

## Research Article

# Almost Periodic Solutions for Wilson-Cowan Type Model with Time-Varying Delays

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Wilson-Cowan model of neuronal population with time-varying delays is considered in this paper. Some sufficient conditions for the existence and delay-based exponential stability of a unique almost periodic solution are established. The approaches are based on constructing Lyapunov functionals and the well-known Banach contraction mapping principle. The results are new, easily checkable, and complement existing periodic ones.

## 1. Introduction

Consider a well-known Wilson-Cowan type model [1, 2] with time-varying delays

$$\begin{aligned} \frac{dX_P(t)}{dt} &= -X_P(t) + [k_P - r_P X_P(t)] \\ &\quad \times G \left[ w_P^1 X_P(t - \tau_P(t)) \right. \\ &\quad \left. - w_N^1 X_N(t - \tau_N(t)) + I_P(t) \right], \\ \frac{dX_N(t)}{dt} &= -X_N(t) + [k_N - r_N X_N(t)] \\ &\quad \times G \left[ w_P^2 X_P(t - \tau_P(t)) \right. \\ &\quad \left. - w_N^2 X_N(t - \tau_N(t)) + I_N(t) \right], \end{aligned} \quad (1)$$

where  $X_P(t), X_N(t)$  represent the proportion of excitatory and inhibitory neurons firing per unit time at the instant  $t$ , respectively.  $r_P$  and  $r_N$  are related to the duration of the refractory period,  $k_P$  and  $k_N$  are constants.  $w_P^1, w_N^1, w_P^2$ , and  $w_N^2$  are the strengths of connections between the populations.  $I_P(t), I_N(t)$  are the external inputs to the excitatory and the inhibitory populations.  $G(\cdot)$  is the response function of neuronal activity and it is always assumed to be sigmoid type.

$\tau_P(t), \tau_N(t)$  correspond to the transmission time-varying delays.

It is interesting to revisit Wilson-Cowan system on the following points.

- (i) The Wilson-Cowan model has a realistic biological background which describes interactions between excitatory and inhibitory populations of neurons [1–3]. It has extensive application such as pattern analysis and image processing [4, 5].
- (ii) There exists rich dynamical behavior in Wilson-Cowan model. Theoretical results about stable limit cycles, equilibria, chaos, and oscillatory activity have been reported in [2, 3, 6–9]. Recently, Decker and Noonburg [8] reported new results about the existence of three periodic solutions when each neuron was stimulated by periodical inputs. However, under time-varying (periodic or almost periodic) inputs, Wilson-Cowan model can have more complex state space and coexistence of divergent solutions and local stable solutions which could not be easily estimated by its boundary. To see this, we can refer to Figure 1 for the phase portrait of solutions of the following

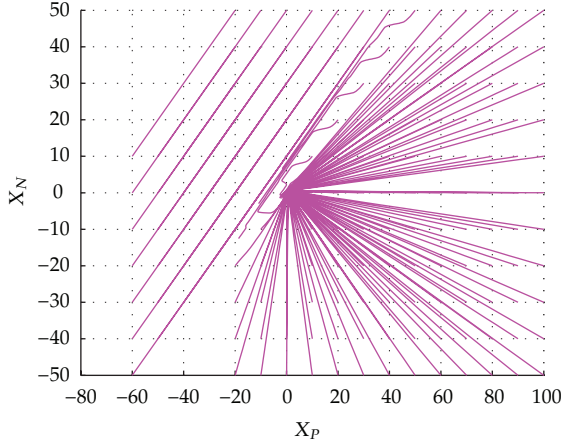


FIGURE 1: Coexistence of divergent and local stable solutions of (2) with almost periodic inputs.

Wilson-Cowan type model with  $G(z) = \tanh(z)$  and almost periodic inputs [10–12]:

$$\begin{aligned} \frac{dX_P(t)}{dt} &= -X_P(t) + [1 - X_P(t)] \\ &\quad \times G[X_P(t) - X_N(t) + 5 \sin \sqrt{2}t], \\ \frac{dX_N(t)}{dt} &= -X_N(t) + [1 - X_N(t)] \\ &\quad \times G[X_P(t) - X_N(t) + 0.5 \cos t \sin t]. \end{aligned} \quad (2)$$

(iii) Few works reported almost periodicity of Wilson-Cowan type model in the literature. Under almost periodic inputs, whether there exists a unique almost periodic solution of (1) which is stable? How to estimate its located boundary? Revealing these results can give a significant insight into the complex dynamical structure of Wilson-Cowan type model.

