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Research Article

The Theory of Falling Shadows Applied to d-Ideals in d-Algebras

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On the basis of the theory of a falling shadow which was first formulated by Wang (1985), the notion of falling d^*-ideals in d-algebras is introduced, and related properties are investigated. Characterizations of a falling d^*-ideal are established. Relations among falling d^*-ideals, falling d-ideals, falling $d^\#$-ideals, falling d-subalgebras, and falling BCK-ideals are discussed.

1. Introduction

In the study of a unified treatment of uncertainty modelled by means of combining probability and fuzzy set theory, Goodman [1] pointed out the equivalence of a fuzzy set and a class of random sets. Wang and Sanchez [2] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. The mathematical structure of the theory of falling shadows is formulated in [3]. Tan et al. [4, 5] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Yuan and Lee [6] considered a fuzzy subgroup (subring, ideal) as the falling shadow of the cloud of the subgroup (subring, ideal). Iséki and Tanaka introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([7, 8]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras; that is, a BCI-algebra is a generalization of a BCK-algebra. As another useful generalization of BCK-algebras, Neggers and Kim [9] introduced the notion of d-algebras. They investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs. After that, some further aspects were studied in [10, 11]. Neggers et al. [12] introduced the concept of d-fuzzy function which generalizes the concept of fuzzy subalgebra to a much larger class of functions in a natural way. In addition, they discussed a method of fuzzification of a wide class of algebraic systems onto $[0,1]$ along with some consequences. Jun et al. [13] discussed implicative ideals of BCK-algebras based on the fuzzy sets and the theory of falling shadows. Also, Jun et al. [14] used the theory of a falling shadow for considering falling d-subalgebras, falling d-ideals, falling $d^{\#}$-ideals, and falling BCK-ideals in d-algebras.

In this paper, we introduce the notion of falling d^{*}-ideals in d-algebras, and investigate several properties. We establish characterizations of falling d^{*}-ideals, and we use these characterizations for considering relations among falling d^{*}-ideals, falling d-ideals, falling $d^{\#}$-ideals, falling d-subalgebras and falling BCK-ideals.

2. Preliminaries

A *d-algebra* is a nonempty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,

(II) $0 * x = 0$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y \in X$.

A *BCK-algebra* is a d-algebra $(X, *, 0)$ satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,

(V) $(x * (x * y)) * y = 0$

for all $x, y, z \in X$.

Any BCK-algebra $(X, *, 0)$ satisfies the following conditions:

(a1) $(\forall x, y \in X)((x * y) * x = 0)$,

(a2) $(\forall x, y, z \in X)((x * z) * (y * z)) * (x * y) = 0)$.

A subset I of a BCK-algebra X is called a *BCK-ideal* of X if it satisfies,

(b1) $0 \in I$.

(b2) $(\forall x \in X)(\forall y \in I)(x * y \in I \Rightarrow x \in I)$.

We now display the basic theory on falling shadows. We refer the reader to the papers [1–5] for further information regarding the theory of falling shadows.

Given a universe of discourse U, let $\mathcal{P}(U)$ denote the power set of U. For each $u \in U$, let

$$\dot{u} := \{E \mid u \in E \text{ and } E \subseteq U\}, \tag{2.1}$$

and for each $E \in \mathcal{P}(U)$, let

$$\dot{E} := \{\dot{u} \mid u \in E\}. \tag{2.2}$$

An ordered pair $(\mathcal{P}(U), \mathcal{B})$ is said to be a hypermeasurable structure on U if $\mathcal{B}$ is a σ-field in $\mathcal{P}(U)$ and $\dot{U} \subseteq \mathcal{B}$. Given a probability space $(\Omega, \mathcal{A}, P)$ and a hypermeasurable structure $(\mathcal{P}(U), \mathcal{B})$ on U, a random set on U is defined to be a mapping $\xi : \Omega \to \mathcal{P}(U)$ which is $\mathcal{A}$-$\mathcal{B}$ measurable, that is,

$$(\forall C \in \mathcal{B})\left(\xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in \mathcal{A}\right). \tag{2.3}$$

Suppose that ξ is a random set on U. Let

$$\widetilde{H}(u) := P(\omega \mid u \in \xi(\omega)) \quad \text{for each } u \in U. \tag{2.4}$$

Then $\widetilde{H}$ is a kind of fuzzy set in U. We call $\widetilde{H}$ a falling shadow of the random set ξ, and ξ is called a cloud of $\widetilde{H}$.

