}_k$ and $\rho \in \mathrm{Sym}_r$ for $k \geq n$ and $r \geq m$. Denote by Perm (Σ) the set of all pairs $(\pi(s), \rho(t))$ in $W_\tau(X)^2$ formed in this way from identities $s \approx t$ in Σ.

Proposition 3.8. *Let V be any variety of type τ. Then* $(X \times X) \cup \mathrm{Id}V \subseteq \mathrm{Perm}\,(\mathrm{Id}V) \subseteq \mathcal{R}_V$.

Proof. First note that any identity $x_j \approx x_k$ in $X \times X$ can be produced by applying two permutations π and ρ to the identity $x_1 \approx x_1$ from $\mathrm{Id}V$, so we have $X \times X \subseteq \mathrm{Perm}\,(\mathrm{Id}V)$. The existence of identity permutations also gives us $\mathrm{Id}V \subseteq \mathrm{Perm}\,(\mathrm{Id}V)$.

Now let $s \approx t$ be an identity of V, with π and ρ permutations on the appropriate sets. We saw above that $s\,\mathcal{R}_V\,t$, and by Proposition 3.6 also $s\,\mathcal{R}_V \pi(s)$ and $t\,\mathcal{R}_V\,\rho(t)$. By the symmetry and transitivity of $\mathcal{R}_V$ we get $\pi(s)\mathcal{R}_V\,\rho(t)$. This shows that $\mathrm{Perm}\,(\mathrm{Id}V) \subseteq \mathcal{R}_V$. □

We note that as a consequence of Proposition 3.8, the equivalence relation $\mathcal{R}_V$ is not in general an equational theory on $W_\tau(X)$. The only equational theory in which any two variables are related is $\mathrm{Id}V$ for V equal to the trivial variety.

Example 3.9. In this example we consider $V = \mathrm{Alg}\,(\tau)$, the variety of all algebras of type τ. It is well-known that for this variety V, $\mathrm{Id}V = \Delta_{W_\tau(X)}$, the identity relation on $W_\tau(X)$; that is, an identity $s \approx t$ holds in V if and only if $s = t$. From Proposition, we know that $\mathrm{Perm}\,(\Delta_{W_\tau(X)})$ is a subset of $\mathcal{R}_V$, and we will show that we have equality in this case. Let s and t be terms of arities n and m, respectively, and suppose that $s\,\mathcal{R}_V\,t$. Without loss of generality, let us assume that $n \geq m$. Then there exist terms $t_1, \ldots, t_m$ and $s_1, \ldots, s_n$ in $W_\tau(X)$ such that

$$s \approx S^m_n(t, t_1, \ldots, t_m) \in \mathrm{Id}V, \qquad t \approx S^n_m(s, s_1, \ldots s_n) \in \mathrm{Id}V. \tag{3.3}$$

The property that $\mathrm{Id}V = \Delta_{W_\tau(X)}$ means that

$$s = S^m_n(t, t_1, \ldots, t_m), \qquad t = S^n_m(s, s_1, \ldots s_n). \tag{3.4}$$

Then we have

$$\begin{aligned} s = \; & s(x_1, \ldots, x_n) = S^m_n(t, t_1, \ldots, t_m) \\ & = S^m_n(S^n_m(s, s_1, \ldots, s_n), t_1, \ldots, t_m) \\ & = S^n_n(s, S^m_n(s_1, t_1, \ldots, t_m), \ldots, S^m_n(s_n, t_1, \ldots, t_m)), \quad \text{by (C1)} \end{aligned} \tag{3.5}$$

This equality forces a strong condition on the entries in the last line. Suppose that the variables occurring in term s are $x_{i_1}, \ldots, x_{i_k}$, with $k \leq n$. Then we must have $S^m_n(s_{i_j}, t_1, \ldots, t_m) = x_{i_j}$ for each $j = 1, 2, \ldots, k$. Then for each index i_j there must exist an index l_j such that $s_{i_j} = x_{l_j}$ and $t_{i_j} = x_{i_j}$. Moreover the indices l_j, for $1 \leq j \leq k$ must be distinct. This means that there is a permutation π on the set $\{1, 2, \ldots, n\}$, such that $\pi(i_j) = l_j$, for $q \leq j \leq k$. Then we have

$$\begin{aligned} t = \; & t(x_1, \ldots, x_m) = S^n_m(s, s_1, \ldots, s_n) \\ & = S^n_m(s, s_1, x_{l_1}, \ldots, x_{l_2}, \ldots, x_{l_k}, \ldots, s_n) \\ & = \pi(s), \end{aligned} \tag{3.6}$$

showing that we can obtain t by variable permutation from s.