Throughout this paper, we always assume that  $k_P, k_N, r_P, r_N, w_P^1, w_P^2, w_N^1, w_N^2$  are positive constants,  $\tau_P(t), \tau_N(t), I_P(t)$ , and  $I_N(t)$  are almost periodic functions [12], and set

$$\begin{aligned} \tau_P^\top &= \sup_{t \in \mathbb{R}} \tau_P(t), & \tau_N^\top &= \sup_{t \in \mathbb{R}} \tau_N(t), \\ I_P^\top &= \sup_{t \in \mathbb{R}} |I_P(t)|, & I_N^\top &= \sup_{t \in \mathbb{R}} |I_N(t)|. \end{aligned} \quad (3)$$

Moreover, we need some basic assumptions in this paper.

(H<sub>1</sub>)  $G(0) = 0$ ,  $\sup_{v \in \mathbb{R}} |G(v)| \leq B_s$  and there exists an  $L > 0$  such that

$$|G(u) - G(v)| \leq L|u - v| \quad \text{for } \forall u, v \in \mathbb{R}. \quad (4)$$

(H<sub>2</sub>) The quadratic equation  $(K + Rx)(I + Wx) = x$  has a positive solution  $\delta$ , where

$$\begin{aligned} W &= \max\{L(w_P^1 + w_N^1), L(w_P^2 + w_N^2)\}, \\ K &= \max\{k_P, k_N\}, \quad R = \max\{r_P, r_N\}, \\ I &= \max\{LI_P^\top, LI_N^\top\}. \end{aligned} \quad (5)$$

(H<sub>3</sub>)  $\tau_P(t), \tau_N(t)$  are bounded and continuously differentiable with  $0 \leq \tau_P(t) \leq \tau_P^\top$ ,

$$\begin{aligned} 0 \leq \tau_N(t) \leq \tau_N^\top, \quad 1 - \dot{\tau}_P(t) > 0, \\ 1 - \dot{\tau}_N(t) > 0 \quad \text{for } t \in \mathbb{R}. \end{aligned} \quad (6)$$

For for all  $u = (u_1, u_2) \in \mathbb{R}^2$ , we define the norm  $\|u\| = \max\{|u_1|, |u_2|\}$ . Let  $B := \{\psi \mid \psi = (\psi_1, \psi_2)\}$ , where  $\psi$  is an almost periodic function on  $\mathbb{R}^2$ . For all  $\psi \in B$ , if we define induced nodule  $\|\psi\|_B = \sup_{t \in \mathbb{R}} \|\psi(t)\|$ , then  $B$  is a Banach space. The initial conditions of system (1) are of the form

$$X_P(s) = \psi_P(s), \quad X_N(s) = \psi_N(s), \quad s \in [-\tau, 0], \quad (7)$$

where  $\tau = \max\{\tau_P^\top, \tau_N^\top\}$  and  $\psi = (\psi_P, \psi_N) \in B$ .

*Definition 1* (see [12]). Let  $u(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  be continuous.  $u(t)$  is said to be almost periodic on  $\mathbb{R}$  if, for any  $\varepsilon > 0$ , it is possible to find a real number  $l = l(\varepsilon) > 0$ , and for any interval with length  $l(\varepsilon)$ , there exists a number  $\delta = \delta(\varepsilon)$  in this interval such that  $|u(t + \delta) - u(t)| < \varepsilon$ , for all  $t \in \mathbb{R}$ .

The remaining part of this paper is organized as follows. In Section 2, we will derive sufficient conditions for checking the existence of almost periodic solutions. In Section 3, we present delay-based exponential stability of the unique almost periodic solution of system (1). In Section 4, we will give an example to illustrate our results obtained in the preceding sections. Concluding remarks are given in Section 5.

## 2. Existence of Almost Periodic Solutions

**Theorem 2.** *Suppose that (H<sub>1</sub>) and (H<sub>2</sub>) hold. If  $KW + R(B_s + \delta W) < 1$ , then there exists a unique almost periodic solution of system (1) in the region*

$$B^* = \{\psi \mid \psi \in B, \|\psi\|_B \leq \delta\}. \quad (8)$$

*Proof.* For for all  $\psi = (\psi_P, \psi_N) \in B^*$ , we consider the almost periodic solution  $X^\psi(t) = (X_P^\psi(t), X_N^\psi(t))$  of the following almost periodic differential equations:

$$\begin{aligned} \frac{dX_P(t)}{dt} &= -X_P(t) + [k_P - r_P \psi_P(t)] \\ &\quad \times G \left[ w_P^1 \psi_P(t - \tau_P(t)) \right. \\ &\quad \left. - w_N^1 \psi_N(t - \tau_N(t)) + I_P(t) \right], \\ \frac{dX_N(t)}{dt} &= -X_N(t) + [k_N - r_N \psi_N(t)] \\ &\quad \times G \left[ w_P^2 \psi_P(t - \tau_P(t)) \right. \\ &\quad \left. - w_N^2 \psi_N(t - \tau_N(t)) + I_N(t) \right]. \end{aligned} \quad (9)$$

