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Research Article

Stabilized Multiscale Nonconforming Finite Element Method for the Stationary Navier-Stokes Equations

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We consider a stabilized multiscale nonconforming finite element method for the two-dimensional stationary incompressible Navier-Stokes problem. This method is based on the enrichment of the standard polynomial space for the velocity component with multiscale function and the nonconforming lowest equal-order finite element pair. Stability and existence uniqueness of the numerical solution are established, optimal-order error estimates are also presented. Finally, some numerical results are presented to validate the performance of the proposed method.

1. Introduction

As the development of science and technology, finite element method has became an important and powerful tool for the complex fluid problems, such as for the Navier-Stokes equations. It is well known that the pressure and velocity pairs satisfy the discrete Inf-Sup condition [1] that plays the key role for simulating the Navier-Stokes equations. However, some unstable mixed finite element pairs which violate the so-called Inf-Sup condition are also popular, see [2–4]. In order to overcome this restriction, various of stabilized methods have been proposed, including the bubble condensation-based methods [5], pressure projection method (PPM) [6–8], the local Gauss integration method (LGIM) [9–11], multiscale method [12, 13], macroelement stabilized method [3, 14], and so on. Most of these stabilized methods necessarily need to introduce the stabilization parameters either explicitly or implicitly. In addition, some of these techniques are conditionally stable or are of suboptimal accuracy. Therefore, the development of mixed finite element methods free from stabilization parameters has become increasingly important.

In 2005, Franca et al. gave a new multiscale method for the reaction-diffusion equation in [15]. The chief characteristic of their method is to use the Petrov-Galerkin approach to split the solution into two parts, and the trial function space is enriched with an unstable bubble-like function, which is the solution to a local problem. Later, Barrenechea and Valentin [13] considered the relationship between the enriched multiscale method and stabilized techniques for generalized Stokes problem based on the P_1-P_1 pair. By enriching the velocity space with an unusual bubble function, Araya et al. established the convergence for the Stokes problem in [16], their method is different from usual residual free bubble method in [5], in which one should choose local basis functions to enrich the standard finite element spaces by solving some local problem analytically. Furthermore, the method proposed in [15] can also be used to treat the unsteady reaction-diffusion problem (see [17]).

Compared with conforming finite element method, the nonconforming finite element methods are more popular due to their simplicity and small support sets of basis functions. Crouzeix and Ravizrt in [18] used the nonconforming piecewise linear velocity and a piecewise constant pressure to solve the Stokes equations. In this paper, motivated by the ideas of [13, 15, 16], we will use the Petrov-Galerkin approach based on the nonconforming velocity space to handle with the steady Navier-Stokes equations. The main differences between [13, 15, 16, 19] and this work lie in the following: (i) the finite element spaces of velocity are different; it is nonconforming element in this paper; (ii) the treated problems are different; we consider the nonlinear problem; (iii) the finite element pairs are different; the NCP_1-P_1 pair is used in this paper.

The outline of this work is arranged as follows. In the following section, the abstract functional setting for steady Navier-Stokes equations is recalled. Section 3 is devoted to derive the general form of enriched multiscale method based on the NCP_1-P_1 pair. After providing the stability and existence uniqueness for the approximation solution, the optimal error estimates are established in Section 4. In Section 5, Some numerical results are presented to verify the established theoretical analysis. Finally, Some conclusions are made in Section 6.

2. Preliminaries

Let Ω be an open bounded domain of $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$ and satisfy a further condition stated in (A1) below. The incompressible stationary Navier-Stokes equations with the homogeneous Dirichlet boundary condition are

$$\begin{aligned} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad &\text{in } \Omega,\\ \operatorname{div} u = 0, \quad &\text{in } \Omega,\\ u|_{\partial\Omega} = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $u = (u_1(x), u_2(x))^T$ represents the velocity, $p = p(x)$ the pressure, $f = f(x) \in L^2(\Omega)^2$ the prescribed body force, $\nu > 0$ the viscosity coefficient.

In order to introduce the variational formulation for problem (2.1), we set

$$\begin{gathered} X = H_0^1(\Omega)^2, \qquad Y = L^2(\Omega)^2, \qquad D(A) = H^2(\Omega)^2 \cap X,\\ M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q\,dx = 0 \right\}. \end{gathered} \tag{2.2}$$

The standard notations of Sobolev space $W^{m,r}(\Omega)$ are used. To simplify, we use $H^m(\Omega)$ instead of $W^{m,r}(\Omega)$ as $r = 2$ and $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$. The spaces $L^2(\Omega)^m$ $(m = 1, 2)$ are endowed with the usual L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. The spaces $H_0^1(\Omega)$ and X are equipped with the scalar product $(\nabla u, \nabla v)$ and the norm $|u|_{1,\Omega}^2$, $u, v \in H_0^1(\Omega)$ (or X).

Define $Au = -\Delta u$ is the operator associated with the Navier-Stokes problem, it is positive self-adjoint operator from $D(A)$ onto Y.

Introducing the bilinear operator

$$B(u,v) = (u\cdot\nabla)v + \frac{1}{2}(\operatorname{div} u)v \quad \forall u, v \in X, \tag{2.3}$$

and defining a trilinear form on $X \times X \times X$ as follows:

$$\begin{aligned} b(u,v,w) = \langle B(u,v), w\rangle_{X'\times X} &= ((u\cdot\nabla)v, w) + \frac{1}{2}((\operatorname{div} u)v, w) \\ &= \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v). \end{aligned} \tag{2.4}$$

The variational formulation of problem (2.1) reads as: find $(u,p) \in (X, M)$ such that for all $(v,q) \in (X, M)$

$$a(u,v) - d(v,p) + d(u,q) + b(u,u,v) = (f,v), \tag{2.5}$$

where

$$\begin{gathered} a(u,v) = \nu(\nabla u, \nabla v), \qquad d(v,q) = -(v, \nabla q) = (q, \operatorname{div} v), \\ B_0((u,p);(v,q)) = a(u,v) - d(v,p) + d(u,q). \end{gathered} \tag{2.6}$$

Clearly, the bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are continuous on $X \times X$ and $X \times M$, respectively. Moreover, $d(\cdot,\cdot)$ also satisfies (see [20]):

$$\sup_{0\neq v\in X} \frac{|d(v,q)|}{|v|_{1,\Omega}} \geq \beta\|q\|_{0,\Omega}, \tag{2.7}$$

where β is a positive constant depending only on Ω.

