Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 808375, 10 pages
http://dx.doi.org/10.1155/2014/808375

Research Article

Analytical and Numerical Study of the Projective Synchronization of the Chaotic Complex Nonlinear Systems with Uncertain Parameters and Its Applications in Secure Communication

Kholod M. Abualnaja[1] and Emad E. Mahmoud[2,3]

[1] *Department of Mathematics, Umm Al-Qura University, P.O. Box 14949, Makkah, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt*
[3] *College of Applied Medical Sciences, Taif University, P.O. Box 21955, Turabah, Saudi Arabia*

Correspondence should be addressed to Emad E. Mahmoud; emad_eluan@yahoo.com

Received 5 November 2013; Revised 18 January 2014; Accepted 2 February 2014; Published 8 April 2014

Academic Editor: Kwok-Wo Wong

The main aim of this research is to find an analytical and numerical study to investigate the projective synchronization of two identical or nonidentical chaotic complex nonlinear systems with uncertain parameters. The secure communication between these systems is achieved based on this study. Based on the adaptive control technique and the Lyapunov function a scheme is designed to achieve projective synchronization of chaotic attractors of these systems. The projective synchronization of two identical complex Chen systems and two different chaotic complex Lü and Lorenz systems is taken as two examples to verify the feasibility of the presented scheme. These chaotic complex systems appear in several applications in physics, engineering, and other applied sciences. Numerical simulations are calculated to demonstrate the effectiveness of the proposed synchronization scheme and verify the theoretical results. The above results will provide theoretical foundation for the secure communication applications based on the proposed scheme.

1. Introduction

In 1999, projective synchronization has been first reported by Mainieri and Rehacek [1] in partially linear systems that the drive and response systems synchronize up to a constant scaling factor δ. Later Xu and Li showed that projective synchronization could be extended to general classes of chaotic systems without partial linearity [2]. Complete synchronization and antisynchronization are the special cases of the projective synchronization where the scaling factor $\delta = 1$ and $\delta = -1$, respectively.

Many researchers had shown the possibility to achieve projective synchronization between two chaotic systems (*with real variables*) with known or unknown parameters [3–6]. There also exist, however, interesting cases of dynamical systems, where the main variables participating in the dynamics are complex [7–25]. The projective synchronization of two identical chaotic complex systems with certain parameters is investigated in [26]. Therefore, it is important to examine the projective synchronization when the master and slave systems (*with complex variables*) are identical or different with fully unknown parameters, which we hope to achieve in this paper.

In applied sciences and engineering there are a lot of problems involving complex variables which are described by these complex systems, for example, when amplitudes of electromagnetic fields and atomic polarization are involved. Increasing the number of variables (or introducing complex variables) is also crucial in chaos synchronization used in secure communications, where one wishes to maximize the content and security of the transmitted information. Increasing the number of variables and parameters in studying projective synchronization of chaotic complex systems is of course crucial in the area of secure communication where one wishes to maximize the content and security of the transmitted information [19, 23]. Secure communication means that two entities are communicating with each other in a way that does not allow anyone else to understand their message. So, we hope to achieve the secure communication based on the proposed scheme of projective synchronization of chaotic complex systems.

A dynamical system is called chaotic if it is deterministic, has a long-term periodic behavior, and exhibits sensitive dependence on the initial conditions. If the system has one positive Lyapunov exponent then the system is called chaotic [27].

Consider the n-dimensional chaotic complex nonlinear system as follows:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})\mathbf{A} + \mathbf{f}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is a state complex vector, $\mathbf{x} = \mathbf{x}^r + j\mathbf{x}^i$, $\mathbf{x}^r = (u_1, u_3, \ldots, u_{2n-1})^T$, $\mathbf{x}^i = (u_2, u_4, \ldots, u_{2n})^T$, $j = \sqrt{-1}$, T denotes transpose, $\mathbf{F}(\mathbf{x})$ is $n \times n$ complex matrix and the elements of this matrix are state complex variables, $\mathbf{A}$ is $n \times 1$ complex (or real) vector of system parameters, $\mathbf{f} = (f_1, f_2, \ldots, f_n)^T$ is a vector of nonlinear complex functions, and superscripts r and i stand for the real and imaginary parts of the state complex vector $\mathbf{x}$.

The purpose of this paper is to investigate the phenomenon of the adaptive projective synchronization of two identical or different systems of the form (1) with fully unknown parameters by designing an adaptive control scheme.

Most of chaotic complex systems can be described by (1), such as complex Lorenz, Chen, and Lü systems [14, 19]. In order to show the results of our scheme of two identical or nonidentical systems of the form (1) we choose, as an example, the chaotic complex Chen, Lorenz, and Lü systems which have been introduced and studied recently in our works [16, 19].

The chaotic complex Chen system is

$$\begin{aligned}\dot{x} &= a(y - x),\\ \dot{y} &= (c - a)x - xz + cy,\\ \dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - bz.\end{aligned} \tag{2}$$

The chaotic complex Lorenz system is

$$\begin{aligned}\dot{x} &= \alpha(y - x),\\ \dot{y} &= \gamma x - xz - y,\\ \dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - \beta z,\end{aligned} \tag{3}$$

while the chaotic complex Lü system is written in the form

$$\begin{aligned}\dot{x} &= \rho(y - x),\\ \dot{y} &= \nu y - xz,\\ \dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - \mu z,\end{aligned} \tag{4}$$

where $\mathbf{x} = (x_1, x_2, x_3)^T = (x, y, z)^T$, a, b, c, α, β, γ, ρ, μ, and ν are positive parameters, $x = u_1 + ju_2$ and $y = u_3 + ju_4$ are complex functions, and u_l $(l = 1, \ldots, 4)$ and $z = u_5$ are real functions. Dots represent derivatives with respect to time and an overbar denotes complex conjugate variables.

The chaotic complex Chen, Lorenz, and Lü systems are a 5-dimensional continuous real autonomous system. System (2) is chaotic when $a = 42$, $b = 4$, and $c = 26$. For the case $\alpha = 14$, $\gamma = 35$, and $\beta = 3.7$ system (3) has chaotic attractor, while system (4) exhibits chaotic behavior when $\rho = 40$, $\nu = 22$, and $\mu = 5$.

The organization of this paper is as follows. Design of the proposed scheme for adaptive projective synchronization of two identical or different n-dimensional chaotic complex nonlinear systems with fully unknown parameters is stated in Section 2. In Section 3 we study projective synchronization of two identical chaotic complex Chen systems as an example for Section 2, while we investigate projective synchronization between the chaotic complex Lorenz system and the chaotic complex Lü system in Section 4. The secure communication based on the results of projective synchronization of two chaotic complex Chen systems is shown in Section 5. Finally, the main conclusions of our investigations are summarized in Section 6.

2. A Scheme for Adaptive Projective Synchronization

We consider two different n-dimensional chaotic complex nonlinear systems of the form (1); one is the master system as

$$\dot{\mathbf{x}}_m = \dot{\mathbf{x}}_m^r + j\dot{\mathbf{x}}_m^i = \mathbf{F}(\mathbf{x}_m)\mathbf{A} + \mathbf{f}(\mathbf{x}_m), \tag{5}$$

and the second is the controlled slave system as

$$\dot{\mathbf{y}}_s = \dot{\mathbf{y}}_s^r + j\dot{\mathbf{y}}_s^i = \mathbf{G}(\mathbf{y}_s)\mathbf{B} + \mathbf{g}(\mathbf{y}_s) + \mathbf{L}, \tag{6}$$

where the additive complex controller $\mathbf{L} = (L_1, L_2, \ldots, L_n)^T = \mathbf{L}^r + j\mathbf{L}^i$, $\mathbf{L}^r = (v_1, v_3, \ldots, v_{2n-1})^T$, and $\mathbf{L}^i = (v_2, v_4, \ldots, v_{2n})^T$.