By almost periodicity of  $\tau_P(t)$ ,  $\tau_N(t)$ ,  $I_P(t)$ , and  $I_N(t)$  and Theorem 3.4 in [12] or [10], (9) has a unique almost periodic solution

$$\begin{aligned} X_P^\psi(t) &= \int_{-\infty}^t \exp(-(t-s)) [k_P - r_P \psi_P(s)] \\ &\quad \times G \left[ w_P^1 \psi_P(s - \tau_P(s)) \right. \\ &\quad \left. - w_N^1 \psi_N(s - \tau_N(s)) + I_P(s) \right] ds, \end{aligned} \quad (10)$$

$$\begin{aligned} X_N^\psi(t) &= \int_{-\infty}^t \exp(-(t-s)) [k_N - r_N \psi_N(s)] \\ &\quad \times G \left[ w_P^2 \psi_P(s - \tau_P(s)) \right. \\ &\quad \left. - w_N^2 \psi_N(s - \tau_N(s)) + I_N(s) \right] ds. \end{aligned}$$

Define a mapping  $F : B^* \rightarrow B$  by setting  $F(\psi)(t) = (F_P(\psi)(t), F_N(\psi)(t)) = X^\psi(t)$ , for all  $\psi \in B^*$ . Now, we prove that  $F$  is a self-mapping from  $B^*$  to  $B^*$ . From (10) and  $(H_1)$ , we obtain

$$\begin{aligned} |F_P(\psi)(t)| &\leq \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^t \exp(-(t-s)) [k_P - r_P \psi_P(s)] \right. \\ &\quad \times G \left[ w_P^1 \psi_P(s - \tau_P(s)) \right. \\ &\quad \left. - w_N^1 \psi_N(s - \tau_N(s)) + I_P(s) \right] ds \Big| \end{aligned}$$

$$\begin{aligned} &\leq \sup_{t \in \mathbb{R}} [k_P + r_P |\psi_P(t)|] \\ &\quad \times G \left| \left[ w_P^1 \psi_P(t - \tau_P(t)) \right. \right. \\ &\quad \left. \left. - w_N^1 \psi_N(t - \tau_N(t)) + I_P(t) \right] \right| \\ &\leq (k_P + r_P \|\psi\|_B) \\ &\quad \times L (w_P^1 \|\psi\|_B + w_N^1 \|\psi\|_B + I_P^\top) \\ &\leq (k_P + r_P \|\psi\|_B) \\ &\quad \times (L (w_P^1 + w_N^1) \|\psi\|_B + LI_P^\top). \end{aligned} \quad (11)$$

By similar estimation, we can get

$$\begin{aligned} |F_N(\psi)| &\leq (k_N + r_N \|\psi\|_B) \\ &\quad \times (L (w_P^2 + w_N^2) \|\psi\|_B + LI_N^\top). \end{aligned} \quad (12)$$

Therefore, by the above estimations and  $(H_2)$ , we get

$$\begin{aligned} \|F(\psi)\|_B &= \sup_{t \in \mathbb{R}} \{|F_P(\psi)|, |F_N(\psi)|\} \\ &\leq (K + R\|\psi\|_B) (W\|\psi\|_B + I) \\ &\leq (K + R\delta) (W\delta + I) = \delta, \end{aligned} \quad (13)$$

which implies that  $F(\psi) \in B^*$ . So, the mapping  $F$  is self-mapping from  $B^*$  to  $B^*$ . Next, we prove that  $F$  is a contraction mapping in the region  $B^*$ . For all  $\psi, \phi \in B^*$ , by (10), we have

$$\begin{aligned} &|F_P(\psi)(t) - F_P(\phi)(t)| \\ &\leq \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^t \exp(-(t-s)) [k_P - r_P \psi_P(s)] \right. \\ &\quad \times G \left[ w_P^1 \psi_P(s - \tau_P(s)) \right. \\ &\quad \left. - w_N^1 \psi_N(s - \tau_N(s)) + I_P(s) \right] ds \\ &\quad - \int_{-\infty}^t \exp(-(t-s)) [k_P - r_P \phi_P(s)] \\ &\quad \times G \left[ w_P^1 \phi_P(s - \tau_P(s)) \right. \\ &\quad \left. - w_N^1 \phi_N(s - \tau_N(s)) + I_P(s) \right] ds \Big|, \end{aligned} \quad (14)$$