It is easy to verify that B_0 satisfies the following important properties for all (u,p), $(v,q) \in (X, M)$ (see [1]):

$$B_0((u,p);(u,p)) = \nu\|u\|_{1,\Omega}^2, \tag{2.8}$$

$$|B_0((u,p);(v,q))| \leq C\left(\|u\|_{1,\Omega} + \|p\|_{0,\Omega}\right)\left(\|v\|_{1,\Omega} + \|q\|_{0,\Omega}\right), \tag{2.9}$$

$$\beta_0\left(\|u\|_{1,\Omega} + \|p\|_{0,\Omega}\right) \leq \sup_{(v,q)\in(X,M)} \frac{|B_0((u,p);(v,q))|}{\|v\|_{1,\Omega} + \|q\|_{0,\Omega}}, \tag{2.10}$$

where $\beta_0 > 0$ is a constant. Here and below, the letter C (with or without subscript) denotes a generic positive constant, depending at most on the data ν, Ω and f. Furthermore, the following estimates about $b(\cdot,\cdot,\cdot)$ are hold [1, 20]:

$$b(u,v,w) = -b(u,w,v), \tag{2.11}$$

$$|b(u,v,w)| \leq \frac{1}{2}c_0 \|u\|_{0,\Omega}{}^{1/2}|u|_{1,\Omega}{}^{1/2}\left(|v|_{1,\Omega}\|w\|_{0,\Omega}^{1/2}|w|_{1,\Omega}^{1/2} + \|v\|_{0,\Omega}^{1/2}|v|_{1,\Omega}^{1/2}|w|_{1,\Omega}\right), \tag{2.12}$$

for all $u, v, w \in X$ and

$$|b(u,v,w)| + |b(v,u,w)| + |b(w,u,v)| \leq C|u|_{1,\Omega}\|Av\|_{0,\Omega}\|w\|_{0,\Omega}, \tag{2.13}$$

for all $u \in X$, $v \in D(A)$, $w \in Y$.

As mentioned above, a further assumption about Ω is needed (see [1]).

(A1) Assume that Ω is regular so that the unique solution $(v,q) \in (X,M)$ of the steady Stokes equations

$$-\nu\Delta v + \nabla q = g, \qquad \operatorname{div} v = 0 \text{ in } \Omega, \qquad v|_{\partial\Omega} = 0, \tag{2.14}$$

for a prescribed $g \in Y$ exists and satisfies

$$\|Av\|_{0,\Omega} + |q|_{1,\Omega} \leq C\|g\|_{0,\Omega}. \tag{2.15}$$

Under the assumption of (A1), if $\partial\Omega$ is of C^2 or Ω is a two-dimensional convex polygon, it has been shown that (see [20])

$$\|v\|_{0,\Omega} \leq \gamma_0|v|_{1,\Omega}, \quad \forall v \in X, \qquad |v|_{1,\Omega} \leq \gamma_0\|Av\|_{0,\Omega}, \quad \forall v \in D(A), \tag{2.16}$$

where γ_0 is a positive constant only depending on Ω.

The following existence and uniqueness results about problem (2.5) are classical (see [1, 20]).

Theorem 2.1. *Assume that ν and $f \in Y$ satisfy the following uniqueness condition:*

$$1 - \frac{c_0\gamma_0^2}{\nu^2}\|f\|_{0,\Omega} > 0. \tag{2.17}$$

Then problem (2.5) *admits a unique solution* $(u,p) \in (D(A), H^1(\Omega) \cap M)$ *with* $\operatorname{div} u = 0$ *such that*

$$|u|_{1,\Omega} \leq \frac{\gamma_0}{\nu}\|f\|_{0,\Omega}, \qquad \|Au\|_{0,\Omega} + |p|_{1,\Omega} \leq C\|f\|_{0,\Omega}. \tag{2.18}$$

3. Enriched Nonconforming Finite Element Method

Let $\mathcal{T}_h$ be a regular triangulation of Ω into element $\{K_j\} : \overline{\Omega} = \cup \overline{K}_j$, that is, $|K_j| \simeq Ch_{K_j}^2$, where $|K_j|$ is the area of the element K_j and h_{K_j} is the diameter of K_j; the mesh parameter h is given by $h = \max\{h_{K_j} : K_j \in \mathcal{T}_h\}$. Denote the boundary segment and the interior boundary by $\gamma_j = \partial\Omega \cap \partial K_j$ and $\gamma_{jk} = \gamma_{kj} = \partial K_j \cap \partial K_k$, respectively. Let Γ_h and Γ_j be the sets of γ_{jk} and γ_j. The centers of γ_j and γ_{jk} are indicated by ξ_j and ξ_{jk}, respectively. The finite element spaces investigated in this paper are the following mixed finite element spaces:

$$\begin{gathered} \mathrm{NCP}_1 = \left\{ v \in Y : v|_{K_j} \in P_1(K_j)^2,\ v(\xi_{jk}) = v(\xi_{kj}),\ v(\xi_j) = 0\ \forall j, k, K_j \in \mathcal{T}_h \right\}, \\ P_1 = \left\{ q \in H^1(\Omega) : q\big|_{K_j} \in P_1(K_j),\ \forall K_j \in \mathcal{T}_h \right\}, \end{gathered} \tag{3.1}$$

where $P_1(K_j)$ is the set of line polynomials on K_j, and noting that the nonconforming finite element space NCP_1 is not a subspace of X. Defining the energy norm

$$\|v\|_{1,h} = \left(\sum_{K_j} |v|_{1,K_j}^2 \right)^{1/2}, \quad \forall v \in \mathrm{NCP}_1. \tag{3.2}$$

The finite element spaces NCP_1 and P_1 satisfy the following approximation property (see [4, 21]): for $(v, q) \in H^2(\Omega) \times H^1(\Omega)$, there are two approximations $v_I \in \mathrm{NCP}_1$ and $q_I \in P_1$ such that

$$\|v - v_I\|_{0,\Omega} + h\left(\|v - v_I\|_{1,h} + \|q - q_I\|_{0,\Omega} \right) \le Ch^2 \left(\|Av\|_{0,\Omega} + |q|_{1,\Omega} \right), \tag{3.3}$$

and the compatibility conditions hold for all j and k:

$$\int_{\gamma_{jk}} [v] ds = 0, \quad \int_{\Gamma_j} v ds = 0 \quad \forall v \in \mathrm{NCP}_1, \tag{3.4}$$

where $[v] = v_{\gamma_{jk}} - v_{\gamma_{kj}}$ denotes the jump of the function v across the boundary γ_{jk}.

Set $\langle \cdot, \cdot \rangle_j = (\cdot, \cdot)_{\partial K_j}$ and $|\cdot|_{m,j} = |\cdot|_{m,K_j}$. Then for all $u, v \in H^1(K_j)^2$, $q \in L^2(\Omega)$, the discrete bilinear forms are

$$a_h(u, v) = \sum_{K_j} \nu (\nabla u, \nabla v)_{K_j}, \qquad d_h(v, p) = \sum_{K_j} (\operatorname{div} v, p)_{K_j}. \tag{3.5}$$

For the nonconforming space NCP_1, we define a local operator

$$\Pi_j : H^1(K_j)^2 \longrightarrow \mathrm{NCP}_1(K_j), \tag{3.6}$$

satisfying

$$\int_{\partial K_j} (v - \Pi_j v) ds = 0. \tag{3.7}$$

Then the local operator Π_j satisfies (see [21])

$$\left|v - \Pi_j v\right|_{1,K_j} \leq Ch^i |v|_{i+1,K_j}, \quad v \in H^{i+1}(K_j),\ i = 0, 1, \qquad \left\|\Pi_j v\right\|_{1,K_j} \leq C\|v\|_{1,K_j}. \tag{3.8}$$

The global operator $\Pi_h : X \to \mathrm{NCP}_1$ is defined as $\Pi_h v|_j = \Pi_j v,\ v \in X$.

As noted, the choice NCP_1-P_1 is an unstable pair that does not satisfy the discrete Inf-Sup condition. Therefore, we need to introduce the enrichment multiscale method to overcome this restriction.