The adaptive synchronization problem is to design a controller $\mathbf{L}$, estimate the unknown parameters of the master and slave systems, and make the slave system follow the master system and become ultimately the same.

Theorem 1. *One may be able to achieve the adaptive projective synchronization between systems* (5) *and* (6) *by a choice of the controller* $\mathbf{L}$ *as*

$$\begin{aligned}\mathbf{L} &= \mathbf{L}^r + j\mathbf{L}^i \\ &= \left[-\mathbf{G}(\mathbf{y}_s)(\widetilde{\mathbf{B}}) + \delta\mathbf{F}(\mathbf{x}_m)(\widetilde{\mathbf{A}})\right] \\ &\quad + \left[-\mathbf{g}(\mathbf{y}_s) + \delta\mathbf{f}(\mathbf{x}_m)\right] - \mathbf{\Psi}\mathbf{e} \\ &= -\mathbf{G}^r(\mathbf{y}_s)(\widetilde{\mathbf{B}}) + \delta\mathbf{F}^r(\mathbf{x}_m)(\widetilde{\mathbf{A}}) \\ &\quad + \left[-\mathbf{g}^r(\mathbf{y}_s) + \delta\mathbf{f}^r(\mathbf{x}_m)\right] - \mathbf{\Psi}\mathbf{e}^r \\ &\quad + j\left(-\mathbf{G}^i(\mathbf{y}_s)(\widetilde{\mathbf{B}}) + \delta\mathbf{F}^i(\mathbf{x}_m)(\widetilde{\mathbf{A}})\right. \\ &\qquad \left. + \left[-\mathbf{g}^i(\mathbf{y}_s) + \delta\mathbf{f}^i(\mathbf{x}_m)\right] - \mathbf{\Psi}\mathbf{e}^i\right),\end{aligned} \tag{7}$$

and the adaptive laws of parameters are selected as

$$\begin{aligned}\dot{\widetilde{\mathbf{A}}} &= \left(-\delta\mathbf{F}^r(\mathbf{x}_m)\right)^T\mathbf{e}^r + \left(-\delta\mathbf{F}^i(\mathbf{x}_m)\right)^T\mathbf{e}^i + \boldsymbol{\zeta}\widehat{\mathbf{A}}, \\ \dot{\widetilde{\mathbf{B}}} &= \left(\mathbf{G}^r(\mathbf{y}_s)\right)^T\mathbf{e}^r + \left(\mathbf{G}^i(\mathbf{y}_s)\right)^T\mathbf{e}^i + \boldsymbol{\zeta}\widehat{\mathbf{B}},\end{aligned} \tag{8}$$

where $\mathbf{e}(t) = \mathbf{y}_s - \delta\mathbf{x}_m = \mathbf{e}^r + j\mathbf{e}^i = (e_1, e_2, \ldots, e_n)^T$ *is the vector of the complex error function* $\mathbf{e}^r = (e_{u_1}, e_{u_3}, \ldots, e_{u_{2n-1}})^T$, $\mathbf{e}^i = (e_{u_2}, e_{u_4}, \ldots, e_{u_{2n}})^T$ *and* δ *is a constant scaling factor. The elements of the vectors* $\widetilde{\mathbf{A}}$ *and* $\widetilde{\mathbf{B}}$ *are the parameters estimations of elements of the vectors* $\mathbf{A}$ *and* $\mathbf{B}$*, respectively; the parameters errors are defined as* $\widehat{\mathbf{A}} = \mathbf{A} - \widetilde{\mathbf{A}}$ *and* $\widehat{\mathbf{B}} = \mathbf{B} - \widetilde{\mathbf{B}}$ *and* $\mathbf{\Psi} = \mathrm{diag}(\psi_1, \psi_2, \ldots, \psi_n)$ *and* $\boldsymbol{\zeta} = \mathrm{diag}(\zeta_1, \zeta_2, \ldots, \zeta_n)$*;* ψ_l *and* ζ_l *are positive constants;* $l = 1, 2, \ldots, n$.

Proof. We subtract (5) from (6) to get

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{e}}^r + j\dot{\mathbf{e}}^i \\ &= \left[\mathbf{G}(\mathbf{y}_s)(\widetilde{\mathbf{B}} + \widehat{\mathbf{B}}) - \delta\mathbf{F}(\mathbf{x}_m)(\widetilde{\mathbf{A}} + \widehat{\mathbf{A}})\right] \\ &\quad + \left[\mathbf{g}(\mathbf{y}_s) - \delta\mathbf{f}(\mathbf{x}_m)\right] + \mathbf{L} \\ &= \mathbf{G}^r(\mathbf{y}_s)(\widetilde{\mathbf{B}} + \widehat{\mathbf{B}}) - \delta\mathbf{F}^r(\mathbf{x}_m)(\widetilde{\mathbf{A}} + \widehat{\mathbf{A}}) \\ &\quad + \left[\mathbf{g}^r(\mathbf{y}_s) - \delta\mathbf{f}^r(\mathbf{x}_m)\right] + \mathbf{L}^r \\ &\quad + j\left(\mathbf{G}^i(\mathbf{y}_s)(\widetilde{\mathbf{B}} + \widehat{\mathbf{B}}) - \delta\mathbf{F}^i(\mathbf{x}_m)(\widetilde{\mathbf{A}} + \widehat{\mathbf{A}})\right. \\ &\qquad \left. + \left[\mathbf{g}^i(\mathbf{y}_s) - \delta\mathbf{f}^i(\mathbf{x}_m)\right] + \mathbf{L}^i\right).\end{aligned} \tag{9}$$

Therefore, we will use Lyapunov function as

$$\begin{aligned}\mathbf{V}(t) &= \frac{1}{2}\left[(\mathbf{e}^r)^T\mathbf{e}^r + (\mathbf{e}^i)^T\mathbf{e}^i + (\mathbf{A} - \widetilde{\mathbf{A}})^T(\mathbf{A} - \widetilde{\mathbf{A}})\right. \\ &\qquad \left. + (\mathbf{B} - \widetilde{\mathbf{B}})^T(\mathbf{B} - \widetilde{\mathbf{B}})\right] \\ &= \frac{1}{2}\left(\sum_{l=1}^{n} e_{u_{2l-1}}^2 + \sum_{l=1}^{n} e_{u_{2l}}^2 + \widehat{\mathbf{A}}^T\widehat{\mathbf{A}} + \widehat{\mathbf{B}}^T\widehat{\mathbf{B}}\right).\end{aligned} \tag{10}$$

The total time derivative of $V(t)$ along the trajectory of the error system (9) is as follows:

$$\begin{aligned}\dot{V}(t) &= (\dot{\mathbf{e}}^r)^T\mathbf{e}^r + (\dot{\mathbf{e}}^i)^T\mathbf{e}^i + \widehat{\mathbf{A}}^T\dot{\widehat{\mathbf{A}}} + \widehat{\mathbf{B}}^T\dot{\widehat{\mathbf{B}}} \\ &= \left(\mathbf{G}^r(\mathbf{y}_s)(\widetilde{\mathbf{B}} + \widehat{\mathbf{B}}) - \delta\mathbf{F}^r(\mathbf{x}_m)(\widetilde{\mathbf{A}} + \widehat{\mathbf{A}})\right. \\ &\qquad \left. + \left[\mathbf{g}^r(\mathbf{y}_s) - \delta\mathbf{f}^r(\mathbf{x}_m)\right] + \mathbf{L}^r\right)^T\mathbf{e}^r \\ &\quad + \left(\mathbf{G}^i(\mathbf{y}_s)(\widetilde{\mathbf{B}} + \widehat{\mathbf{B}}) - \delta\mathbf{F}^i(\mathbf{x}_m)(\widetilde{\mathbf{A}} + \widehat{\mathbf{A}})\right. \\ &\qquad \left. + \left[\mathbf{g}^i(\mathbf{y}_s) - \delta\mathbf{f}^i(\mathbf{x}_m)\right] + \mathbf{L}^i\right)^T\mathbf{e}^i \\ &\quad + \widehat{\mathbf{A}}^T\left(-\dot{\widetilde{\mathbf{A}}}\right) + \widehat{\mathbf{B}}^T\left(-\dot{\widetilde{\mathbf{B}}}\right),\end{aligned} \tag{11}$$

where $\dot{\widehat{\mathbf{A}}} = -\dot{\widetilde{\mathbf{A}}}$ and $\dot{\widehat{\mathbf{B}}} = -\dot{\widetilde{\mathbf{B}}}$.