which leads to

$$\begin{aligned}
& |F_P(\psi)(t) - F_P(\phi)(t)| \\
& \leq \sup_{t \in \mathbb{R}} \int_{-\infty}^t \exp(-(t-s)) \\
& \quad \times \left[ k_P L \left| w_P^1 \psi_P(s - \tau_P(s)) - w_N^1 \psi_N(s - \tau_N(s)) \right. \right. \\
& \quad \quad \left. \left. - w_P^1 \phi_P(s - \tau_P(s)) + w_N^1 \phi_N(s - \tau_N(s)) \right| \right. \\
& \quad \left. + r_P \left| \psi_P(s) G \left[ w_P^1 \psi_P(s - \tau_P(s)) \right. \right. \right. \\
& \quad \quad \quad \left. \left. - w_N^1 \psi_N(s - \tau_N(s)) \right. \right. \\
& \quad \quad \quad \left. \left. + I_P(s) \right] \right. \\
& \quad \left. - \phi_P(s) G \left[ w_P^1 \phi_P(s - \tau_P(s)) \right. \right. \\
& \quad \quad \quad \left. \left. - w_N^1 \phi_N(s - \tau_N(s)) \right. \right. \\
& \quad \quad \quad \left. \left. + I_P(s) \right] \right| \Big] ds \\
& \leq k_P L \left( w_P^1 \sup_{t \in \mathbb{R}} |\psi_P(t) - \phi_P(t)| + w_N^1 \sup_{t \in \mathbb{R}} |\psi_N(t) - \phi_N(t)| \right) \\
& \quad + \sup_{t \in \mathbb{R}} \int_{-\infty}^t \exp(-(t-s)) r_P \\
& \quad \times \left[ \left| \psi_P(s) G \left( w_P^1 \psi_P(s - \tau_P(s)) \right. \right. \right. \\
& \quad \quad \left. \left. - w_N^1 \psi_N(s - \tau_N(s)) + I_P(s) \right) \right. \\
& \quad \left. - \psi_P(s) G \left( w_P^1 \phi_P(s - \tau_P(s)) \right. \right. \\
& \quad \quad \left. \left. - w_N^1 \phi_N(s - \tau_N(s)) \right. \right. \\
& \quad \quad \left. \left. + I_P(s) \right) \right| \\
& \quad + \left| \psi_P(s) G \left( w_P^1 \phi_P(s - \tau_P(s)) \right. \right. \\
& \quad \quad \left. \left. - w_N^1 \phi_N(s - \tau_N(s)) + I_P(s) \right) \right. \\
& \quad \left. - \phi_P(s) G \left( w_P^1 \phi_P(s - \tau_P(s)) \right. \right. \\
& \quad \quad \left. \left. - w_N^1 \phi_N(s - \tau_N(s)) + I_P(s) \right) \right| \Big] ds \\
& \leq k_P L \left( w_P^1 + w_N^1 \right) \|\psi - \phi\|_B \\
& \quad + r_P \left( B_s \|\psi - \phi\|_B + \delta L \left( w_P^1 + w_N^1 \right) \|\psi - \phi\|_B \right). \tag{15}
\end{aligned}$$

By similar argument, we can get

$$\begin{aligned}
& |F_N(\psi)(t) - F_N(\psi)(t)| \\
& \leq k_N L \left( w_P^2 + w_N^2 \right) \|\psi - \phi\|_B \\
& \quad + r_N \left( B_s \|\psi - \phi\|_B + \delta L \left( w_P^2 + w_N^2 \right) \|\psi - \phi\|_B \right). \tag{16}
\end{aligned}$$

From (15) and (16), we have

$$\begin{aligned}
& \|F(\psi) - F(\phi)\|_B \\
& = \sup_{t \in \mathbb{R}} \{ |F_P(\psi) - F_P(\phi)|, |F_N(\psi) - F_N(\phi)| \} \\
& \leq KW \|\psi - \phi\| \\
& \quad + R(B_s \|\psi - \phi\| + \delta W \|\psi - \phi\|) \\
& = (KW + RB_s + R\delta W) \|\psi - \phi\|. \tag{17}
\end{aligned}$$

Since  $KW + RB_s + R\delta W \in (0, 1)$ , it is clear that the mapping  $F$  is a contraction. Therefore the mapping  $F$  possesses a unique fixed point  $X^* \in B^*$  such that  $FX^* = X^*$ . By (9),  $X^*$  is an almost periodic solution of system (1) in  $B^*$ . The proof is complete.  $\square$