Let E_h be a finite dimensional space, called multiscale space, such that

$$E_h \in H^1(\mathcal{T}_h)^2, \quad E_h \cap \mathrm{NCP}_1 = \{0\}, \quad \text{where } H^1(\mathcal{T}_h)^2 = \left\{v \in Y : v|_{K_j} \in H^1(K_j)^2\right\}. \tag{3.9}$$

The discrete weak formulation of the Stokes equations is to find $u_h + u_e \in \mathrm{NCP}_1 \oplus E_h$ and $p_h \in P_1$, such that

$$a_h(u_h + u_e, v_h) - d_h(v_h, p_h) + d_h(u_h + u_e, q_h) = (f, v)_\Omega, \tag{3.10}$$

for all $v_h \in \mathrm{NCP}_1 \oplus H_0^1(\mathcal{T}_h)^2$ and $q_h \in P_1$. Let $u_e|_{K_j} = u_e^{K_j} + u_e^{\partial K_j}$, we can solve it through the following local problem:

$$\begin{gathered}
-\nu\Delta u_e^{K_j} = f + \nu\Delta u_h - \nabla p_h \quad \text{in } K_j, \qquad u_e^{K_j}\Big|_{\partial K_j} = 0, \\
-\nu\Delta u_e^{\partial K_j} = 0 \quad \text{in } K_j, \qquad u_e^{\partial K_j} = g_e \quad \text{on } \partial K_j, \\
-\nu\partial_{ss} g_e = \frac{1}{h_e}\left[\nu\partial_n u_h + p_h I \cdot n\right]_E, \quad g_e = 0 \ \text{ at the nodes,}
\end{gathered} \tag{3.11}$$

where h_e denotes the length of the edge $E \in \partial K_j$; n the normal outward vector on ∂K_j; ∂_s, ∂_n are the tangential and normal derivative operators, respectively; I is the $\mathbb{R}^{2\times 2}$ identity matrix. Equation (3.11) is well posed, that is, u_e can be expressed by u_h, p_h, and f on each element K_j. For convenience, we define two local operators $\mathcal{M}_{K_j} : L^2(K_j)^2 \to H_0^1(K_j)^2$ and $\mathcal{L}_{K_j} : L^2(\partial K_j)^2 \to H^1(K_j)^2$ by

$$\begin{gathered}
u_e^{K_j} = \frac{1}{\nu}\mathcal{M}_{K_j}(f + \nu\Delta u_h - \nabla p_h), \quad \forall K_j \in \mathcal{T}_h, \\
u_e^{\partial K} = \frac{1}{\nu}\mathcal{L}_{K_j}\left(\left[\nu\partial_n u_h + p_h I \cdot n\right]_E\right), \quad \forall K_j \in \mathcal{T}_h,\ E \in \Gamma_h.
\end{gathered} \tag{3.12}$$

With Green formulation and (3.12), for all $(u_h, p_h), (v_h, q_h) \in \mathrm{NCP}_1 \times P_1$, (3.10) can be rewritten as

$$\begin{aligned}
&\sum_{K_j}\Big[\nu(\nabla u_h, \nabla v_h)_{K_j} - (p_h, \nabla\cdot v_h)_{K_j} + (q_h, \nabla\cdot u_h)_{K_j}\Big] \\
&\quad + \sum_{K_j}\frac{1}{\nu}\Big(\mathcal{M}_{K_j}(-\nu\Delta u_h + \nabla p_h) - \mathcal{H}_{K_j}\big([\nu\partial_n u_h + p_h I\cdot n]_E\big), \nu\Delta v_h + \nabla q_h\Big)_{K_j} \\
&\quad + \sum_{E\in\Gamma_h}\frac{1}{\nu}\Big(\mathcal{H}_{K_j}\big([\nu\partial_n u_h + p_h I\cdot n]_E\big), \nu\partial_n v_h + q_h I\cdot n\Big)_E \\
&\quad = \sum_{K_j}\left[(f, v_h)_{K_j} + \frac{1}{\nu}\Big(\mathcal{M}_{K_j}(f), \nu\Delta v_h + \nabla q_h\Big)_{K_j}\right].
\end{aligned} \tag{3.13}$$

With the help of (3.13), the enriched nonconforming finite element method for the stationary Navier-Stokes equations (2.1) is rewritten as follows: find $(u_h, p_h) \in \mathrm{NCP}_1 \times P_1$ such that

$$B((u_h, p_h); (v_h, q_h)) + b(u_h, u_h, v_h) = F(v_h, q_h) \tag{3.14}$$

for all $(v_h, q_h) \in \mathrm{NCP}_1 \times P_1$, where

$$\begin{aligned}
B((u_h, p_h); (v_h, q_h)) &= B_h((u_h, p_h); (v_h, q_h)) + \sum_{K_j}\frac{1}{\nu}\Big(\mathcal{M}_{K_j}(\nabla p_h), \nabla q_h\Big)_{K_j} \\
&\quad + \sum_{E\in\Gamma_h}\frac{1}{\nu}\Big(\mathcal{H}_{K_j}([\nu\partial_n u_h]), [\nu\partial_n v_h]\Big)_E \\
&\triangleq B_h((u_h, p_h); (v_h, q_h))_\Omega + \sum_{K_j}\tau_{K_j}(\nabla p_h, \nabla q_h)_{K_j} \\
&\quad + \sum_{E\in\Gamma_h}\tau_E([\nu\partial_n u_h], [\nu\partial_n v_h])_E, \\
F(v_h, q_h) &= \sum_{K_j}\left[(f, v_h)_{K_j} + \frac{1}{\nu}\Big(\mathcal{M}_{K_j}(f), \nabla q_h\Big)_{K_j}\right], \\
B_h((u_h, p_h); (v_h, q_h)) &= a_h(u_h, v_h) - d_h(v_h, p_h) + d_h(u_h, q_h).
\end{aligned} \tag{3.15}$$

By applying the technique to one used in [16], we can obtain that $(b_{K_j}, 1)_{K_j}/|K_j| \backsimeq \tilde{C}h^2_{K_j}$, $(a_{K_j}, 1)_E/h_e \backsimeq h_e/12$, $\tau_{K_j} \backsimeq \tilde{C}h^2_{K_j}$ and $\tau_E \backsimeq h_e/(12\nu)$. Moreover, if f is a piecewise constant, then we have $\mathcal{M}_{K_j}(f) = b_{K_j} f$,

$$\Big(\mathcal{M}_{K_j}(f), \nabla q_h\Big)_{K_j} = \frac{\big(b_{K_j}, 1\big)_{K_j}}{|K_j|}(f, \nabla q_h)_{K_j} \simeq \tilde{C}h^2_{K_j}(f, \nabla q_h)_{K_j}. \tag{3.16}$$

Define the mesh-dependent norms as follows:

$$|||u|||_h^2 = \nu\|u\|_{1,h}^2 + \sum_{E\in\Gamma_h}\tau_E\|[\nu\partial_n u]\|_{0,E}^2, \qquad \|q\|_h^2 = \sum_{K_j}\tau_{K_j}|q|_{1,K_j}^2. \tag{3.17}$$

Remark 3.1. The assumption of piecewise constant f is made simply to analyze the problem (3.14), but this assumption does not affect the precision of this method, and (3.14) may be implemented as it is presented for a general function $f \in L^2(\Omega)^2$. Here, we do not give the detail proof about this fact; readers can visit Appendix B of the paper [16] for $f \in H^1(\Omega)^2$.