By substituting from (7) and (8) into (11) we obtain

$$\begin{aligned}\dot{V}(t) &= (\mathbf{e}^r)^T\mathbf{e}^r + (\mathbf{e}^i)^T\mathbf{e}^i + \widehat{\mathbf{A}}^T\dot{\widehat{\mathbf{A}}} + \widehat{\mathbf{B}}^T\dot{\widehat{\mathbf{B}}} \\ &= \left(\mathbf{G}^r(\mathbf{y}_s)(\widehat{\mathbf{B}}) - \delta\mathbf{F}^r(\mathbf{x}_m)(\widehat{\mathbf{A}}) - \mathbf{\Psi}\mathbf{e}^r\right)^T\mathbf{e}^r \\ &\quad + \left(\mathbf{G}^i(\mathbf{y}_s)(\widehat{\mathbf{B}}) - \delta\mathbf{F}^i(\mathbf{x}_m)(\widehat{\mathbf{A}}) - \mathbf{\Psi}\mathbf{e}^i\right)^T\mathbf{e}^i \\ &\quad + \widehat{\mathbf{A}}^T\left(\delta(\mathbf{F}^r(\mathbf{x}_m))^T\mathbf{e}^r + \left(\delta\mathbf{F}^i(\mathbf{x}_m)\right)^T\mathbf{e}^i - \boldsymbol{\zeta}\widehat{\mathbf{A}}\right) \\ &\quad + \widehat{\mathbf{B}}^T(-\mathbf{G}^r(\mathbf{y}_s))^T\mathbf{e}^r + \left(-\mathbf{G}^i(\mathbf{y}_s)\right)^T\mathbf{e}^i - \boldsymbol{\zeta}\widehat{\mathbf{B}} \\ &= -\left[(\mathbf{\Psi}\mathbf{e}^r)^T\mathbf{e}^r + (\mathbf{\Psi}\mathbf{e}^i)^T\mathbf{e}^i\right] - \widehat{\mathbf{B}}^T(\boldsymbol{\zeta}\widehat{\mathbf{B}}) - \widehat{\mathbf{A}}^T(\boldsymbol{\zeta}\widehat{\mathbf{A}}) \\ &= -\left(\sum_{l=1}^{n}\psi_{2l-1}e_{u_{2l-1}}^2 + \sum_{l=1}^{n}\psi_l e_{u_{2l}}^2\right) \\ &\quad - \widehat{\mathbf{B}}^T(\boldsymbol{\zeta}\widehat{\mathbf{B}}) - \widehat{\mathbf{A}}^T(\boldsymbol{\zeta}\widehat{\mathbf{A}}).\end{aligned} \tag{12}$$

Since $V(t)$ is a positive definite function and its derivative is negative definite, therefore, Lyapunov's direct method implies that the equilibrium point $e_{u_{2l}}$ and $e_{u_{2l-1}} = 0$; $l = 1, \ldots, n$. Consequently, the states of the slave system and the master system will be globally synchronized asymptotically. This completes the proof. □

Remark 2. If systems (5) and (6) satisfy $\mathbf{f}(\cdot) = \mathbf{g}(\cdot)$ and $\mathbf{F}(\cdot) = \mathbf{G}(\cdot)$, then the structure of system (5) and system (6) is identical. Therefore, our scheme is also applicable to the adaptive synchronization of two identical chaotic systems with fully unknown parameters and the adaptive laws of parameters are selected as

$$\begin{aligned}\dot{\widetilde{\mathbf{A}}} = \dot{\widetilde{\mathbf{B}}} &= \left[(\mathbf{G}^r(\mathbf{y}_s(t)))^T - (\delta\mathbf{F}^r(\mathbf{x}_m(t)))^T\right]\mathbf{e}^r \\ &\quad + \left[\left(\mathbf{G}^i(\mathbf{y}_s(t))\right)^T - \left(\delta\mathbf{F}^i(\mathbf{x}_m(t))\right)^T\right]\mathbf{e}^i + \boldsymbol{\zeta}\widehat{\mathbf{A}}.\end{aligned} \tag{13}$$

Finally, our scheme is illustrated by applying it for two identical Chen systems in Section 3 and two different chaotic complex Lorenz and Lü systems in Section 4.

3. Projective Synchronization between Two Complex Chen Systems

3.1. Analytical Formula of Controller. Let us now investigate the projective synchronization of two identical chaotic complex Chen systems with uncertain parameters as an example for Section 2. The master and the slave systems are thus defined, respectively, as follows:

$$\begin{aligned}\dot{x}_m &= a\left(y_m - x_m\right),\\ \dot{y}_m &= (c-a)\,x_m - x_m z_m + c y_m,\\ \dot{z}_m &= \frac{1}{2}\left(\overline{x}_m y_m + x_m \overline{y}_m\right) - b z_m,\end{aligned} \tag{14}$$

$$\begin{aligned}\dot{x}_s &= a\left(y_s - x_s\right) + L_1,\\ \dot{y}_s &= (c-a)\,x_s - x_s z_s + c y_s + L_2,\\ \dot{z}_s &= \frac{1}{2}\left(\overline{x}_s y_s + x_s \overline{y}_s\right) - b z_s + L_3,\end{aligned} \tag{15}$$

where $L_1 = v_1 + jv_2$, $L_2 = v_3 + jv_4$ and $L_3 = v_5$, $L_4 = v_7$ are complex and real control functions, respectively, which are to be determined.

The complex systems (14) and (15) can be formed, respectively, as

$$\begin{pmatrix}\dot{x}_m\\ \dot{y}_m\\ \dot{z}_m\end{pmatrix} = \begin{pmatrix} y_m - x_m & 0 & 0\\ -x_m & x_m + y_m & 0\\ 0 & 0 & -z_m\end{pmatrix}\begin{pmatrix}a\\ c\\ b\end{pmatrix} + \begin{pmatrix}0\\ -x_m z_m\\ \frac{1}{2}\left(\overline{x}_m y_m + x_m\overline{y}_m\right)\end{pmatrix},$$

$$\begin{pmatrix}\dot{x}_s\\ \dot{y}_s\\ \dot{z}_s\end{pmatrix} = \begin{pmatrix} y_s - x_s & 0 & 0\\ -x_s & x_s + y_s & 0\\ 0 & 0 & -z_s\end{pmatrix}\begin{pmatrix}a\\ c\\ b\end{pmatrix} + \begin{pmatrix}0\\ -x_s z_s\\ \frac{1}{2}\left(\overline{x}_s y_s + x_s\overline{y}_s\right)\end{pmatrix} + \begin{pmatrix}L_1\\ L_2\\ L_3\end{pmatrix}. \tag{16}$$