Example 3.10. A nontrivial variety V of type τ is said to be normal if it does not satisfy any identity of the form $x_j \approx t$, where x_j is a variable and t is a nonvariable term. For each type τ, there is a smallest normal variety N_τ, which is defined by the set of identities $\{s \approx t \mid s, t \in W_\tau(X) \setminus X\}$. That is, any two nonvariable terms are related by $\mathrm{Id}N_\tau$, while each variable is related only to itself. Using the fact that $(X \times X) \cup \mathrm{Id}V$ is always contained in $\mathcal{R}_V$, we see that $\mathcal{R}_{N_\tau} = (X \times X) \cup W_\tau(X)^2 = \mathrm{Perm}\,(\mathrm{Id}N_\tau)$. This gives another example of a variety V for which $\mathcal{R}_V = \mathrm{Perm}\,(\mathrm{Id}V)$.

We can use the relation $\mathcal{R}_V$ to characterize when a variety V is normal.

Proposition 3.11. *A variety V of type τ is normal if and only if no variable is $\mathcal{R}_V$-related to a nonvariable term.*

Proof. When V is a normal variety, we have $N_\tau \subseteq V$ and so by Lemma 3.4 $\mathcal{R}_V \subseteq \mathcal{R}_{N_\tau}$. By the characterization of $\mathcal{R}_{N_\tau}$ from Example 3.10 this means that no variable can be $\mathcal{R}_V$-related to a nonvariable term. Conversely, suppose that $\mathcal{R}_V$ has the property that a variable can only be related to another variable. Since $\mathrm{Id}V \subseteq \mathcal{R}_V$, this means that $\mathrm{Id}V$ cannot contain any identity of the form $x_j \approx t$ for x_j a variable and t a nonvariable term; in other words, V must be normal. □

4. The relation $\mathcal{R}_V$ for varieties of semigroups

In this section we describe the relations $\mathcal{R}_V$ and $\mathcal{L}_V$ when V is a variety of semigroups, that is, a variety of type (2) satisfying the associative identity. We denote by *Sem* the variety $\mathrm{Mod}\{x(yz) \approx (xy)z\}$ of all semigroups. For any variety V, we use $\mathcal{L}(V)$ for the lattice of subvarieties of V; in particular $\mathcal{L}(\mathrm{Sem})$ is the lattice of all semigroup varieties.

We will follow the convention for semigroup varieties of denoting the binary operation by juxtaposition, and of omitting brackets from terms. In this way, any term can be represented by a semigroup "word" consisting of a string of variable symbols as letters; for instance, the term $f(x_1, f(x_2, f(x_2, x_1)))$ becomes the word $x_1x_2x_2x_1$. We use this idea to define several properties of terms and identities. The *length* of a term is its length as a word, the total number of occurrences of variables in the term. An identity $s \approx t$ is called *regular* if the two terms s and t contain exactly the same variable symbols. A set of identities is said to be regular if all the identities in the set are regular, and a variety V is called regular if the set $\mathrm{Id}V$ of all its identities is regular. A semigroup identity $s \approx t$ is called *periodic* if $s = x^a$ and $t = x^b$ for some variable x and some natural numbers $a \neq b$. A variety of semigroups is called *uniformly periodic* if it satisfies a periodic identity. A variety is not uniformly periodic if and only if all its identities $s \approx t$ have the property that s and t have equal lengths. For more information on uniformly periodic varieties, see [4].