Remark 3.2. Generally speaking, the following linear algebra equations can be obtained from the discrete system of original problem:

$$\begin{pmatrix} A & -D \\ D^T & 0 \end{pmatrix}\begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}, \tag{3.18}$$

where the matrices A and D are deduced from the diffusion, convection, and incompressible terms; F is the variation of the source term. The norm of matrix A gets smaller as the convection increases; therefore, some unnecessary oscillations will be created. In order to eliminate these oscillations, we introduce the stabilized term, in this case, the coefficient matrix of discrete formulation transforms into

$$\begin{pmatrix} A & -D \\ D^T & G \end{pmatrix}, \tag{3.19}$$

where G is derived from the stabilized term, that is, the term of $(\nabla p_h, \nabla q_h)$. As the considered problem has strong convection, in order to obtain a good behavior of matrix $\begin{pmatrix} A & -D \\ D^T & G \end{pmatrix}$, we should choose a proper G. In this way, the singularly perturbed problem can be treated effectively. The reason that we treat the convection term not use enriched function technique is to simply the theoretical analysis and computation, and the discrete convection term has no influence about the stabilized term $G(\cdot,\cdot)$.

Lemma 3.3. *Let* $(v_h, q_h) \in \mathrm{NCP}_1 \times \mathrm{P}_1$, *then,*

$$B((v_h, q_h); (v_h, q_h)) = |||v_h|||_h^2 + \|q_h\|_h^2. \tag{3.20}$$

Proof. The results follow from the definition of (3.15) and the mesh-dependent norms in each $K \in \mathcal{T}_h$.

Before establishing the stability of scheme (3.14), we introduce the local trace theorem (see [1]). There exists $C > 0$, independent of h, such that

$$\|u\|_{0,\partial K_j}^2 \le C\left(h_{K_j}^{-1}\|u\|_{0,K_j}^2 + h_{K_j}|u|_{1,K_j}^2\right), \quad \forall u \in H^1(K_j). \tag{3.21}$$

□

Theorem 3.4. *There exist two positive constants C, β depending on ν, for all (u_h, p_h), $(v_h, q_h) \in \mathrm{NCP}_1 \times \mathrm{P}_1$ such that*

$$\left|B\big((u_h, p_h); (v_h, q_h)\big)\right| \leq C\Big(\|u_h\|_{1,h} + \|p_h\|_{0,\Omega}\Big)\Big(\|v_h\|_{1,h} + \|q_h\|_{0,\Omega}\Big), \tag{3.22}$$

$$\sup_{0\neq (v_h,q_h)\in(\mathrm{NCP}_1,P_1)} \frac{\left|B\big((u_h, p_h); (v_h, q_h)\big)\right|}{\|v_h\|_{1,h} + \|q_h\|_{0,\Omega}} \geq \beta\Big(\|u_h\|_{1,h} + \|p_h\|_{0,\Omega}\Big). \tag{3.23}$$

Proof. It follows from (u_h, p_h), $(v_h, q_h) \in \mathrm{NCP}_1 \times P_1$, inverse inequality, (3.15), and (3.21) that

$$\begin{aligned}
&\left|B\big((u_h, p_h); (v_h, q_h)\big)\right| \\
&\quad\leq \nu\|u_h\|_{1,h}\|v_h\|_{1,h} + \|u_h\|_{1,h}\|q_h\|_{0,\Omega} + \|p_h\|_{0,\Omega}\|v_h\|_{1,h} \\
&\qquad + \sum_{K_j}\tau_{K_j}|p_h|_{1,K_j}|q_h|_{1,K_j} + \sum_{E\in\Gamma_h}\tau_E\|[\nu\partial_n u_h]\|_{0,E}\|[\nu\partial_n v_h]\|_{0,E} \\
&\quad\leq \nu\|u_h\|_{1,h}\|v_h\|_{1,h} + \|u_h\|_{1,h}\|q_h\|_{0,\Omega} + \|p_h\|_{0,\Omega}\|v_h\|_{1,h} + C_1\|p_h\|_{0,\Omega}\|q_h\|_{0,\Omega} \\
&\qquad + C_2\sum_{E\in\Gamma_h}\tau_E\Big(h_{K_j}^{-1/2}\|\nabla u_h\|_{0,K_j} + h_{K_j}^{1/2}|\nabla u_h|_{1,K_j}\Big)\Big(h_{K_j}^{-1/2}\|\nabla v_h\|_{0,K_j} + h_{K_j}^{1/2}|\nabla v_h|_{1,K_j}\Big) \\
&\quad\leq C(\nu)\|u_h\|_{1,h}\|v_h\|_{1,h} + \|u_h\|_{1,h}\|q_h\|_{0,\Omega} + \|p_h\|_{0,\Omega}\|v_h\|_{1,h} + C_1\|p_h\|_{0,\Omega}\|q_h\|_{0,\Omega} \\
&\quad\leq C\Big(\|u_h\|_{1,h} + \|p_h\|_{0,\Omega}\Big)\Big(\|v_h\|_{1,h} + \|q_h\|_{0,\Omega}\Big),
\end{aligned} \tag{3.24}$$

that is, the continuity result (3.22) holds.

From the properties of the nonconforming finite element given in [18], for all $p_h \in L^2(K_j)$, there exists a function $w \in H^1(K_j)^2$, such that $\|w\|_{1,h} = \|p_h\|_0$ and

$$\big(\nabla\cdot w, p_h\big)_{K_j} = \|p_h\|_{0,K_j}^2, \qquad \|w\|_{1,h} \leq C\|p_h\|_{0,\Omega}. \tag{3.25}$$

Using the Cauchy-Schwartz inequality and (3.25), we have

$$\begin{aligned}
&\left|B\big((u_h, p_h); (-w, 0)\big)\right| \\
&\quad= -\nu(\nabla u_h, \nabla w) + \big(p_h, \nabla\cdot w\big) - \sum_{E\in\Gamma_h}\tau_E([\nu\partial_n u_h], [\nu\partial_n w])_E \\
&\quad\geq -\nu\|u_h\|_{1,h}\|w\|_{1,h} + C_0\|w\|_{1,h}\|p_h\|_{0,\Omega} - \sum_{E\in\Gamma_h}\tau_E\|[\nu\partial_n u_h]\|_{0,E}\|[\nu\partial_n w]\|_{0,E} \\
&\quad\geq -\left(\nu\|u_h\|_{1,h}^2 + \sum_{E\in\Gamma_h}\tau_E\left\|[\nu\partial_n u_h]_{0,E}^2\right\|\right)^{1/2}\left(\nu\|w\|_{1,h}^2 + \sum_{E\in\Gamma_h}\tau_E\|[\nu\partial_n w]\|_{0,E}^2\right)^{1/2} \\
&\qquad + C_0\|w\|_{1,h}\|p_h\|_{0,h}.
\end{aligned} \tag{3.26}$$