*Remark 3.* Obviously, quadratic curve  $\mathcal{C}(v) := RWv^2 + (KW + RI - 1)v + KI$  satisfies with  $\mathcal{C}(0) \geq 0$ . So,  $\Delta := (KW + RI - 1)^2 - 4RWKI > 0$  and  $KW + RI < 1$  guarantees the existence of  $\delta$  in  $(H_2)$  and  $\delta$  lies in the following interval:

$$\left[ \frac{1 - KW - RI - \sqrt{\Delta}}{2RW}, \frac{1 - KW - RI + \sqrt{\Delta}}{2RW} \right]. \tag{18}$$

By Theorem 2, we know that the unique almost periodic solution depends on  $KW + R(B_s + \delta W) < 1$ . Choosing  $\delta = (1 - (KW + RI))/2RW$ , we get a simple assumption as follows:

$$(\widehat{H}_2) \quad KW + RI < 1 - 2\sqrt{KWR I}, RB_s < (1 - (KW - RI))/2,$$

and hence it leads to a parameter-based result.

**Corollary 4.** *Suppose that  $(H_1)$  and  $(\widehat{H}_2)$  hold. Then there exists a unique almost periodic solution of system (1) in the region  $B^* = \{\psi \mid \psi \in B, \|\psi\|_B \leq (1 - (KW + RI))/2RW\}$ .*

### 3. Delay-Based Stability of the Almost Periodic Solution

In this section, we establish locally exponential stability of the unique almost periodic solution of system (1) in the region  $B^*$ , which is delay dependent.

**Theorem 5.** *Suppose that  $(H_1)$ – $(H_3)$  hold. If  $KW + R(B_s + \delta W) < 1$  and there exist constants  $\ell_1 > 0$ ,  $\ell_2 > 0$  such that*

$$\begin{aligned}
(1 - r_P B_s) \ell_1 & > \sup_{t \in \mathbb{R}} \frac{C_P}{1 - \tau_P(v_P^{-1}(t))}, \\
(1 - r_N B_s) \ell_2 & > \sup_{t \in \mathbb{R}} \frac{C_N}{1 - \tau_N(v_N^{-1}(t))}, \tag{19}
\end{aligned}$$

where  $C_P := L[(k_P + r_P \delta) \ell_1 w_P^1 + (k_N + r_N \delta) \ell_2 w_P^2]$  and  $C_N := L[(k_P + r_P \delta) \ell_1 w_N^1 + (k_N + r_N \delta) \ell_2 w_N^2]$ ,  $v_P^{-1}(t)$  and  $v_N^{-1}(t)$  are the inverse functions of  $v_P(t) = t - \tau_P(t)$  and  $v_N(t) = t - \tau_N(t)$ , then system (1) has exactly one almost periodic solution  $X^*(t)$  in the region  $B^*$  which is locally exponentially stable.

*Proof.* From Theorem 2, system (1) has a unique almost periodic solution  $X^*(t) \in B^*$ . Let  $X(t) = (X_P(t), X_N(t))$  be an arbitrary solution of system (1) with initial value  $\psi = (\psi_P, \psi_N) \in B^*$ . Set  $X(t) = X_P(t) - X_P^*(t)$ ,  $Y(t) = X_N(t) - X_N^*(t)$ . By system (1), we get

$$\begin{aligned} \frac{dX(t)}{dt} &= -X(t) \\ &+ [k_P - r_P(X(t) + X_P^*(t))] \\ &\times G[w_P^1(X(t - \tau_P(t)) + X_P^*(t - \tau_P(t))) \\ &\quad - w_N^1(Y(t - \tau_N(t)) + X_N^*(t - \tau_N(t)))I_P(t)] \\ &- [k_P - r_P X_P^*(t)] \\ &\times G[w_P^1 X_P^*(t - \tau_P(t)) \\ &\quad - w_N^1 X_N^*(t - \tau_N(t)) + I_P(t)], \\ \frac{dY(t)}{dt} &= -Y(t) \\ &+ [k_N - r_N(Y(t) + X_N^*(t))] \\ &\times G[w_P^2(X(t - \tau_P(t)) + X_P^*(t - \tau_P(t))) \\ &\quad - w_N^2(Y(t - \tau_N(t)) + X_N^*(t - \tau_N(t))) + I_N(t)] \\ &- [k_N - r_N X_N^*(t)] \\ &\times G[w_P^2 X_P^*(t - \tau_P(t)) \\ &\quad - w_N^2 X_N^*(t - \tau_N(t)) + I_N(t)]. \end{aligned} \quad (20)$$

Construct the auxiliary functions  $F_P(u)$ ,  $F_N(u)$  defined on  $[0, +\infty)$  as follows:

$$\begin{aligned} F_P(u) &:= (u - 1 + r_P B_s) \ell_1 + \sup_{t \in \mathbb{R}} \frac{C_P e^{u \tau_P^\top}}{1 - \dot{\tau}_P(v_P^{-1}(t))}, \\ F_N(u) &:= (u - 1 + r_N B_s) \ell_2 + \sup_{t \in \mathbb{R}} \frac{C_N e^{u \tau_N^\top}}{1 - \dot{\tau}_N(v_N^{-1}(t))}. \end{aligned} \quad (21)$$