So, by comparing the complex systems (16) with the form of systems (5) and (6), respectively, we find

$$\begin{gathered}\mathbf{F}\left(\mathbf{x}_m\right) = \begin{pmatrix} y_m - x_m & 0 & 0\\ -x_m & x_m + y_m & 0\\ 0 & 0 & -z_m\end{pmatrix},\\ \mathbf{G}\left(\mathbf{y}_s\right) = \begin{pmatrix} y_s - x_s & 0 & 0\\ -x_s & x_s + y_s & 0\\ 0 & 0 & -z_s\end{pmatrix},\\ \mathbf{A} = \mathbf{B} = \begin{pmatrix}a\\ c\\ b\end{pmatrix}, \qquad \mathbf{f}\left(\mathbf{x}_m\right) = \begin{pmatrix}0\\ -x_m z_m\\ \frac{1}{2}\left(\overline{x}_m y_m + x_m\overline{y}_m\right)\end{pmatrix},\\ \mathbf{g}\left(\mathbf{y}_s\right) = \begin{pmatrix}0\\ -x_s z_s\\ \frac{1}{2}\left(\overline{x}_s y_s + x_s\overline{y}_s\right)\end{pmatrix}.\end{gathered} \tag{17}$$

According to Theorem 1, the controller is designed as

$$\begin{gathered}\mathbf{L} = \left[-\mathbf{G}\left(\mathbf{y}_s\right)\left(\widetilde{\mathbf{B}}\right) + \delta\mathbf{F}\left(\mathbf{x}_m\right)\left(\widetilde{\mathbf{A}}\right)\right] + \left[-\mathbf{g}\left(\mathbf{y}_s\right) + \delta\mathbf{f}\left(\mathbf{x}_m\right)\right] - \mathbf{\Psi}\mathbf{e},\\ \begin{pmatrix}L_1\\ L_2\\ L_3\end{pmatrix} = \begin{pmatrix} -\widehat{a}\left(y_s - x_s\right) + \delta\widehat{a}\left(y_m - x_m\right) - \psi_1 e_1\\ -\widehat{c}\left(y_s + x_s\right) + \widehat{a}x_s + x_s z_s + \delta\left[\widehat{c}\left(y_m + x_m\right) - \widehat{a}x_m - x_m z_m\right] - \psi_2 e_2\\ \widehat{b}z_s - \frac{1}{2}\left(\overline{x}_s y_s + x_s\overline{y}_s\right) + \delta\left[-\widehat{b}z_m + \frac{1}{2}\left(\overline{x}_m y_m + x_m\overline{y}_m\right)\right] - \psi_3 e_3\end{pmatrix},\\ \mathbf{L} = \begin{pmatrix} -\widehat{a}\left(u_{3s} - u_{1s} - \delta u_{3m} + \delta u_{1m}\right) - \psi_1 e_{u_1}\\ -\widehat{c}\left(u_{3s} + u_{1s}\right) + \widehat{a}u_{1s} + u_{1s}u_{5s} + \delta\left[\widehat{c}\left(u_{3m} + u_{1m}\right) - \widehat{a}u_{1m} - u_{1m}u_{5m}\right] - \psi_2 e_{u_3}\\ \widehat{b}\left(u_{5s} - \delta u_{5m}\right) - u_{1s}u_{3s} + \delta u_{1m}u_{3m} - u_{2s}u_{4s} + \delta u_{2m}u_{4m} - \psi_3 e_{u_5}\end{pmatrix}\\ + j\begin{pmatrix} -\widehat{a}\left(u_{4s} - u_{2s} - \delta u_{4m} + \delta u_{2m}\right) - \psi_1 e_{u_2}\\ -\widehat{c}\left(u_{4s}(t) + u_{2s}\right) + \widehat{a}u_{2s} + u_{2s}u_{5s} + \delta\left[\widehat{c}\left(u_{4m} + u_{2m}\right) - \widehat{a}u_{2m} - u_{2m}u_{5m}\right] - \psi_2 e_{u_4}\\ 0\end{pmatrix},\end{gathered} \tag{18}$$

where $e_{u_l} = u_{ls} - \delta u_{lm}$; $l = 1, 2, 3, 4, 5, 7$.

Since $\mathbf{A} = \mathbf{B} = (a, c, b)^T$ we can calculate the adaptive laws of parameters by using (13) as

$$\dot{\tilde{\mathbf{A}}} = \dot{\tilde{\mathbf{B}}} = \begin{pmatrix} \dot{\tilde{a}} \\ \dot{\tilde{c}} \\ \dot{\tilde{b}} \end{pmatrix} = \begin{pmatrix} -e_{u1}^2 - \delta_{u2}^2 + \zeta_1 \hat{a} \\ \left(e_{u_3} + e_{u_1}\right) e_{u_3} + \left(e_{u_4} + e_{u_2}\right) e_{u_2} + \zeta_2 \hat{c} \\ e_{u_5}^2 + \zeta_3 \hat{b} \end{pmatrix}. \tag{19}$$

3.2. Numerical Results. To verify and demonstrate the feasibility of the proposed scheme, we discuss the simulation results of the projective synchronization between two identical chaotic complex Chen systems (14) and (15). Systems (14) and (15) with the controller (18) are solved numerically, and the parameters are chosen as $a = 42$, $b = 4$, and $c = 26$. The initial condition of the master system state vector, the initial value of the slave system state vector, and the diagonal constant matrices are taken as $(x_m(0), y_m(0), z_m(0))^T = (1 + 2j, 3 + 4j, 5, 6)^T$ and $(x_s(0), y_s(0), z_s(0), w_s(0))^T = (6 + 8j, 3 + 4j, 8, 1)^T$ and $\boldsymbol{\Psi} = \text{diag}(12, 15, 11)$ and $\boldsymbol{\zeta} = \text{diag}(6, 9, 10, 7)$. The initial values of estimate for unknown parameters vector are considered as $(\hat{a}(0), \hat{c}(0), \hat{b}(0))^T = (3, 4, 5)^T$. The results are depicted in Figures 1 and 2.

In Figure 1 the solutions of (14) and (15) are plotted subject to different initial conditions and show that projective synchronization is indeed achieved. In Figure 1(a) we select $\delta = -1$ and the attractors in (u_1, u_3, u_5) space of master system (14) and slave system (15) have the same size but opposite shape. But when we choose $\delta = -2$ in Figure 1(b) the attractors of (14) and (15) have the opposite shape in (u_1, u_3, u_5) space, but the size of the attractor of the slave system is twice as big as of the master system. Figure 2 shows that the estimated values of the unknown parameters $\hat{a}(t)$, $\hat{c}(t)$, and $\hat{b}(t)$ converge to 42, 26, and 4, respectively. These results ensure that our scheme is suitable for effecting adaptive projective synchronization of two identical chaotic complex nonlinear systems.

4. Projective Synchronization between Complex Lorenz and Lü Systems

4.1. Analytical Formula of Controller. This subsection is devoted to test the validity of the Scheme of Section 2 by applying it to the chaotic complex Lorenz as a master system and Lü complex as a salve system with fully unknown parameters.

The master and slave systems are described by the following equations, respectively:

$$\begin{aligned} \dot{x}_m &= \alpha \left(y_m - x_m\right), \\ \dot{y}_m &= \gamma x_m - y_m - x_m z_m, \\ \dot{z}_m &= \frac{1}{2}\left(\overline{x}_m y_m + x_m \overline{y}_m\right) - \beta z_m, \end{aligned} \tag{20}$$

$$\begin{aligned} \dot{x}_s &= \rho \left(y_s - x_s\right) + L_1, \\ \dot{y}_s &= \nu y_s - x_s z_s + L_2, \\ \dot{z}_s &= \frac{1}{2}\left(\overline{x}_s y_s + x_s \overline{y}_s\right) - \mu z_s + L_3, \end{aligned} \tag{21}$$

where $x_m = u_{1m} + j u_{2m}$, $y_m = u_{3m} + j u_{4m}$, $z_m = u_{5m}$, $x_s = u_{1s} + j u_{2s}$, $y_s = u_{3s} + j u_{4s}$, $z_s = u_{5s}$, $L_1 = v_1 + j v_2$, $L_2 = v_3 + j v_4$, and $L_3 = v_5$ are complex and real control functions, respectively, which we need to determine.