Let $s = s(x_1, \ldots, x_n)$ be a term of some arity $n \geq 1$, and let π be a permutation from Sym_m for some $m \geq n$. In Section 3 we defined $\pi(s)$ to be the term $s(x_{\pi(x_1)}, \ldots, x_{\pi(x_n)})$ formed from s by permutation of the variables in s according to π. An important feature of this process is that the term $\pi(s)$ has the same structure as the term s, in the sense that the semantic tree of the term $\pi(s)$ is isomorphic as a graph to the semantic tree for s. In particular, the term $\pi(s)$ has the same length and the same number of distinct variables occurring in it as s does. Which variables occur need not be the same; for instance, $s = x_1x_2$ can be permuted into $\pi(s) = x_3x_4$, changing the arity of the term and which variables occur. As a result, a regular identity such as

$x_1x_2 \approx x_2x_1$ can be permuted by two different permutations π and ρ into a nonregular identity such as $x_3x_4 \approx x_5x_6$. Thus the set Perm (IdV) from Section 3 need not be regular even when IdV is regular. This motivates a new definition. We will call an identity $s \approx t$ permutation-regular if the number of distinct variables occurring in s and t is the same. As usual, a set of identities will be called permutation-regular if all the identities in the set are permutation-regular. We will make use of the following basic fact.

Lemma 4.1. *Let V be a variety of semigroups. If V is regular, then* Perm (IdV) *is permutation-regular.*

We saw in Section 3 that any two terms of the same arity $n \geq 2$ are $\mathcal{L}_V$-related, for any variety V, and that only terms of the same arity can be $\mathcal{L}_V$-related. Thus the only thing of interest for $\mathcal{L}_V$ when V is a variety of semigroups is which unary terms are related to each other. Let T_1 denote the set of unary semigroup terms, so that $T_1 = \{x^i \mid i \geq 1\}$.

Proposition 4.2. *For any variety V of semigroups, $\mathcal{L}_V \cap T_1^2 = \mathcal{R}_V \cap T_1^2$. That is, two unary terms are $\mathcal{L}_V$-related if and only if they are $\mathcal{R}_V$-related.*

Proof. Let x^i and x^j be two unary terms, for $i, j \geq 1$, with $i \neq j$. Then $x^i \mathcal{R}_V x^j$ if and only if $x^i \approx S_1^1(x^j, x^p)$ and $x^j \approx S_1^1(x^i, x^q)$ both hold in IdV, for some unary terms x^p and x^q. These identities hold if and only if $x^i \approx x^{jp}$ and $x^j \approx x^{iq}$ hold in V. Similarly, $x^i \mathcal{L}_V x^j$ if and only if $x^i \approx S_1^1(x^p, x^j)$ and $x^j \approx S_1^1(x^q, x^i)$ both hold in IdV, for some unary terms x^p and x^q, which is also equivalent to having both $x^i \approx x^{jp}$ and $x^j \approx x^{iq}$ in IdV. □

This result allows us to completely characterize the relation $\mathcal{L}_V$ for V a variety of semigroups, and begins our description of $\mathcal{R}_V$. Moreover, we have proved the following useful characterization of when two unary terms are $\mathcal{R}_V$-related.

Corollary 4.3. *Let V be a variety of semigroups and let x^i and x^j be unary terms with $i \neq j$.Then $x^i \mathcal{R}_V x^j$ if and only if the identities $x^i \approx x^{pj}$ and $x^j \approx x^{qi}$ hold in V for some natural numbers p, $q \geq 1$.*

Now we describe how the relations $\mathcal{R}_V$ behave, starting with unary terms.