One can easily show that  $F_P(u)$ ,  $F_N(u)$  are well defined and continuous. Assumption (19) implies that  $F_P(0) < 0$ ,  $F_P(u) \rightarrow +\infty$  as  $u \rightarrow +\infty$  and  $F_N(0) < 0$ ,  $F_N(u) \rightarrow +\infty$  as  $u \rightarrow +\infty$ . It follows that there exists a common  $\lambda > 0$  such that  $F_P(\lambda) < 0$  and  $F_N(\lambda) < 0$ .

Consider the Lyapunov functional

$$\begin{aligned} V(t) &= [\ell_1 |X(t)| + \ell_2 |Y(t)|] e^{\lambda t} \\ &+ \int_{t-\tau_P(t)}^t \frac{C_P}{1 - \dot{\tau}_P(v_P^{-1}(s))} |X(s)| e^{\lambda(s+\tau_P^\top)} ds \\ &+ \int_{t-\tau_N(t)}^t \frac{C_N}{1 - \dot{\tau}_N(v_N^{-1}(s))} |Y(s)| e^{\lambda(s+\tau_N^\top)} ds. \end{aligned} \quad (22)$$

Calculating the upper right derivative of  $V(t)$  along system (1), one has

$$\begin{aligned} D^+V(t) &\leq \lambda [\ell_1 |X(t)| + \ell_2 |Y(t)|] e^{\lambda t} \\ &+ e^{\lambda t} \ell_1 [-|X(t)| \\ &\quad + |[k_P - r_P(X(t) + X_P^*(t))] \\ &\quad \times G[w_P^1(X(t - \tau_P(t)) + X_P^*(t - \tau_P(t))) \\ &\quad - w_N^1(Y(t - \tau_N(t)) + X_N^*(t - \tau_N(t))) \\ &\quad + I_P(t)] \\ &\quad - [k_P - r_P X_P^*(t)] \\ &\quad \times G[w_P^1 X_P^*(t - \tau_P(t)) \\ &\quad - w_N^1 X_N^*(t - \tau_N(t)) + I_P(t)]] \\ &+ e^{\lambda t} \ell_2 [-|Y(t)| \\ &\quad + |[k_N - r_N(Y(t) + X_N^*(t))] \\ &\quad \times G[w_P^2(X(t - \tau_P(t)) + X_P^*(t - \tau_P(t))) \\ &\quad - w_N^2(Y(t - \tau_N(t)) + X_N^*(t - \tau_N(t))) \\ &\quad + I_N(t)] \\ &\quad - [k_N - r_N X_N^*(t)] \\ &\quad \times G[w_P^2 X_P^*(t - \tau_P(t)) \\ &\quad - w_N^2 X_N^*(t - \tau_N(t)) + I_N(t)]] \\ &+ \frac{C_P}{1 - \dot{\tau}_P(v_P^{-1}(t))} |X(t)| e^{\lambda(t+\tau_P^\top)} \\ &- C_P |X(t - \tau_P(t))| \exp[\lambda(t - \tau_P(t) + \tau_P^\top)] \\ &+ \frac{C_N}{1 - \dot{\tau}_N(v_N^{-1}(t))} |Y(t)| e^{\lambda(t+\tau_N^\top)} \\ &- C_N |Y(t - \tau_N(t))| \exp[\lambda(t - \tau_N(t) + \tau_N^\top)], \end{aligned} \quad (23)$$

which leads to

$$\begin{aligned} D^+V(t) &\leq [(\lambda - 1) \ell_1 e^{\lambda t} |X(t)| + (\lambda - 1) \ell_2 e^{\lambda t} |Y(t)|] \\ &+ e^{\lambda t} \ell_1 [k_P L (w_P^1 |X(t - \tau_P(t))| \\ &\quad + w_N^1 |Y(t - \tau_N(t))|) + r_P |X(t)| B_s \\ &+ r_P \delta L (w_P^1 |X(t - \tau_P(t))| \\ &\quad + w_N^1 |Y(t - \tau_N(t))|)] \end{aligned}$$