For example, $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\widetilde{H}$ be a fuzzy set in U, and let $\widetilde{H}_t := \{u \in U \mid \widetilde{H}(u) \geq t\}$ be a t-cut of $\widetilde{H}$. Then

$$\xi : [0,1] \longrightarrow \mathcal{P}(U), \quad t \longmapsto \widetilde{H}_t \tag{2.5}$$

is a random set and ξ is a cloud of $\widetilde{H}$. We will call ξ defined above as the cut-cloud of $\widetilde{H}$ (see [1]).

3. Falling d^*-Ideals

In what follows let X denote a d-algebra unless otherwise specified.

A nonempty subset S of X is called a *d-subalgebra* of X (see [11]) if $x * y \in S$ whenever $x \in S$ and $y \in S$.

A subset I of X is called a *BCK-ideal* of X (see [11]) if it satisfies conditions (b1) and (b2).

A subset I of X is called a *d-ideal* of X (see [11]) if it satisfies condition (b2) and

(b3) $(\forall x, y \in X)(x \in I \Rightarrow x * y \in I)$.

A d-ideal I of X is called a *$d^{\#}$-ideal* of X (see [11]) if, for arbitrary $x, y, z \in X$,

(b4) $x * z \in I$ whenever $x * y \in I$ and $y * z \in I$.

Definition 3.1 (see [14]). Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let

$$\xi : \Omega \longrightarrow \mathcal{P}(X) \tag{3.1}$$

be a random set. If $\xi(\omega)$ is a d-subalgebra (BCK-ideal, d-ideal and $d^{\#}$-ideal, resp.) of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$, then the falling shadow $\widetilde{H}$ of the random set ξ, that is,

$$\widetilde{H}(x) = P(\omega \mid x \in \xi(\omega)) \tag{3.2}$$

is called a *falling d-subalgebra* (*falling BCK-ideal, falling d-ideal* and *falling $d^{\#}$-ideal*,resp.) of X.

Lemma 3.2 (see [14]). *Let $\widetilde{H}$ be a falling shadow of a random set ξ on X. Then $\widetilde{H}$ is a falling d-ideal of X if and only if the following conditions are valid:*

(a) $(\forall x, y \in X)(\Omega(x * y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi))$,

(b) $(\forall x, y \in X)(\Omega(x; \xi) \subseteq \Omega(x * y; \xi))$.

Lemma 3.3 (see [14]). *If $\widetilde{H}$ is a falling d-ideal of X, then*

$$(\forall x, y \in X)\left(y * x = 0 \Longrightarrow \Omega(x; \xi) \subseteq \Omega(y; \xi)\right). \tag{3.3}$$

Proposition 3.4. *For a falling shadow $\widetilde{H}$ of a random set ξ on X, if $\widetilde{H}$ is a falling d-ideal of X, then*

$$(\forall x, y, z \in X)\left((x * y) * z = 0 \Longrightarrow \Omega(y; \xi) \cap \Omega(z; \xi) \subseteq \Omega(x; \xi)\right). \tag{3.4}$$

Proof. Let $x, y, z \in X$ be such that $(x * y) * z = 0$. Using Lemma 3.3, we have $\Omega(z; \xi) \subseteq \Omega(x * y; \xi)$. It follows from Lemma 3.2(a) that

$$\Omega(y; \xi) \cap \Omega(z; \xi) \subseteq \Omega(y; \xi) \cap \Omega(x * y; \xi) \subseteq \Omega(x; \xi). \tag{3.5}$$

This completes the proof. □

A fuzzy set μ on X is called a *fuzzy d-ideal* of X (see [10]) if it satisfies

(i) $(\forall x, y \in X)(\mu(x) \geq \min\{\mu(x * y), \mu(y)\})$,

(ii) $(\forall x, y \in X)(\mu(x * y) \geq \mu(x))$.