Proposition 4.4. *Let V be a variety of semigroups which is not uniformly periodic. Then $\mathcal{L}_V \cap T_1^2 = \mathcal{R}_V \cap T_1^2 = \Delta_{T_1}$; that is, two unary terms are related by $\mathcal{R}_V$ if and only if they are equal.*

Proof. Let x^i and x^j be two unary terms which are $\mathcal{L}_V$- or $\mathcal{R}_V$-related. By Corollary 4.3, this forces identities of the form $x^i \approx x^{pj}$ and $x^j \approx x^{qi}$ to hold in V, for some natural numbers p and q. But when V is not uniformly periodic, an identity of the form $x^a \approx x^b$ can hold in V if and only if $a = b$. Thus we must have $i = pj$ and $j = qi$. This can only happen if $i = j$, and the terms x^i and x^j are in fact equal. □

What happens with unary terms for uniformly periodic varieties depends on the particular variety. We recall from Section 3 that Perm (IdV) $\subseteq \mathcal{R}_V$. We will show that if V is both regular and uniformly periodic, then Id$V \cap T_1^2$ = Perm (IdV) $\cap T_1^2$, but $\mathcal{R}_V \cap T_1^2$ can be larger.

Lemma 4.5. *If V is a variety of semigroups which is both regular and uniformly periodic, then* Id$V \cap T_1^2$ = Perm (IdV) $\cap T_1^2$.

Proof. Since Id$V \subseteq$ Perm (IdV) by definition, we know that Id$V \cap T_1^2 \subseteq$ Perm (IdV) $\cap T_1^2$. For the opposite inclusion, suppose that $x^i \approx x^j$ is in Perm (IdV) for some unary terms x^i and x^j. Then there exist some identity $s \approx t$ in IdV and some permutations π and ρ such that $x^i = \pi(s)$ and $x^j = \rho(t)$. Since permutations do not change the number of variables occurring or the length of a term, both s and t must look like x_k^i and x_m^j, respectively, for some variables x_k and x_m. Since V is regular and $s \approx t$ is in IdV, the variables x_k and x_m must in fact be the same. Therefore $x^i \approx x^j$ is actually in IdV. □

Any uniformly periodic variety V must satisfy an identity of the form $x^a \approx x^{a+b}$ for some natural numbers a and b. We denote by $B_{a,b}$ the variety Mod$\{x(yz) \approx (xy)z, x^a \approx x^{a+b}\}$, known as a Burnside variety. Thus any uniformly periodic variety of semigroups is a subvariety of $B_{a,b}$ for some $a, b \geq 1$. An important fact about the identities of the variety $B_{a,b}$ is the following: an identity of the form $x^u \approx x^v$ holds in this variety if and only if either $u = v$ or both $u, v \geq a$ and $u \equiv v$ modulo b. Combining this fact with Corollary 4.3 allows us to describe which unary terms are $\mathcal{R}_V$-related for the variety $V = B_{a,b}$.

Corollary 4.6. *Let $V = B_{a,b}$, for $a, b \geq 1$. Then $x^i\, \mathcal{R}_V\, x^j$ if and only if both $i, j \geq a$ and the congruences $ip \equiv j$ modulo b and $jq \equiv i$ modulo b have solutions $p, q \geq 1$.*

Some basic number theory now provides us with some examples. Let us note that in $V = B_{a,b}$, the unary terms are (up to equivalence modulo IdV and hence equivalence in $\mathcal{R}_V$ as well) $x, x^2, \ldots, x^{a+b-1}$. In the case $a = b = 1$, we have all unary terms equivalent, and $\mathcal{R}_V \cap T_1^2 = T_1^2$. For $V = B_{a,1}$ or $V = B_{a,2}$, for any $a \geq 1$, it is easy to see that $\mathcal{R}_V \cap T_1^2$ is just Id$V \cap T_1^2$. But for $V = B_{1,a}$ when a is a prime number, the terms $x, x^2, \ldots, x^{a-1}$ are all $\mathcal{R}_V$-related to each other, but not to x^a; in this case more terms are related by $\mathcal{R}_V$ than those related by IdV. For $V = B_{2,5}$, we can show that there are 3 distinct classes of terms under $\mathcal{R}_V$: $\{x\}$, $\{x^2, x^3, x^4, x^6\}$ and $\{x^5\}$. This shows that for this choice of V, we have Perm (IdV) $\subset \mathcal{R}_V \subset \nabla_{W_\tau(X)}$.

Finally, we consider the relation $\mathcal{R}_V$ for terms of arbitrary arity. Here too, uniformly periodic varieties behave differently from those which are not uniformly periodic.