Considering systems (20) and (21) are equivalent to systems (5) and (6), respectively, so

$$\begin{gathered} \mathbf{F}\left(\mathbf{x}_m\right) = \begin{pmatrix} y_m - x_m & 0 & 0 \\ 0 & x_m & 0 \\ 0 & 0 & -z_m \end{pmatrix}, \\ \mathbf{G}\left(\mathbf{y}_s\right) = \begin{pmatrix} y_s - x_s & 0 & 0 \\ 0 & y_s & 0 \\ 0 & 0 & -z_s \end{pmatrix}, \\ \mathbf{A} = \begin{pmatrix} \alpha \\ \gamma \\ \beta \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \rho \\ \nu \\ \mu \end{pmatrix}, \\ \mathbf{f}\left(\mathbf{x}_m\right) = \begin{pmatrix} 0 \\ -y_m - x_m z_m \\ \frac{1}{2}\left(\overline{x}_m y_m + x_m \overline{y}_m\right) \end{pmatrix}, \\ \mathbf{g}\left(\mathbf{y}_s\right) = \begin{pmatrix} 0 \\ -x_s z_s \\ \frac{1}{2}\left(\overline{x}_s y_s + x_s \overline{y}_s\right) \end{pmatrix}. \end{gathered} \tag{22}$$

According to (7), the adaptive controller is calculated as

$$\begin{aligned} L_1 &= v_1 + j v_2 \\ &= -\tilde{\rho}\left(\left(u_{3s} - u_{1s}\right) + j\left(u_{4s} - u_{2s}\right)\right) \\ &\quad + \tilde{\alpha}\delta\left(\left(u_{3m} - u_{1m}\right) + j\left(u_{4m} - u_{2m}\right)\right) \\ &\quad - \psi_1\left(e_{u_1} - j e_{u_2}\right), \\ L_2 &= v_3 + j v_4 \\ &= -\tilde{\nu}\left(u_{3s} + j u_{4s}\right) + u_{5s}\left(u_{1s} + j u_{2s}\right) \\ &\quad + \delta\tilde{\gamma}\left(u_{1m} + j u_{2m}\right) - \delta\left(u_{3m} + j u_{4m}\right) \\ &\quad - \delta u_{5m}\left(u_{1m} + j u_{2m}\right) - \psi_2\left(e_{u_3} - j e_{u_4}\right), \\ L_3 = v_5 &= \tilde{\mu} u_{5s} - \delta\tilde{\beta} u_{5m} - u_{3s} u_{1s} - u_{4s} u_{2s} \\ &\quad + \delta u_{3m} u_{1m} + \delta u_{4m} u_{2m} - \psi_3 e_{u_5}, \end{aligned} \tag{23}$$

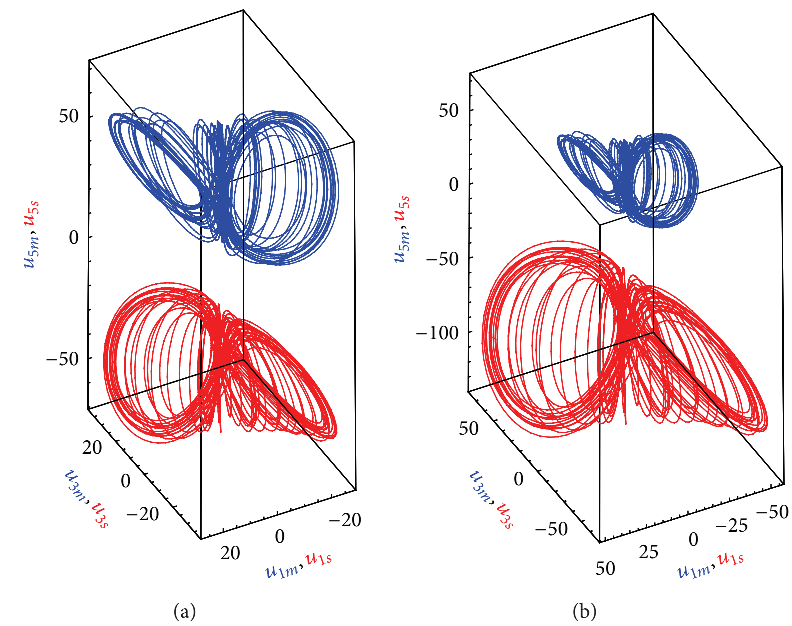

FIGURE 1: Chaotic attractors of the master system (14) (blue color) and the slave system (15) (red color): (a) when $\delta = -1$, (b) when $\delta = -2$.

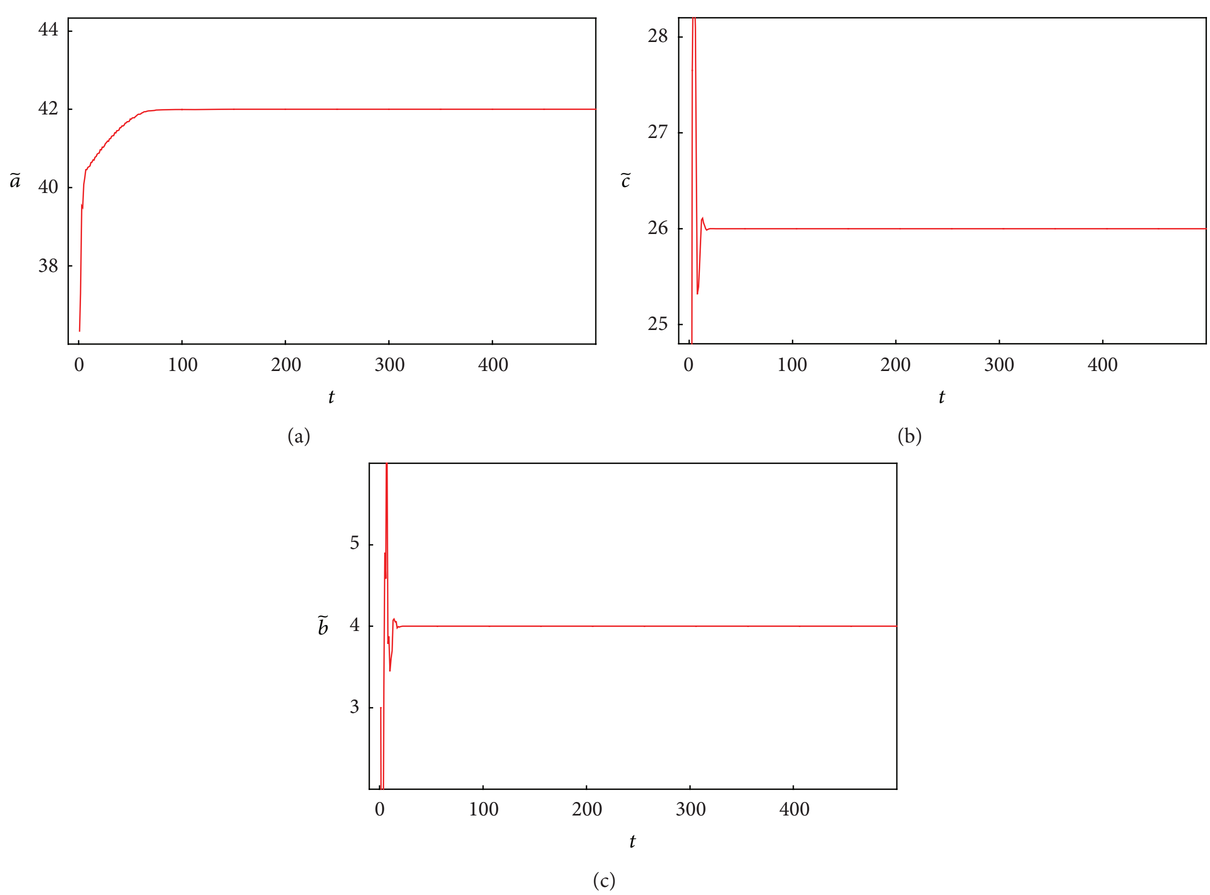

FIGURE 2: Adaptive parameters estimation laws $\tilde{a}(t)$, $\tilde{c}(t)$, and $\tilde{b}(t)$ of the master system (14) and slave system (15) versus t. (a) $(\tilde{a}(t), t)$ diagram, (b) $(\tilde{c}(t), t)$ diagram, and (c) $(\tilde{b}(t), t)$ diagram.