Using (3.21) and inverse inequality, we obtain that

$$\begin{aligned}\tau_E \|[\nu\partial_n w]\|_{0,E}^2 &\le \frac{h_e}{12\nu}\left(h_K^{-1}\|\nu\nabla w\cdot n\|_{0,K}^2 + h_K|\nu\nabla w\cdot n|_{1,K}^2\right)\\ &\le \frac{h_e\nu}{12h_K}|w|_{1,K}^2 + \frac{C_K\nu h_e}{12h_K}|w|_{1,K}^2 \le \frac{\nu(1+C_K)}{12}|w|_{1,K}^2.\end{aligned} \tag{3.27}$$

Combining (3.26) with (3.27) yields

$$\begin{aligned}&\left|B((u_h,p_h);(-w,0))\right|\\ &\ge -\sqrt{C\nu}\|w\|_{1,h}\left(\nu|u_h|_{1,h}^2 + \sum_{E\in\Gamma_h}\tau_E\|[\nu\partial_n u_h]\|_{0,E}^2\right)^{1/2} + C_0\|w\|_{1,h}\left\|p_h\right\|_{0,\Omega}\\ &= -\sqrt{C\nu}\|w\|_{1,h}|||u_h|||_h + C_0\left\|p_h\right\|_{0,\Omega}^2\\ &\ge -C\nu\gamma_1^{-1}|||u_h|||_h^2 + (C_0-\gamma_1)\left\|p_h\right\|_{0,\Omega}^2,\end{aligned} \tag{3.28}$$

where $C = (1+C_0)/12$ with $C_0 = \max_{k\in\mathcal{T}_h} C_K$, and γ_1 is chosen small enough. Let

$$(v_h, q_h) = (u_h - \delta w, p_h), \quad \delta > 0. \tag{3.29}$$

Using (3.26) and Lemma 3.3 we have

$$\begin{aligned}\left|B((u_h,p_h);(v_h,q_h))\right| &= \left|B((u_h,p_h);(u_h,p_h)) + \delta B((u_h,p_h);(-w,0))\right|\\ &\ge |||u_h|||_h^2 + \left\|p_h\right\|_h^2 + \delta\left(-C\nu\gamma_1^{-1}|||u_h|||_h^2 + (C_0-\gamma_1)\left\|p_h\right\|_{0,\Omega}^2\right)\\ &\ge \left(1 - C\delta\nu\gamma_1^{-1}\right)|||u_h|||_h^2 + \left\|p_h\right\|_h^2 + \delta(C_0-\gamma_1)\left\|p_h\right\|_{0,\Omega}^2\\ &\ge \nu\left(1 - C\delta\nu\gamma_1^{-1}\right)\|u_h\|_{1,h}^2 + \delta(C_0-\gamma_1)\left\|p_h\right\|_{0,\Omega}^2,\end{aligned} \tag{3.30}$$

provided that $0 < \delta < \gamma_1/(C\nu)$ and $0 < \gamma_1 < C_0$. Denote

$$C(\nu) \triangleq \min\left\{\nu\left(1 - C\delta\nu\gamma_1^{-1}\right), \delta(C_0-\gamma_1)\right\}, \qquad C(\delta) \triangleq \max\left\{2, 1+2\delta^2\right\}. \tag{3.31}$$

Then we have

$$\begin{aligned}\|v_h\|_{1,h}^2 + \left\|q_h\right\|_{0,\Omega}^2 &= \|u_h - \delta w\|_{1,h}^2 + \left\|p_h\right\|_{0,\Omega}^2\\ &\le 2\|u_h\|_{1,h}^2 + \left(1+2\delta^2\right)\left\|p_h\right\|_{0,\Omega}^2\\ &\le C(\delta)\left(\|u_h\|_{1,h}^2 + \left\|p_h\right\|_{0,\Omega}^2\right).\end{aligned} \tag{3.32}$$

Taking $\beta = C(\nu)/(C(\delta))$, we obtain the desired result (3.23). □

Theorem 3.5. *Under the assumptions of Theorem 2.1 and the following condition:*

$$\text{the strong uniqueness condition: } 1 - c_0\gamma_0 \frac{\gamma_0 + \nu^{1/2}}{\nu^2} \|f\|_{0,\Omega} > 0. \tag{3.33}$$

Problem (3.14) *admits a unique solution* $(u_h, p_h) \in (\mathrm{NCP}_1, \mathrm{P}_1)$, *and satisfying*

$$\begin{aligned} \|u_h\|_{1,h} &\leq \frac{\gamma_0 + \nu^{1/2}}{\nu} \|f\|_{0,\Omega}, \\ \|p_h\|_{0,\Omega} &\leq \beta^{-1} \left[\left(\gamma_0 + \frac{C}{\nu} \right) \|f\|_{0,\Omega} + c_0\gamma_0\nu^{-2} \left(\gamma_0 + \nu^{1/2} \right)^2 \|f\|_{0,\Omega}^2 \right]. \end{aligned} \tag{3.34}$$

Proof. Let Hilbert space $H_h = (\mathrm{NCP}_1, P_1)$ be with the scalar product and norm

$$((v,q);(w,r))_{H_h} = \sum_{K_j} (\nabla v, \nabla w)_{K_j} + (q,r), \tag{3.35}$$

and K_h be a nonvoid, convex, and compact subset of H_h defined by

$$\begin{aligned} K_h = \Bigg\{ (v,q) \in H_h : \|v\|_{1,h} &\leq \frac{\gamma_0 + \nu^{1/2}}{\nu} \|f\|_0, \\ \|q\|_{0,\Omega} &\leq \frac{\nu\gamma_0 + C}{\beta\nu} \|f\|_{0,\Omega} + \frac{c_0\gamma_0 \left(\gamma_0 + \nu^{1/2}\right)^2}{\beta\nu^2} \|f\|_{0,\Omega}^2 \Bigg\}. \end{aligned} \tag{3.36}$$

Defining a continuous mapping from K_h into H_h as follows: given $(\overline{v}, \overline{q}) \in K_h$, for all $(w,r) \in H_h$, find $(v,q) = F(\overline{v}, \overline{q})$ such that

$$B((v,q);(w,r)) + b(\overline{v}, v, w) = (f,w) + \sum_{K_j} \tau_{K_j} (f, \nabla r)_{K_j}. \tag{3.37}$$

Taking $(w,r) = (v,q)$, using (2.8)–(2.13) and inverse inequality yields

$$\begin{aligned} \nu\|v\|_{1,h}^2 + \|q\|_h^2 \leq |||v|||_h^2 + \|q\|_h^2 &\leq \gamma_0 \|f\|_{0,\Omega} \|v\|_{1,h} + \sum_{K_j} \tau_{K_j} \|f\|_{0,K_j} \|\nabla q\|_{0,K_j} \\ &\leq \left(\frac{\gamma_0^2}{2\nu} + \frac{h^2}{2} \right) \|f\|_{0,\Omega}^2 + \frac{\nu}{2} \|v\|_{1,h}^2 + \frac{1}{2} \sum_{K_j} \tau_{K_j} \|\nabla q\|_{0,K_j}^2 \\ &= \left(\frac{\gamma_0^2}{2\nu} + \frac{h^2}{2} \right) \|f\|_{0,\Omega}^2 + \frac{\nu}{2} \|v\|_{1,h}^2 + \frac{1}{2} \|q\|_h^2. \end{aligned} \tag{3.38}$$

As a consequence, we have

$$\|v\|_{1,h} \le \frac{\gamma_0 + \nu^{1/2}}{\nu} \|f\|_{0,\Omega}. \tag{3.39}$$