Lemma 3.5 (see [10]). *A fuzzy set μ on X is a fuzzy d-ideal of X if and only if, for every $\lambda \in [0, 1]$, $\mu_\lambda := \{x \in X \mid \mu(x) \geq \lambda\}$ is a d-ideal of X when it is nonempty.*

Theorem 3.6. *If we take the probability space $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on $[0, 1]$ and m is the usual Lebesgue measure, then every fuzzy d-ideal of X is a falling d-ideal of X.*

Proof. Let μ be a fuzzy d-ideal of X. Then $\mu_\lambda (\neq \emptyset)$ is a d-ideal of X for all $\lambda \in [0, 1]$ by Lemma 3.5. Let

$$\xi : \Omega \longrightarrow \mathcal{P}(X) \tag{3.6}$$

be a random set and $\xi(\lambda) = \mu_\lambda$ for every $\lambda \in \Omega$. Then μ is a falling d-ideal of X. □

We provide an example to show that the converse of Theorem 3.6 is not true.

Example 3.7. Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the Cayley table as follows:

$$\begin{array}{c|cccc} * & 0 & a & b & c \\ \hline 0 & 0 & 0 & 0 & 0 \\ a & a & 0 & a & a \\ b & b & b & 0 & 0 \\ c & c & c & a & 0 \end{array} \tag{3.7}$$

Let $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and define a random set

$$\xi : \Omega \longrightarrow \mathcal{P}(X), \qquad \omega \longmapsto \begin{cases} \{0\} & \text{if } \omega \in [0, 0.2), \\ \emptyset & \text{if } \omega \in [0.2, 0.3), \\ \{0, a\} & \text{if } \omega \in [0.3, 0.6), \\ \{0, b\} & \text{if } \omega \in [0.6, 0.85), \\ X & \text{if } \omega \in [0.85, 1]. \end{cases} \tag{3.8}$$

Then the falling shadow $\widetilde{H}$ of ξ is a falling d-ideal of X, and it is represented as follows:

$$\widetilde{H}(x) = \begin{cases} 0.9 & \text{if } x = 0, \\ 0.45 & \text{if } x = a, \\ 0.4 & \text{if } x = b, \\ 0.15 & \text{if } x = c. \end{cases} \tag{3.9}$$

We know that $\widetilde{H}$ is not a fuzzy d-ideal of X since

$$\widetilde{H}(c) = 0.15 \ngeq 0.4 = \min\left\{\widetilde{H}(c * b), \widetilde{H}(b)\right\}. \tag{3.10}$$

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let

$$F(X) := \{f \mid f : \Omega \longrightarrow X \text{ is a mapping}\}. \tag{3.11}$$

Define an operation $\circledast$ on $F(X)$ by

$$(\forall \omega \in \Omega)\big((f \circledast g)(\omega) = f(\omega) * g(\omega)\big) \tag{3.12}$$

for all $f, g \in F(X)$. Let $\theta \in F(X)$ be defined by $\theta(\omega) = 0$ for all $\omega \in \Omega$. Then $(F(X); \circledast, \theta)$ is a d-algebra [14]. For any subset A of X and $f \in F(X)$, let

$$\begin{gathered} A_f := \{\omega \in \Omega \mid f(\omega) \in A\}, \\ \xi : \Omega \longrightarrow \wp(F(X)), \qquad \omega \longmapsto \{f \in F(X) \mid f(\omega) \in A\}. \end{gathered} \tag{3.13}$$

Then $A_f \in \mathcal{A}$.

Theorem 3.8. *If A is a d-ideal of X, then*

$$\xi(\omega) = \{f \in F(X) \mid f(\omega) \in A\} \tag{3.14}$$

is a d-ideal of $F(X)$.