$$\begin{aligned}
& + e^{\lambda t} \ell_2 \left[ k_N L \left( w_P^2 |X(t - \tau_P(t))| \right. \right. \\
& \quad \left. \left. + w_N^2 |Y(t - \tau_N(t))| \right) \right. \\
& \quad \left. + r_N |Y(t)| B_s + r_N \delta L \left( w_P^2 |X(t - \tau_P(t))| \right. \right. \\
& \quad \quad \left. \left. + w_N^2 |Y(t - \tau_N(t))| \right) \right] \\
& + \frac{C_P}{1 - \dot{\tau}_P(v_P^{-1}(t))} |X(t)| e^{\lambda(t + \tau_P^\top)} \\
& - C_P |X(t - \tau_P(t))| e^{\lambda t} \\
& + \frac{C_N}{1 - \dot{\tau}_N(v_N^{-1}(t))} |Y(t)| e^{\lambda(t + \tau_N^\top)} \\
& - C_N |Y(t - \tau_N(t))| e^{\lambda t} \\
& \leq \left[ (\lambda - 1 + r_P B_s) \ell_1 + \sup_{t \in \mathbb{R}} \frac{C_P e^{\lambda \tau_P^\top}}{1 - \dot{\tau}_P(v_P^{-1}(t))} \right] \\
& \quad \times e^{\lambda t} |X(t)| \\
& + \left[ (\lambda - 1 + r_N B_s) \ell_2 + \sup_{t \in \mathbb{R}} \frac{C_N e^{\lambda \tau_N^\top}}{1 - \dot{\tau}_N(v_N^{-1}(t))} \right] \\
& \quad \times e^{\lambda t} |Y(t)| \\
& \leq -c_1 [|X(t)| + |Y(t)|] e^{\lambda t},
\end{aligned} \tag{24}$$

where  $c_1 := -(1/2) \max\{F_P(\lambda), F_N(\lambda)\} > 0$ . We have from the above that  $V(t) \leq V(t_0)$  and

$$\min\{\ell_1, \ell_2\} e^{\lambda t} (|X(t)| + |Y(t)|) \leq V(t) \leq V(0), \quad t \geq 0. \tag{25}$$

Note that

$$\begin{aligned}
V(0) = & \left[ (\ell_1 + \ell_2) + \tau_P^\top \sup_{s \in [-\tau_P^\top, 0]} \frac{C_P}{1 - \dot{\tau}_P(v_P^{-1}(s))} e^{\lambda \tau_P^\top} \right. \\
& \left. + \tau_N^\top \sup_{s \in [-\tau_N^\top, 0]} \frac{C_N}{1 - \dot{\tau}_N(v_N^{-1}(s))} e^{\lambda \tau_N^\top} \right] \|\psi - \psi^*\|,
\end{aligned} \tag{26}$$

where  $\psi^*(s) = X^*(s)$ ,  $s \in [-\tau, 0]$ . Then there exists a positive constant  $M > 1$  such that

$$\begin{aligned}
|X_P(t) - X_P^*(t)| & \leq M \|\psi - \psi^*\| e^{-\lambda t}, \\
|X_N(t) - X_N^*(t)| & \leq M \|\psi - \psi^*\| e^{-\lambda t}.
\end{aligned} \tag{27}$$

The proof is complete.  $\square$

Set  $\delta = (1 - (KW + RI))/2RW$ . It follows from Corollary 4 and Theorem 5 the following.

**Corollary 6.** Suppose that  $(H_1)$ ,  $(\widehat{H}_2)$ , and  $(H_3)$  hold. If there exist constants  $\ell_1 > 0$ ,  $\ell_2 > 0$  such that

$$\begin{aligned}
(1 - r_P B_s) \ell_1 & > \sup_{t \in \mathbb{R}} \frac{L(\alpha_P \ell_1 w_P^1 + \alpha_N \ell_2 w_P^2)}{1 - \dot{\tau}_P(v_P^{-1}(t))}, \\
(1 - r_N B_s) \ell_2 & > \sup_{t \in \mathbb{R}} \frac{L(\alpha_P \ell_1 w_N^1 + \alpha_N \ell_2 w_N^2)}{1 - \dot{\tau}_N(v_N^{-1}(t))},
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\alpha_P & := k_P + r_P \frac{1 - (KW + RI)}{2RW}, \\
\alpha_N & := k_N + r_N \frac{1 - (KW + RI)}{2RW},
\end{aligned} \tag{29}$$

$v_P^{-1}(t)$  and  $v_N^{-1}(t)$  are defined as Theorem 5, then system (1) has exactly one almost periodic solution  $X^*(t)$  in the region

$$B^* = \left\{ \psi \mid \psi \in B, \|\psi\|_B \leq \frac{1 - (KW + RI)}{2RW} \right\}, \tag{30}$$

which is locally exponentially stable.