Using again (2.17), (3.23), (3.37), and inverse inequality, we arrive at

$$\begin{aligned} \beta\left(\|v\|_{1,h} + \|q\|_{0,\Omega}\right) &\le \frac{|f,w| + \left|\sum_{K_j} \tau_{K_j}(f,\nabla r)_{K_j}\right|}{\|w\|_{1,h} + \|r\|_0} + c_0\gamma_0\|\overline{v}\|_{1,h}\|v\|_{1,h} \\ &\le \gamma_0\|f\|_{0,\Omega} + \frac{Ch}{\nu}\|f\|_{0,\Omega} + c_0\gamma_0\nu^{-2}\left(\gamma_0 + \nu^{1/2}\right)^2\|f\|_{0,\Omega}^2 \\ &\le \left(\frac{\gamma_0\nu + Ch}{\nu}\right)\|f\|_{0,\Omega} + c_0\gamma_0\nu^{-2}\left(\gamma_0 + \nu^{1/2}\right)^2\|f\|_{0,\Omega}^2. \end{aligned} \tag{3.40}$$

Hence, the two estimates imply $(v,q) = F(\overline{v},\overline{q}) \in K_h$, thanks to the fixed point theorem, the mapping $(v,q) = F(\overline{v},\overline{q}) \in K_h$ has at least one fixed point $(u_h,p_h) \in K_h$; namely, $(u_h,p_h) \in K_h$ is a numerical solution of problem (3.14).

Next, we shall prove that the problem (3.14) has a unique solution (u_h,p_h). In fact, if (v_h,q_h) also satisfies (3.14), then for all $(w,r) \in (\mathrm{NCP}_1, P_1)$ we have

$$B\left((u_h - v_h, p_h - q_h);(w,r)\right) = b(v_h - u_h, u_h, w) + b(v_h, v_h - u_h, w). \tag{3.41}$$

Taking $(w,r) = (u_h - v_h, p_h - q_h)$ in (3.41) and using again (2.8)–(2.13), Lemma 3.3, it follows that

$$\nu\|u_h - v_h\|_{1,h}^2 \le c_0\gamma_0\|u_h\|_1\|u_h - v_h\|_{1,h}^2 \le c_0\gamma_0\frac{\gamma_0 + \nu^{1/2}}{\nu}\|f\|_{0,\Omega}\|u_h - v_h\|_{1,h}^2, \tag{3.42}$$

Which, together with the strong uniqueness condition

$$\nu - c_0\gamma_0\frac{\gamma_0 + \nu^{1/2}}{\nu}\|f\|_{0,\Omega} = \nu\left(1 - c_0\gamma_0\frac{\gamma_0 + \nu^{1/2}}{\nu^2}\|f\|_{0,\Omega}\right) > 0, \tag{3.43}$$

gives $u_h = v_h$. Using again (2.13), (3.23), and (3.41), we obtain $\beta\|p_h - q_h\|_{0,\Omega}^2 \le 0$ which implies $p_h = q_h$. □

4. Error Estimates

In order to derive the error estimates of the numerical solution (u_h,p_h), we introduce the Galerkin projection $(R_h,Q_h):(X,M) \to (\mathrm{NCP}_1,P_1)$ defined as follows: for all $(v_h,q_h) \in (\mathrm{NCP}_1,P_1)$

$$B\left((R_h(v,q),Q_h(v,q));(v_h,q_h)\right) = B_0\left((v,q);(v_h,q_h)\right). \tag{4.1}$$

Noting the Theorem 3.4, $(R_h(v,q),Q_h(v,q))$ is well defined.

By using a similar argument to the one used in [14, 22], we have the following lemma.

Lemma 4.1. *Let $(u,p) \in D(A) \times (H^1(\Omega) \cap M)$; under the assumptions of Theorems 3.4 and 3.5, the projection operator (R_h, Q_h) satisfies*

$$\left\|u - R_h(u,p)\right\|_{0,\Omega} + h\left(\left\|u - R_h(u,p)\right\|_{1,h} + \left\|p - Q_h(u,p)\right\|_{0,\Omega}\right) \leq Ch^2\left(\|Au\|_{0,\Omega} + |p|_{1,\Omega}\right). \tag{4.2}$$

Proof. From $(u,p) \in [H^2(\Omega)^2 \cap X] \times [H^1(\Omega) \cap M]$, we have $[\nu\partial_n u]_E = 0$. For all $(v_h, q_h) \in \mathrm{NCP}_1 \times P_1$, using (4.1) yields

$$\begin{aligned} B((R_h(u,p), Q_h(u,p))(v_h, q_h)) &= B_0((u,p); (v_h, q_h)) \\ &= B((u,p); (v_h, q_h)) - \sum_{K_j} \tau_{K_j} (\nabla p, \nabla q_h)_{K_j}. \end{aligned} \tag{4.3}$$

From the definition of $(R_h(u,p), Q_h(u,p))$, (3.3), combining Theorem 3.4, (4.3), the triangular with inverse inequalities, we arrive at

$$\begin{aligned} &\left\|u - R_h(u,p)\right\|_{1,h} + \left\|p - Q_h(u,p)\right\|_{0,\Omega} \\ &\leq \|u - u_I\|_{1,h} + \left\|p - p_I\right\|_{0,\Omega} + \left\|u_I - R_h(u,p)\right\|_{1,h} + \left\|p_I - Q_h(u,p)\right\|_{0,\Omega} \\ &\leq \|u - u_I\|_{1,h} + \left\|p - p_I\right\|_{0,\Omega} + \beta^{-1} \sup_{(v_h,q_h)\in \mathrm{NCP}_1 \times P_1} \frac{\left|B((u_I - R_h(u,p), p_I - Q_h(u,p)); (v_h, q_h))\right|}{\|v_h\|_{1,h} + \left\|q_h\right\|_{0,\Omega}} \\ &\leq \beta^{-1} \sup_{(v_h,q_h)\in \mathrm{NCP}_1 \times P_1} \frac{\left|B((u_I - u, p_I - p); (v_h, q_h))\right| + \left|\sum_{K_j} \tau_{K_j} (\nabla p, \nabla q_h)_{K_j}\right|}{\|v_h\|_{1,h} + \left\|q_h\right\|_{0,\Omega}} \\ &\quad + \|u - u_I\|_{1,h} + \left\|p - p_I\right\|_{0,\Omega}. \end{aligned} \tag{4.4}$$

It is easy to check that

$$\left|B((u_I - u, p_I - p); (v_h, q_h))\right| \leq C\left(\|u - u_I\|_{1,h} + \left\|p - p_I\right\|_{0,\Omega} + h|p|_1\right)\left(\|v_h\|_{1,h} + \left\|q_h\right\|_{0,\Omega}\right). \tag{4.5}$$