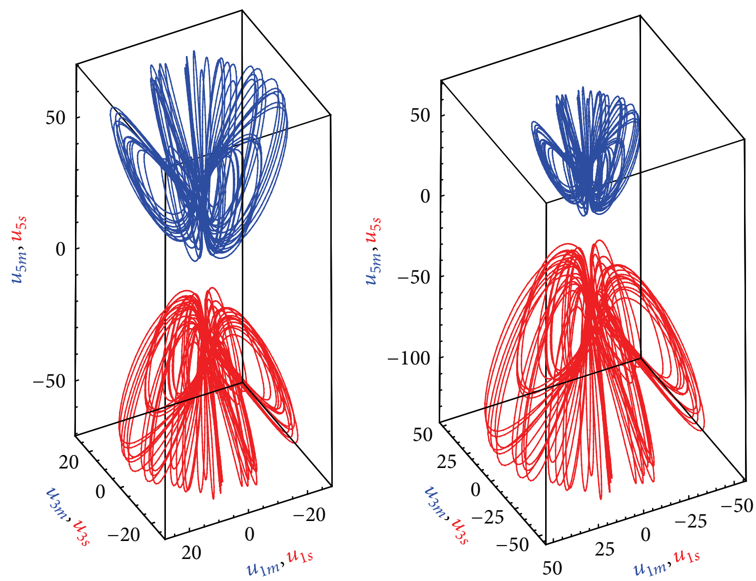

FIGURE 3: Chaotic attractors of the master system (20) (blue color) and the slave system (21) (red color): (a) when $\delta = -1$, (b) when $\delta = -2$.

and we can calculate the adaptive laws of parameters $\tilde{\alpha}$, $\tilde{\gamma}$, $\tilde{\beta}$, $\tilde{\rho}$, $\tilde{\nu}$, and $\tilde{\mu}$ by using (8) as

$$\begin{aligned}\dot{\tilde{\alpha}} &= -e_{u_1}(\delta u_{3m} - \delta u_{1m}) - e_{u_2}(\delta u_{4m} - \delta u_{2m}) + \zeta_1 \hat{\alpha},\\ \dot{\tilde{\gamma}} &= -\delta e_{u_3} u_{1m} - \delta e_{u_4} u_{2m} + \zeta_2 \hat{\gamma},\\ \dot{\tilde{\beta}} &= \delta e_{u_5} u_{5m} + \zeta_3 \hat{\beta},\\ \dot{\tilde{\rho}} &= e_{u_1}(u_{3s} - u_{1s}) + e_{u_2}(u_{4s} - u_{2s}) + \zeta_1 \hat{\rho},\\ \dot{\tilde{\nu}} &= e_{u_3} u_{3s} + e_{u_4} u_{4s} + \zeta_2 \hat{\nu}, \qquad \dot{\tilde{\mu}} = -e_{u_5} u_{5s} + \zeta_3 \hat{\mu}.\end{aligned} \tag{24}$$

4.2. Numerical Results. In this subsection, we solve systems (20) and (21) with (23) and (24) numerically (using, e.g., Mathematica 7 software) with the initial conditions $t_0 = 0$, $u_{1m}(0) = 1$, $u_{2m}(0) = 2$, $u_{3m}(0) = 3$, $u_{4m}(0) = 4$, $u_{5m}(0) = 5$, $u_{1s}(0) = 5$, $u_{2s}(0) = -3$, $u_{3s}(0) = 13$, $u_{4s}(0) = 2$, and $u_{5s}(0) = 8$. We choose $\mathbf{\Psi} = \text{diag}(12, 15, 11)$ and $\boldsymbol{\zeta} = \text{diag}(6, 9, 10)$. The initial values of the parameters estimation laws are $\tilde{\alpha}(0) = 1$, $\tilde{\gamma}(0) = 2$, $\tilde{\beta}(0) = 3$, $\tilde{\rho}(0) = 1$, $\tilde{\nu}(0) = 4$, and $\tilde{\mu}(0) = 5$. The results of adaptive projective synchronization of two different chaotic complex Lorenz and Lü systems are shown in Figure 3. In Figures 3(a) and 3(b) we plotted hyperchaotic attractors for different values of δ as $\delta = -1$ and -2, respectively. It is clear that, from Figure 3(a), the attractors in (u_1, u_3, u_5) plane of master system (20) and slave system (21) have the same size but opposite shape. In Figure 3(b), for $\delta = -2$, the two attractors have opposite shape and the size of the attractor of (20) is one half the slave system. In Figure 4 it can be seen that the synchronization errors will converge to zero after small value of t. Figure 4 shows the estimations of $\tilde{\alpha}(t)$, $\tilde{\gamma}(t)$, $\tilde{\beta}(t)$, $\tilde{\rho}(t)$, $\tilde{\nu}(t)$, and $\tilde{\mu}(t)$ of the unknown parameters of master and slave systems (20) and (21) which converge to $\alpha = 14$, $\gamma = 35$, $\beta = 3.7$, $\rho = 40$, $\nu = 22$, and $\mu = 5$, respectively, as $t \to \infty$.

5. The Application in Secure Communications

In this section, secure communications scheme based on projective synchronization between two identical chaotic complex Chen systems is investigated. We consider the two chaotic complex Chen systems as transmitter and receiver systems. The message signal $r(t)$ and chaotic signals of the transmitter system are encrypted by means of an invertible nonlinear function $\Xi = \phi(r, u_{1m}, u_{2m}, u_{3m}, u_{4m}, u_{5m})$. Then we add the signal Ξ to one of the five variables u_{1m}, u_{2m}, u_{3m}, u_{4m}, u_{5m}; for instance, we inject it into the variable u_{3m} so the combined signal is $\Delta(t) = \Xi + u_{3m}$. Then, chaotic signals of the transmitter system and combined signal are transmitted to the receiver side. In the receiver, the controller $\mathbf{L}$ can be constructed by (18), so the projective synchronization between two chaotic complex Chen systems will be achieved after some time and the states of $\mathbf{X}$ will approach $\mathbf{Y}/\delta$ where δ is a nonzero constant scaling factor and increases the content and security of the transmitted message. At a certain time the receiver starts to recover Ξ through a simple transformation $\Xi = \Delta(t) - u_{3s}/\delta$. Finally, since the nonlinear function ϕ is invertible, the message signal can be recovered as $\check{r}(t) = \phi^{-1}(u_{1s}, u_{2s}, u_{3s}, u_{4s}, u_{5s}, \Xi)$.