Proof. Assume that A is a d-ideal of X, and let $\omega \in \Omega$. Let $f, g \in F(X)$ be such that $g \in \xi(\omega)$ and $f \circledast g \in \xi(\omega)$. Then $g(\omega) \in A$ and $f(\omega) * g(\omega) = (f \circledast g)(\omega) \in A$. Since A is a d-ideal of X, it follows from (b2) that $f(\omega) \in A$ so that $f \in \xi(\omega)$. For any $f \in F(X)$, if $f \in \xi(\omega)$ then $f(\omega) \in A$. It follows that from (b3) that $(f \circledast g)(\omega) = f(\omega) * g(\omega) \in A$ for all $g \in F(X)$. Hence $f \circledast g \in \xi(\omega)$ for all $g \in F(X)$. Therefore $\xi(\omega)$ is a d-ideal of $F(X)$. □

Theorem 3.9. *If $\widetilde{H}$ is a falling d-ideal of X, then*

(a) $(\forall x, y \in X)(\widetilde{H}(x * y) \geq \widetilde{H}(x))$,

(b) $(\forall x, y \in X)(\widetilde{H}(x) \geq T_m(\widetilde{H}(x * y), \widetilde{H}(y)))$,

where $T_m(s,t) = \max\{s + t - 1, 0\}$ for any $s, t \in [0,1]$.

Proof. (a) It is clear.

(b) By Definition 3.1, $\xi(\omega)$ is a d-ideal of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Hence

$$\{\omega \in \Omega \mid x * y \in \xi(\omega)\} \cap \{\omega \in \Omega \mid y \in \xi(\omega)\} \subseteq \{\omega \in \Omega \mid x \in \xi(\omega)\}, \tag{3.15}$$

and thus

$$\begin{aligned} \widetilde{H}(x) &= P(\omega \mid x \in \xi(\omega)) \\ &\geq P(\{\omega \mid x * y \in \xi(\omega)\} \cap \{\omega \mid y \in \xi(\omega)\}) \\ &\geq P(\omega \mid x * y \in \xi(\omega)) + P(\omega \mid y \in \xi(\omega)) - P(\omega \mid x * y \in \xi(\omega) \text{or } y \in \xi(\omega)) \\ &\geq \widetilde{H}(x * y) + \widetilde{H}(y) - 1. \end{aligned} \tag{3.16}$$

Hence

$$\widetilde{H}(x) \geq \max\left\{\widetilde{H}(x * y) + \widetilde{H}(y) - 1, 0\right\} = T_m\left(\widetilde{H}(x * y), \widetilde{H}(y)\right). \tag{3.17}$$

This completes the proof. □

A d-algebra X is called a d^**-algebra* (see [11]) if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

If a $d^\#$-ideal I of X satisfies

(b5) $x * y \in I$ and $y * x \in I$ imply $(x * z) * (y * z) \in I$ and $(z * x) * (z * y) \in I$ for all $x, y, z \in X$, then we say that I is a d^**-ideal* of X (see [11]).

Definition 3.10. For a a probability space $(\Omega, \mathcal{A}, P)$ and a random set ξ on X, if $\xi(\omega)$ is a d^*-ideal of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$, then the falling shadow $\widetilde{H}$ of the random set ξ is called a *falling* d^**-ideal* of X.

Example 3.11. Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

(3.18)

Let $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and define a random set

$$\xi : \Omega \longrightarrow \mathcal{P}(X), \qquad \omega \longmapsto \begin{cases} \{0, a\} & \text{if } \omega \in [0, 0.6), \\ \emptyset & \text{if } \omega \in [0.6, 0.7), \\ X & \text{if } \omega \in [0.7, 1]. \end{cases} \tag{3.19}$$

Then the falling shadow $\widetilde{H}$ of ξ is a falling d^*-ideal of X.

Obviously, every falling d^*-ideal is a falling $d^\#$-ideal, but the converse does not hold in general.