Combining (4.4), (4.5), and inverse inequality yields

$$\begin{aligned}
&\left\|u - R_h(u,p)\right\|_{1,h} + \left\|p - Q_h(u,p)\right\|_{0,\Omega} \\
&\quad\leq C\left(\|u - u_I\|_{1,h} + \left\|p - p_I\right\|_{0,\Omega} + h\left|p\right|_1\right) + \beta^{-1} \sup_{(v_h,q_h)\in \mathrm{NCP}_1\times P_1} \frac{\left|\sum_{K_j} \tau_{K_j}\left(\nabla p, \nabla q_h\right)_{K_j}\right|}{\|v_h\|_{1,h} + \left\|q_h\right\|_{0,\Omega}} \\
&\quad\leq C_1 h\left(\|Au\|_{0,\Omega} + \left|p\right|_{1,\Omega}\right) + C_2 h \sup_{(v_h,q_h)\in \mathrm{NCP}_1\times P_1} \frac{\sum_{K_j} h_{K_j}\left\|\nabla p\right\|_{0,K_j}\left\|\nabla q_h\right\|_{0,K_j}}{\|v_h\|_{1,h} + \left\|q_h\right\|_{0,\Omega}} \\
&\quad\leq Ch\left(\|Au\|_{0,\Omega} + \left\|p\right\|_{1,\Omega}\right).
\end{aligned} \tag{4.6}$$

In order to derive the estimate in the L^2-norm, we consider the following dual problem with $(e,\eta) = (u - R_h(u,p), p - Q_h(u,p))$:

$$-\Delta\Phi + \nabla\Psi = e \quad \text{in } \Omega, \tag{4.7}$$

$$\operatorname{div}\Phi = 0 \quad \text{in } \Omega, \tag{4.8}$$

$$\Phi|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega. \tag{4.9}$$

Based on the assumption of (A1), (4.7)–(4.9) have a unique solution and satisfy

$$\|A\Phi\|_{0,\Omega} + |\Psi|_{1,\Omega} \leq C\left\|u - R_h(u,p)\right\|_{0,\Omega}. \tag{4.10}$$

Multiplying (4.7) and (4.8) by e and η, respectively, integrating over Ω, and using (4.3) with $(v_h, q_h) = (\Phi_I, \Psi_I)$, we see that

$$\begin{aligned}
&\left\|u - R_h(u,p)\right\|_0^2 \\
&\quad= B_0\left(\left(u - R_h(u,p), p - Q_h(u,p)\right); (\Phi,\Psi)\right) \\
&\qquad- \sum_{K_j}\left\langle \frac{\partial\Phi}{\partial n}, u - R_h(u,p)\right\rangle_j + \sum_{K_j}\left\langle\left(u - R_h(u,p)\right)\cdot n, \Psi\right\rangle_j \\
&\quad= B\left(\left(u - R_h(u,p), p - Q_h(u,p)\right); (\Phi,\Psi)\right) - \sum_{K_j}\tau_{K_j}\left(\nabla\left(p - Q_h(u,p)\right), \nabla\Psi\right)_{K_j} \\
&\qquad- \sum_{K_j}\left\langle \frac{\partial\Phi}{\partial n}, u - R_h(u,p)\right\rangle_j + \sum_{K_j}\left\langle\left(u - R_h(u,p)\right)\cdot n, \Psi\right\rangle_j
\end{aligned}$$

$$\begin{aligned}
&= B((u - R_h(u,p), p - Q_h(u,p)); (\Phi - \Phi_I, \Psi - \Psi_I)) \\
&\quad - \sum_{K_j} \tau_{K_j} (\nabla (p - Q_h(u,p)), \nabla \Psi)_{K_j} + \sum_{K_j} \tau_{K_j} (\nabla p, \nabla \Psi_I)_{K_j} \\
&\quad - \sum_{K_j} \left\langle \frac{\partial \Phi}{\partial n}, u - R_h(u,p) \right\rangle_j + \sum_{K_j} \langle (u - R_h(u,p)) \cdot n, \Psi \rangle_j,
\end{aligned} \tag{4.11}$$

where (Φ_I, Ψ_I) is the finite element interpolation of (Φ, Ψ) in (NCP_1, P_1) and satisfies (3.3). For each $E \in \partial K_j$, we define the mean value of $u - R_h(u,p)$ and Ψ on E

$$\overline{u - R_h(u,p)} = \frac{1}{h_e} \int_E (u - R_h(u,p)) |_{K_j} ds; \qquad \overline{\Psi} = \frac{1}{h_e} \int_E \Psi|_{K_j} ds. \tag{4.12}$$

Note that each interior edge appears twice in the sum of (4.11); $\overline{u - R_h(u,p)}$ and $\overline{\Psi}$ are constants. Then it follows from (4.11) that

$$\begin{aligned}
&\|u - R_h(u,p)\|_{0,\Omega}^2 \\
&\quad = B((u - R_h(u,p), p - Q_h(u,p)); (\Phi - \Phi_I, \Psi - \Psi_I)) + \sum_{K_j} \tau_{K_j} (\nabla p, \nabla \Psi_I)_{K_j} \\
&\qquad - \sum_{K_j} \tau_{K_j} (\nabla (p - Q_h(u,p)), \nabla \Psi)_{K_j} + \sum_{K_j} \sum_{E \in \partial K_j} \left((u - R_h(u,p)) \cdot n, \Psi - \overline{\Psi} \right)_E \\
&\qquad - \sum_{K_j} \sum_{E \in \partial K_j} \left(\frac{\partial \Phi}{\partial n}, u - R_h(u,p) - \overline{u - R_h(u,p)} \right)_E.
\end{aligned} \tag{4.13}$$

Combining (3.3) with Lemma 4.1, we deduce that

$$\begin{aligned}
&B((u - R_h(u,p), p - Q_h(u,p)); (\Phi - \Phi_I, \Psi - \Psi_I)) \\
&\qquad \le Ch^2 \left(\|Au\|_{0,\Omega} + |p|_{1,\Omega} \right) (\|A\Phi\|_{0,\Omega} + |\Psi|_{1,\Omega}),
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\sum_{K_j} \tau_{K_j} (\nabla p, \nabla \Psi_I)_{K_j} - \sum_{K_j} \tau_{K_j} (\nabla (p - Q_h(u,p)), \nabla \Psi)_{K_j} \\
&\qquad \le \sum_{K_j} \tau_{K_j} |p|_{1,K_j} |\Psi_I|_{1,K_j} + \sum_{K_j} \tau_{K_j} |p - Q_h(u,p)|_{1,K_j} |\Psi|_{1,K_j}.
\end{aligned} \tag{4.15}$$

With the help of (4.12), we have

$$\int_E \left[(u - R_h(u,p)) - \overline{u - R_h(u,p)} \right] ds = 0. \tag{4.16}$$

Combining the definition of Π_j, (4.16), and local trace theorem (3.21) with the standard argument for the nonconforming element (see [21]), we see that

$$\begin{aligned}
&\sum_{K_j}\sum_{E\in\partial K_j}\left(\frac{\partial\Phi}{\partial n}, u-R_h(u,p)-\overline{u-R_h(u,p)}\right)_E \\
&\quad=\sum_{K_j}\sum_{E\in\partial K_j}\left(\frac{\partial\Phi}{\partial n}-\frac{\partial(\Pi_j\Phi)}{\partial n}, u-R_h(u,p)-\overline{u-R_h(u,p)}\right)_E \\
&\quad\leq\sum_{K_j}\sum_{E\in\partial K_j}\|\nabla(\Phi-\Pi_j\Phi)\|_{L^2(E)}\left\|u-R_h(u,p)-\overline{u-R_h(u,p)}\right\|_{L^2(E)} \\
&\quad\leq Ch^2\left(\|Au\|_{0,\Omega}+|p|_{1,\Omega}\right)\|A\Phi\|_{0,\Omega}.
\end{aligned}\tag{4.17}$$