Proposition 4.7. *If V is a variety of semigroups which is not uniformly periodic, then $\mathcal{R}_V$ =* Perm (IdV).

Proof. This proof is a modification of the argument from Example 3.9. First, by Proposition 3.8 we have Perm (IdV) $\subseteq \mathcal{R}_V$, so we need to show the opposite inclusion. Let s and t be terms of arities n and m, respectively, with $n \geq m$, and suppose that $s\, \mathcal{R}_V\, t$. Then there exist terms $t_1, \ldots, t_m$ and $s_1, \ldots, s_n$ in $W_\tau(X)$ such that

$$s \approx S_n^m(t, t_1, \ldots, t_m) \in \mathrm{Id}V, \qquad t \approx S_m^n(s, s_1, \ldots s_n) \in \mathrm{Id}V. \tag{4.1}$$

Then we have

$$\begin{aligned} s \approx\ & S_n^m(t, t_1, \ldots, t_m) \\ \approx\ & S_n^m(S_m^n(s, s_1, \ldots, s_n), t_1, \ldots, t_m) \\ \approx\ & S_n^n(s, S_n^m(s_1, t_1, \ldots, t_m), \ldots, S_n^m(s_n, t_1, \ldots, t_m)), \quad \text{by (C1)} \end{aligned} \tag{4.2}$$

Where in Example 3.9 we have equality of terms, we now have only equivalence modulo IdV. However, the condition that V is not uniformly periodic means that the term

$S_n^n(s, S_n^m(s_1, t_1, \ldots, t_m), \ldots, S_n^m(s_n, t_1, \ldots, t_m))$ must have the same length as s. This is sufficient to force the same requirement for variable entries as before to produce our permutation π. Let the variables occurring in term s be $x_{i_1}, \ldots, x_{i_k}$, with $k \leq n$. Then we must have $S_n^m(s_{i_j}, t_1, \ldots, t_m) = x_{i_j}$ for each $j = 1, 2, \ldots, k$. Then for each index i_j there must exist an index l_j such that $s_{i_j} = x_{l_j}$ and $t_{i_j} = x_{i_j}$. Moreover the indices l_j, for $1 \leq j \leq k$ must be distinct. This means that there is a permutation π on the set $\{1, 2, \ldots, n\}$, such that $\pi(i_j) = l_j$, for $q \leq j \leq k$. Then we have

$$\begin{aligned} t = t(x_1, \ldots, x_m) &= S_m^n(s, s_1, \ldots, s_n) \\ &= S_m^n(s, s_1, x_{l_1}, \ldots, x_{l_2}, \ldots, x_{l_k}, \ldots, s_n) \\ &= \pi(s). \end{aligned} \tag{4.3}$$

This shows that $t = \pi(s)$ for some permutation π, and hence that $\mathcal{R}_V \subseteq \text{Perm}\,(\text{Id}V)$. □

The converse of this proposition is not however true. As an example we consider the smallest normal variety of type (2), the variety $Zero$ of zero semigroups defined by $xy \approx zw$. This is a uniformly periodic but not regular variety, but the relation $\mathcal{R}_V$ for this variety V is equal to Perm (IdV), from Example 3.10.

At the other extreme is the variety $B_{1,1}$ of idempotent semigroups or bands. The lattice $\mathcal{L}(B_{1,1})$ of band varieties is known to be countably infinite and its structure has been completely described by Birjukov [5], Fennemore [6, 7], Gerhard [8], and Gerhard and Petrich [9]. Our next result shows that varieties of bands are the only semigroup varieties for which $\mathcal{R}_V$ is the total relation $\nabla_{W_\tau(X)}$ on $W_\tau(X)$.

Theorem 4.8. *Let V be a variety of semigroups. Then $\mathcal{R}_V = \nabla_{W_\tau(X)}$ if and only if V is a subvariety of the variety $B_{1,1}$ of bands.*

Proof. First let V be a variety of bands, so $V \subseteq B_{1,1}$. Then it is easy to show by induction on the complexity of terms that for any two terms s and t, of any arities n and m, respectively, we have $s(t, t, \ldots, t) \approx t \in \text{Id}V$. This means that we can always write $t \approx S_m^n(s, t, \ldots, t) \in \text{Id}V$ and $s \approx S_n^m(t, s, \ldots, s) \in \text{Id}V$, making $s\mathcal{R}_V t$.