#### 4. An Example

In this section, we give an example to demonstrate the results obtained in previous sections. Consider a Wilson-Cowan type model with time-varying delays as follows:

$$\begin{aligned}
\frac{dX_P(t)}{dt} & = -X_P(t) + [k_P - r_P X_P(t)] \\
& \quad \times G \left[ w_P^1 X_P(t - \tau_P(t)) \right. \\
& \quad \left. - w_N^1 X_N(t - \tau_N(t)) + I_P(t) \right], \\
\frac{dX_N(t)}{dt} & = -X_N(t) + [k_N - r_N X_N(t)] \\
& \quad \times G \left[ w_P^2 X_P(t - \tau_P(t)) \right. \\
& \quad \left. - w_N^2 X_N(t - \tau_N(t)) + I_N(t) \right],
\end{aligned} \tag{31}$$

where  $k_P = k_N = 1$ ,  $r_P = r_N = 0.01$ ,  $w_P^1 = w_P^2 = w_N^1 = w_N^2 = 0.1$ ,  $\tau_P(t) = \tau_N(t) = 0.1t + 10$ ,  $I_P(t) = 7 \sin \sqrt{7}t$ ,  $G(v) = \tanh(v)$ , and  $I_N(t) = 7 \cos \sqrt{2}t$ . It is easy to calculate that

$$\begin{aligned}
K = 1, \quad W = 0.2, \quad R = 0.01, \quad I = 7, \\
v_P^{-1}(t) = v_N^{-1}(t) = \frac{(t + 10)}{0.9},
\end{aligned} \tag{32}$$

$$L = B_s = 1, \quad \delta = 182.5.$$

It is easy to check that  $(H_1)$  and  $(\widehat{H}_2)$  hold. By Corollary 4, (31) has a unique almost periodic solution  $X^*(t)$  in region  $B^* = \{\psi \mid \psi \in B, \|\psi\|_B \leq 182.5\}$ . Setting  $\ell_1 = \ell_2 = 0.5$ , we can check that  $(H_3)$  and (28) hold and hence  $X^*(t)$  is exponentially stable in  $B^*$ . Figure 2 shows the transient behavior of the unique almost periodic solution  $(X_P(t), X_N(t))$  in  $B^*$ . Phase portrait of attractivity of  $X_P$  and  $X_N$  is illustrated in Figure 3.

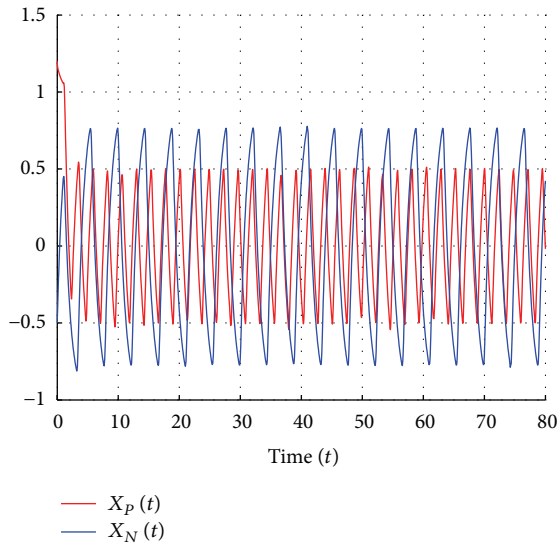


FIGURE 2: Transient behavior of the almost periodic solution of (31) located in  $B^*$ .

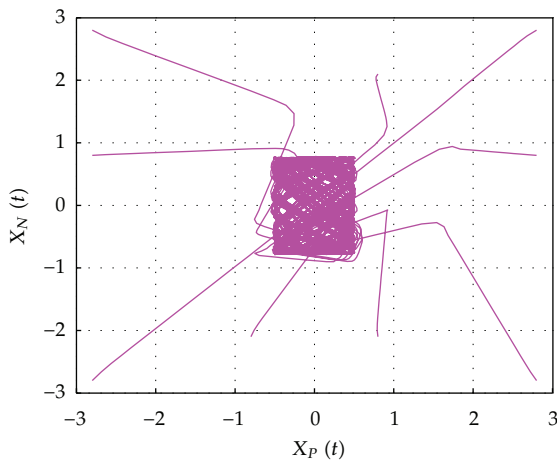


FIGURE 3: Phase portrait of attractivity of the unique almost periodic solution of (31) located in  $B^*$ .

### 5. Concluding Remarks

In this paper, we investigate Wilson-Cowan type model and obtain the existence of a unique almost periodic solution and its delay-based local stability in a convex subset. Our results are new and can reduce to periodic case, hence, complement existing periodic ones [7, 8]. We point out that there will exist multiple periodic (almost periodic) solution for system (1) under suitable parameter configuration. However, it is difficult to analyze its multistability of almost periodic solution by the existing method [10, 11, 13, 14]. We leave it for interested readers.

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