In the following numerical simulations the system parameters, initial conditions of the transmitter, and receiver systems are chosen as the same values as those in Section 3 and the constant scaling factor $\delta = 0.7$. We choose the invertible function as $\Xi = u_{1m} + \arctan(r(t))$; $r(t) = \cos(2\pi t)$ and we assume that the signal Ξ is added to the variable u_{3m}. The numerical simulation for the application of projective synchronization in secure communication is

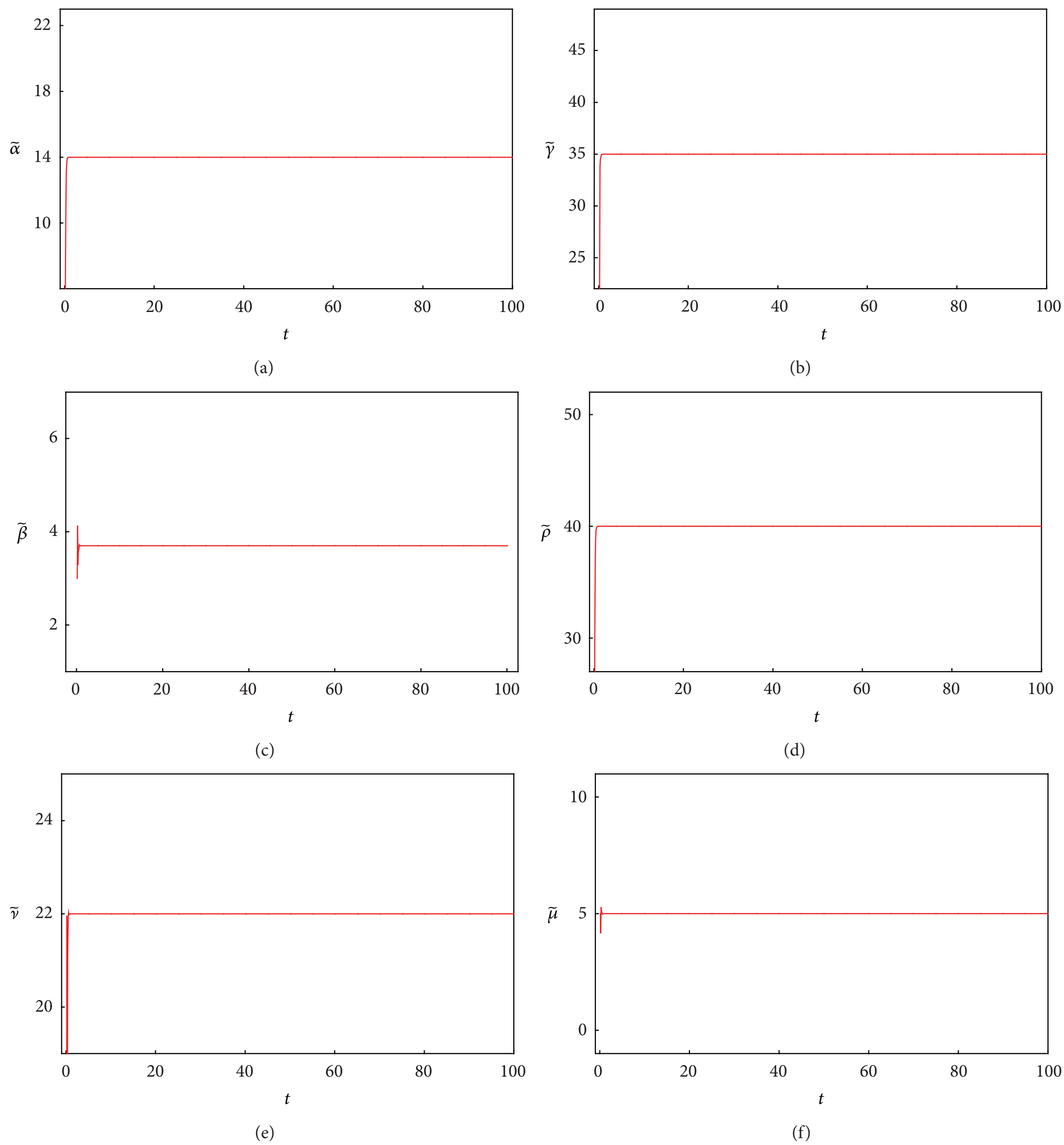

FIGURE 4: Adaptive parameters estimation laws $\tilde{\alpha}(t)$, $\tilde{\gamma}(t)$, $\tilde{\beta}(t)$, $\tilde{\rho}(t)$, $\tilde{\nu}(t)$, and $\tilde{\mu}(t)$ of the master system (20) and slave system (21) versus t. (a) $(\tilde{\alpha}(t), t)$ diagram, (b) $(\tilde{\gamma}(t), t)$ diagram, (c) $(\tilde{\beta}(t), t)$ diagram, (d) $(\tilde{\rho}(t), t)$ diagram, (e) $(\tilde{\nu}(t), t)$ diagram, and (f) $(\tilde{\mu}(t), t)$ diagram.

shown in Figure 5. The message $r(t)$ and the transmitted signal $\Delta(t)$ are shown in Figures 5(a) and 5(b), respectively. Figure 5(c) displays the recovered message $\check{r}(t)$. The error between the original message and the recovered one is shown in Figure 5(d). From Figure 4(d), it is easy to find that the information signal $s(t)$ is recovered accurately after a short transient.

6. Conclusion

Synchronization and control are important topics which have been studied to date primarily on dynamical systems described by real variables in applied nonlinear sciences. There also exist, however, interesting cases of dynamical systems where the main variables participating in the dynamics