Example 3.12. Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the Cayley table as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	c	b	0	c
c	c	b	b	0

(3.20)

For a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, define a random set

$$\xi : \Omega \longrightarrow \mathcal{P}(X), \qquad \omega \longmapsto \begin{cases} \{0, a\} & \text{if } \omega \in [0, 0.3), \\ X & \text{if } \omega \in [0.3, 0.8), \\ \emptyset & \text{if } t \in [0.8, 1]. \end{cases} \tag{3.21}$$

Then the falling shadow $\widetilde{H}$ of ξ is a falling $d^{\#}$-ideal of X, but not a falling d^{*}-ideal of X because if $\omega \in [0, 0.3)$ then $\xi(\omega) = \{0, a\}$ is not a d^{*}-ideal of X.

A characterization of a falling $d^{\#}$-ideal is established as follows.

Lemma 3.13 (see [14]). *For a falling shadow $\widetilde{H}$ of a random set ξ on X, the following are equivalent:*

(a) *$\widetilde{H}$ is a falling $d^{\#}$-ideal of X,*

(b) *$\widetilde{H}$ is a falling d-ideal of X that satisfies the following inclusion:*

$$(\forall x, y, z \in X)\big(\Omega(x * y; \xi) \cap \Omega(y * z; \xi) \subseteq \Omega(x * z; \xi)\big). \tag{3.22}$$

We provide characterizations of a falling d^{*}-ideal.

Theorem 3.14. *For a falling shadow $\widetilde{H}$ of a random set ξ on X, $\widetilde{H}$ is a falling d^{*}-ideal of X if and only if the following conditions are valid for every $x, y, z \in X$:*

(a) $\Omega(x * y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi)$,

(b) $\Omega(x; \xi) \subseteq \Omega(x * y; \xi)$,

(c) $\Omega(x * y; \xi) \cap \Omega(y * z; \xi) \subseteq \Omega(x * z; \xi)$,

(d) $\Omega(x * y; \xi) \cap \Omega(y * x; \xi) \subseteq \Omega((x * z) * (y * z); \xi) \cap \Omega((z * x) * (z * y); \xi)$.

Proof. Assume that $\widetilde{H}$ is a falling d^{*}-ideal of X. Then $\widetilde{H}$ is a falling $d^{\#}$-ideal of X, and so conditions (a), (b), and (c) are valid by Lemmas 3.2 and 3.13. Let $x, y, z \in X$ and $\omega \in \Omega$. If $\omega \in \Omega(x * y; \xi) \cap \Omega(y * x; \xi)$, then $x * y \in \xi(\omega)$ and $y * x \in \xi(\omega)$. Since $\xi(\omega)$ is a d^{*}-ideal of X, it follows from (b5) that $(x * z) * (y * z) \in \xi(\omega)$ and $(z * x) * (z * y) \in \xi(\omega)$ so that

$$\omega \in \Omega\big((x * z) * (y * z); \xi\big) \cap \Omega\big((z * x) * (z * y); \xi\big) \tag{3.23}$$

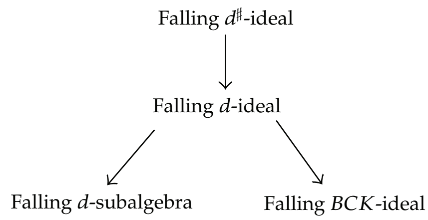
for all $x, y, z \in X$. Therefore (d) holds.

Conversely, suppose that conditions (a), (b), (c), and (d) are valid. Three conditions (a), (b), and (c) imply that $\widetilde{H}$ is a falling $d^{\#}$-ideal of X by Lemmas 3.2 and 3.13. Finally, let $x, y, z \in X$ and $\omega \in \Omega$ be such that $x * y \in \xi(\omega)$ and $y * x \in \xi(\omega)$. Using the condition (d), we have

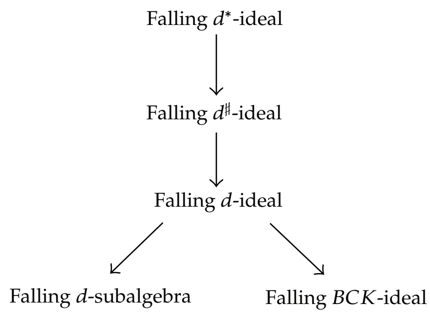
$$\omega \in \Omega(x * y; \xi) \cap \Omega(y * x; \xi) \subseteq \Omega\big((x * z) * (y * z); \xi\big) \cap \Omega\big((z * x) * (z * y); \xi\big), \tag{3.24}$$

which implies that $(x * z) * (y * z) \in \xi(\omega)$ and $(z * x) * (z * y) \in \xi(\omega)$. Therefore $\widetilde{H}$ is a falling d^{*}-ideal of X. □