Conversely, suppose that V has the property that any two terms (of any arities) are related by $\mathcal{R}_V$. Then the term x is related to the term x^2, so we must be able to express $x \approx S_1^1(x^2, p) \in \text{Id}V$ for some unary term $p = x^c$, for some $c \geq 1$. In particular, our variety V must satisfy an identity of the form $x \approx x^a$ for some $a \geq 1$. If $a = 1$, we have $x \approx x^2 \in \text{Id}V$, and we have shown that V is a variety of bands. If $a > 1$, then we can deduce the following identities from $x \approx x^a$:

$$\begin{aligned} x &\approx x^a \approx x^{2a-1} \approx x^{3a-2} \approx \cdots \in \text{Id}V \\ x^2 &\approx x^{a+1} \approx x^{2a} \approx x^{3a-1} \approx \cdots \in \text{Id}V \\ x^3 &\approx x^{a+2} \approx x^{2a+1} \approx x^{3a} \approx \cdots \in \text{Id}V \\ &\vdots \qquad\qquad\qquad \vdots \\ x^{a-1} &\approx x^{2(a-1)} \approx x^{3(a-1)} \approx x^{4(a-1)} \approx \cdots \in \text{Id}V. \end{aligned} \tag{4.4}$$

Now we also know that x is $\mathcal{R}_V$-related to x^{a-1}, which means that we can write $x \approx S_1^1(x^{a-1}, q) \in \text{Id}V$ for some unary term $q = x^k$, for some k. Therefore, we get $x \approx x^{k(a-1)} \in \text{Id}V$. A similar argument applied to $x^2\mathcal{R}_V x^{a-1}$ then gives $x^2 \approx x^{m(a-1)} \in \mathrm{Id}V$ for some m. Since $x^{k(a-1)} \approx x^{m(k-1)}$ is in $\mathrm{Id}V$ from above, we see that by transitivity we have $x \approx x^2$ in $\mathrm{Id}V$, and V is a variety of bands. □

Theorem 4.9. *Let $V = B_{a,b}$ for some $a, b \geq 1$. Let t be any term of arity $n \geq 2$ which has at least one variable x_k occurring in it a number of times which is congruent to* 1 *modulo b . Then $x^a\, \mathcal{R}_V\, t^a$.*

Proof. We can always write $t^a = S_n^1(a^a, p)$ for some n-ary term p, by taking $p = t$. But we also need to be able to write $x^a \approx S_1^n(t^a, q_1, \ldots, q_n) \in \mathrm{Id}V$ for some unary terms $q_1, \ldots, q_n$. Let x_k be a variable which occurs in t exactly v times, where v is congruent to 1 modulo b. For the term q_k, we use x, and for all the other terms $q_1, \ldots, q_n$, we use x^b. Then $S_1^n(t^a, q_1, \ldots, q_n) = t^a(x^b, \ldots, x^b, x, x^b, \ldots, x^b) = (x^{qb+1})^a$ for some natural number q. Then in $B_{a,b}$ we have $(x^{qb+1})^a \approx x^{aqb+a} \approx x^a$, as required. □

Corollary 4.10. *Let $V = B_{a,b}$ for some $a, b \geq 1$ with $a + b \geq 3$. Then* Perm $(\mathrm{Id}V)$ *is a proper subset of $\mathcal{R}_V$, which is a proper subset of $\nabla_{W_\tau(X)}$ on $W_\tau(X)$.*

Proof. By the previous theorem, we have $x^a\mathcal{R}_V\,(xy)^a$. Since the terms x^a and $(xy)^a$ contain different numbers of variables, and V is regular, the identity $x^a \approx (xy)^a$ cannot be in Perm $(\mathrm{Id}V)$. Thus Perm $(\mathrm{Id}V)$ is a proper subset of $\mathcal{R}_V$. The remaining claim follows from Theorem 4.8. □

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