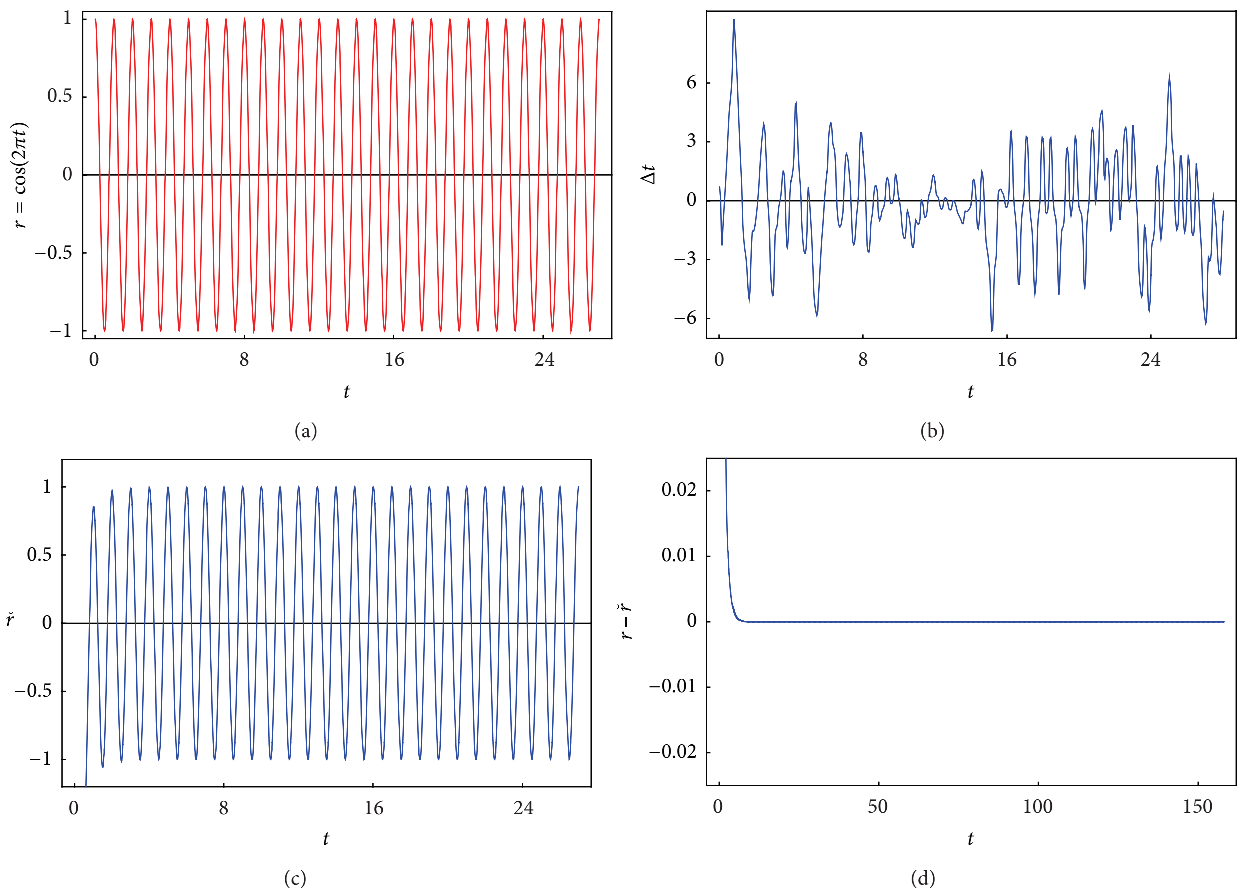

Figure 5: Simulation results of secure communication using projective synchronization of two identical chaotic complex Chen systems when the message signal is $r(t) = \cos 2\pi t$. (a) The original message $r(t)$. (b) The transmitted signal $\Delta(t)$. (c) The recovered message $\check{r}(t)$. (d) The error signal $r(t) - \check{r}(t)$.

are complex as, for example, when amplitudes of electromagnetic fields are involved. Our goal in this paper is to study and investigate projective synchronization of chaotic attractors of complex systems with uncertain parameters. A scheme is designed to achieve projective synchronization of two identical or different chaotic complex nonlinear systems with uncertain parameters based on Lyapunov functions. Through this scheme we determined analytically the control complex functions and adaptive laws of parameters to achieve projective synchronization. Illustrative examples are given to verify the correctness of our scheme. The secure communications by using projective synchronization in two chaotic complex Chen systems are implemented.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors would like to thank Institute of Scientific Research and Revival of Islamic Heritage at Umm Al-Qura University (Project ID 43305003) for the financial support.

References

[1] R. Mainieri and J. Rehacek, "Projective synchronization in three-dimensional chaotic systems," *Physical Review Letters*, vol. 82, no. 15, pp. 3042–3045, 1999.

[2] D. Xu and Z. Li, "Controlled projective synchronization in nonpartially-linear chaotic systems," *International Journal of Bifurcation and Chaos*, vol. 12, no. 6, pp. 1395–1402, 2002.

[3] G.-H. Li, "Generalized projective synchronization between Lorenz system and Chen's system," *Chaos, Solitons and Fractals*, vol. 32, pp. 1454–1458, 2007.

[4] J. Yan and C. Li, "Generalized projective synchronization of a unified chaotic system," *Chaos, Solitons and Fractals*, vol. 26, no. 4, pp. 1119–1124, 2005.

[5] Q. Jia, "Projective synchronization of a new hyperchaotic Lorenz system," *Physics Letters A*, vol. 370, no. 1, pp. 40–45, 2007.

[6] M. Hu and Z. Xu, "Adaptive feedback controller for projective synchronization," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 3, pp. 1253–1260, 2008.

[7] A. C. Fowler, J. D. Gibbon, and M. J. McGuinness, "The real and complex Lorenz equations and their relevance to physical systems," *Physica D*, vol. 7, no. 1–3, pp. 126–134, 1983.

[8] C.-Z. Ning and H. Haken, "Detuned lasers and the complex Lorenz equations: subcritical and supercritical Hopf bifurcations," *Physical Review A*, vol. 41, no. 7, pp. 3826–3837, 1990.

[9] E. E. Mahmoud, "Adaptive anti-lag synchronization of two identical or non-identical hyperchaotic complex nonlinear systems with uncertain parameters," *Journal of the Franklin Institute*, vol. 349, no. 3, pp. 1247–1266, 2012.

[10] E. E. Mahmoud, "Dynamics and synchronization of new hyperchaotic complex Lorenz system," *Mathematical and Computer Modelling*, vol. 55, no. 7-8, pp. 1951–1962, 2012.

[11] E. E. Mahmoud, "Modified projective phase synchronization of chaotic complex nonlinear systems," *Mathematics and Computers in Simulation*, vol. 89, pp. 69–85, 2013.

[12] E. E. Mahmoud, "Lag synchronization of hyperchaotic complex nonlinear systems via passive control," *Applied Mathematics & Information Sciences*, vol. 7, no. 4, pp. 1429–1436, 2013.

[13] E. E. Mahmoud, "Complex complete synchronization of two non-identical hyperchaotic complex nonlinear systems," *Mathematical Methods in the Applied Sciences*, vol. 37, no. 3, pp. 321–328, 2014.

[14] P. Liu and S. Liu, "Anti-synchronization between different chaotic complex systems," *Physica Scripta*, vol. 83, no. 6, Article ID 065006, 2011.

[15] S. Liu and P. Liu, "Adaptive anti-synchronization of chaotic complex nonlinear systems with unknown parameters," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 6, pp. 3046–3055, 2011.

[16] G. M. Mahmoud, M. A. Al-Kashif, and S. A. Aly, "Basic properties and chaotic synchronization of complex Lorenz system," *International Journal of Modern Physics C*, vol. 18, no. 2, pp. 253–265, 2007.

[17] G. M. Mahmoud and E. E. Mahmoud, "Complex modified projective synchronization of two chaotic complex nonlinear systems," *Nonlinear Dynamics*, vol. 73, no. 4, pp. 2231–2240, 2013.

[18] G. M. Mahmoud, E. E. Mahmoud, and A. A. Arafa, "Passive control of *n*-dimensional chaotic complex nonlinear systems," *Journal of Vibration and Control*, vol. 19, pp. 1061–1071, 2013.

[19] G. M. Mahmoud, T. Bountis, and E. E. Mahmoud, "Active control and global synchronization of the complex Chen and Lü systems," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 17, no. 12, pp. 4295–4308, 2007.

[20] S. Wang and H. Yao, "The effect of control strength on lag synchronization of nonlinear coupled complex networks," *Abstract and Applied Analysis*, vol. 2012, Article ID 810364, 11 pages, 2012.

[21] X. Zhou, L. Xiong, W. Cai, and X. Cai, "Adaptive synchronization and antisynchronization of a hyperchaotic complex Chen system with unknown parameters based on passive control," *Journal of Applied Mathematics*, vol. 2013, Article ID 845253, 8 pages, 2013.

[22] G. M. Mahmoud, E. E. Mahmoud, and A. A. Arafa, "Controlling hyperchaotic complex systems with unknown parameters based on adaptive passive method," *Chinese Physics B*, vol. 22, Article ID 060508, 2013.

[23] G. M. Mahmoud, E. E. Mahmoud, and A. A. Arafa, "On projective synchronization of hyperchaotic complex nonlinear systems based on passive theory for secure communications," *Physica Scripta*, vol. 87, Article ID 055002, 2013.

[24] G. M. Mahmoud and E. E. Mahmoud, "Modified projective lag synchronization of two nonidentical hyperchaotic complex nonlinear systems," *International Journal of Bifurcation and Chaos*, vol. 21, no. 8, pp. 2369–2379, 2011.

[25] E. E. Mahmoud and K. M. Abualnaja, "Complex lag synchronization of two identical chaotic complex nonlinear systems," *Central European Journal of Physics*, vol. 12, pp. 63–69, 2014.

[26] G. M. Mahmoud and E. E. Mahmoud, "Synchronization and control of hyperchaotic complex Lorenz system," *Mathematics and Computers in Simulation*, vol. 80, no. 12, pp. 2286–2296, 2010.

[27] A. H. Nayfeh and B. Balachandran, *Applied Nonlinear Dynamics*, John Wiley & Sons, New York, NY, USA, 1995.