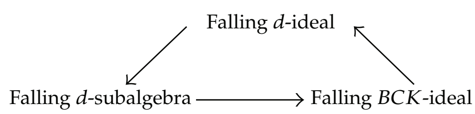
The following relation is described in [14]



Combining this relation and the fact that every falling d^*-ideal is a falling $d^{\#}$-ideal, we have the following relation:



In this diagram, the reverse implications are not true, and we need additional conditions for considering the reverse implications. Jun et al. [14] showed that the following relation holds in d^*-algebras:



Lemma 3.15 (see [14]). *For a falling shadow $\widetilde{H}$ of a random set ξ on X, if $\widetilde{H}$ is a falling BCK-ideal of X, then*

(a) $(\forall x, y \in X)(x * y = 0 \Rightarrow \Omega(y;\xi) \subseteq \Omega(x;\xi))$,

(b) $(\forall x, y \in X)(\Omega(x * y;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x;\xi))$.

Theorem 3.16. *If X is a BCK-algebra, then every falling BCK-ideal of X is a falling d^*-ideal of X.*

Proof. Let $\widetilde{H}$ be a falling BCK-ideal of a BCK-algebra X. Then

$$\Omega(x * y;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x;\xi) \tag{3.25}$$

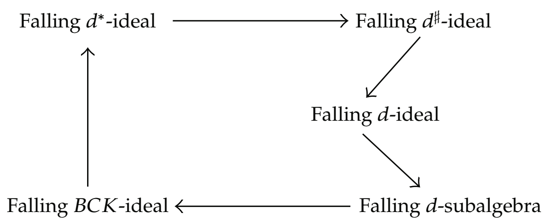
for all $x, y \in X$ by Lemma 3.15(b). Using (a1), we have $(x*y)*x = 0$ for all $x, y \in X$. Hence, by Lemma 3.15(a), we get $\Omega(x;\xi) \subseteq \Omega(x * y;\xi)$ for all $x, y \in X$. If $\omega \in \Omega(x * y;\xi) \cap \Omega(y * z;\xi)$, then $x * y \in \xi(\omega)$ and $y * z \in \xi(\omega)$. Note that $((x * z) * (y * z)) * (x * y) = 0 \in \xi(\omega)$. Since $\xi(\omega)$ is a BCK-ideal of X, it follows from (b2) that $x * z \in \xi(\omega)$ so that $\omega \in \Omega(x * z;\xi)$. Thus $\Omega(x * y;\xi) \cap \Omega(y * z;\xi) \subseteq \Omega(x * z;\xi)$. Let $\omega \in \Omega(x * y;\xi) \cap \Omega(y * x;\xi)$. Then $x * y \in \xi(\omega)$ and $y * x \in \xi(\omega)$. By (IV) and (a2), we have $((z * x) * (z * y)) * (y * x) = 0 \in \xi(\omega)$ and

$((x * z) * (y * z)) * (x * y) = 0 \in \xi(\omega)$. It follows from (b2) that $(z * x) * (z * y) \in \xi(\omega)$ and $(x * z) * (y * z) \in \xi(\omega)$ so that $\omega \in \Omega((x * z) * (y * z); \xi) \cap \Omega((z * x) * (z * y); \xi)$. Hence

$$\Omega(x * y; \xi) \cap \Omega(y * x; \xi) \subseteq \Omega((x * z) * (y * z); \xi) \cap \Omega((z * x) * (z * y); \xi). \tag{3.26}$$

Using Theorem 3.14, we conclude that $\widetilde{H}$ is a falling d^*-ideal of X. □

Note that every BCK-algebra is a d^*-algebra (see [11]). Therefore, the above diagrams together with Theorem 3.16 induce the following diagram in BCK-algebras:



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