In a similar way, we have

$$\sum_{K_j}\sum_{E\in\partial K_j}\left((u-R_h(u,p))\cdot n,\Psi-\overline{\Psi}\right)_E\leq Ch^2(\|Au\|_{0,\Omega}+|p|_1)|\Psi|_{1,\Omega}.\tag{4.18}$$

By combining (4.13)–(4.15) with (4.17)-(4.18), we deduce that

$$\|u-R_h(u,p)\|_{0,\Omega}\leq Ch^2\left(\|Au\|_{0,\Omega}+|p|_{1,\Omega}\right),\tag{4.19}$$

which, together with (4.6). We finish the proof. □

Theorem 4.2. *Assume that the conditions of Theorems 3.4 and 3.5 are valid; let* (u,p), (u_h,p_h) *be the solutions of* (2.1) *and* (3.14), *respectively, then*

$$\|u-u_h\|_{1,h}+\|p-p_h\|_{0,\Omega}\leq Ch.\tag{4.20}$$

Proof. We get the following error equation by combining (2.1) with (3.14), for all $(v_h,q_h)\in(\mathrm{NCP}_1,P_1)$

$$\begin{aligned}
&B((u-u_h,p-p_h);(v_h,q_h))+b(u-u_h,u,v_h)+b(u_h,u-u_h,v_h)-\sum_{K_j}\left\langle\frac{\partial u}{\partial n},v_h\right\rangle_j \\
&\quad+\sum_{K_j}\langle v_h\cdot n,p\rangle_j=\sum_{K_j}\tau_{K_j}(\nabla p,\nabla q_h)_{K_j}-\sum_{K_j}\tau_{K_j}(f,\nabla q_h)_{K_j}.
\end{aligned}\tag{4.21}$$

With (4.3), (4.21) can be rewritten as

$$\begin{aligned}
&B((e_h,\eta_h);(v_h,q_h))+b(u-R_h(u,p)+e_h,u,v_h)+b(u_h,u-R_h(u,p)+e_h,v_h) \\
&\quad-\sum_{K_j}\left\langle\frac{\partial u}{\partial n},v_h\right\rangle_j+\sum_{K_j}\langle v_h\cdot n,p\rangle_j=-\sum_{K_j}\tau_{K_j}(f,\nabla q_h)_{K_j},
\end{aligned}\tag{4.22}$$

where $e_h=R_h(u,p)-u_h$ and $\eta_h=Q_h(u,p)-p_h$.

From Theorem 3.5 and (4.22), we get that

$$
\begin{aligned}
\|\eta_h\|_{0,\Omega} &\le \beta^{-1} \sup_{0\neq(v_h,q_h)\in(\mathrm{NCP}_1,P_1)} \frac{\left|B((e_h,\eta_h);(v_h,q_h))\right|}{\|v_h\|_{1,h}+\|q_h\|_{0,\Omega}} \\
&\le \beta^{-1} \sup_{0\neq(v_h,q_h)\in(\mathrm{NCP}_1,P_1)} \frac{1}{\|v_h\|_{1,h}+\|q_h\|_{0,\Omega}} \\
&\quad\times\left(\left|b(u-R_h(u,p)+e_h,u,v_h)\right| + \left|\sum_{K_j}\tau_{K_j}(f,\nabla q_h)_{K_j}\right|\right. \\
&\quad\left. +\left|b(u_h,u-R_h(u,p)+e_h,v_h)\right| + \left|\sum_{K_j}\left\langle\frac{\partial u}{\partial n},v_h\right\rangle_j\right| + \left|\sum_{K_j}\langle v_h\cdot n,p\rangle_j\right|\right).
\end{aligned}
\tag{4.23}
$$

Again, with (2.13), Theorem 2.1, inverse inequality, and Lemma 4.1, we have

$$
\begin{aligned}
&\left|b(u-R_h(u,p),u,v_h)\right| + \left|b(u_h,u-R_h(u,p),v_h)\right| \\
&\quad\le c_0\gamma_0\left(|u|_{1,\Omega}+\|u_h\|_{1,h}\right)\left\|u-R_h(u,p)\right\|_{1,h}\|v_h\|_{1,h} \le Ch\|v_h\|_{1,h},
\end{aligned}
\tag{4.24}
$$

$$
|b(e_h,u,v_h)| + |b(u_h,e_h,v_h)| \le c_0\gamma_0\left(|u|_{1,\Omega}+\|u_h\|_{1,h}\right)\|e_h\|_{1,h}\|v_h\|_{1,h},
\tag{4.25}
$$

$$
\begin{aligned}
\sum_{K_j}\tau_{K_j}\left|(f,\nabla q_h)_{K_j}\right| &\le \sum_{K_j}\tau_{K_j}\|f\|_{0,K_j}\|\nabla q_h\|_{0,K_j} \\
&\le \sum_{K_j}\tilde{C}h_{K_j}^2\|f\|_{0,K_j}\|\nabla q_h\|_{0,K_j} \\
&\le C(\nu)h\|f\|_{0,\Omega}\|q_h\|_{0,\Omega},
\end{aligned}
\tag{4.26}
$$

and using the similar arguments as for (4.17)-(4.18) yields

$$
\left|\sum_{K_j}\left\langle\frac{\partial u}{\partial n},v_h\right\rangle_j\right| + \left|\sum_{K_j}\langle v_h\cdot n,p\rangle_j\right| \le Ch^2\left(\|Au\|_{0,\Omega}+|p|_{1,\Omega}\right)\|v_h\|_{1,h}.
\tag{4.27}
$$

Combining (4.23)–(4.27) with Theorem 3.5, we arrive at

$$
\|\eta_h\|_{0,\Omega} \le Ch + 2c_0\gamma_0\frac{\gamma_0+\nu^{1/2}}{\nu}\|f\|_{0,\Omega}\|e_h\|_{1,h}.
\tag{4.28}
$$

Choosing $(v_h, q_h) = (e_h, \eta_h)$ in (4.22), we obtain that

$$\begin{aligned} B\big((e_h,\eta_h);(e_h,\eta_h)\big) + b(e_h,u,e_h) + \sum_{K_j}\tau_{K_j}(f,\nabla\eta_h)_{K_j} \\ = -b(u-R_h(u,p),u,e_h) - b(u_h,u-R_h(u,p),e_h) \\ + \sum_{K_j}\left\langle\frac{\partial u}{\partial n},v_h\right\rangle_j - \sum_{K_j}\langle e_h\cdot n,p\rangle_j. \end{aligned} \tag{4.29}$$

Using (2.13), (2.17), Theorem 2.1, and Lemma 3.3, we get

$$\begin{aligned} B\big((e_h,\eta_h);(e_h,\eta_h)\big) - b(e_h,u,e_h) &= |||e_h|||_h^2 + \|\eta_h\|_h^2 - c_0\gamma_0|u|_1\|e_h\|_{1,h}^2 \\ &\geq \nu\|e_h\|_{1,h}^2 - c_0\gamma_0|u|_1\|e_h\|_{1,h}^2 \\ &\geq \nu\Big(1 - c_0\gamma_0^2\nu^{-2}\|f\|_0\Big)\|e_h\|_{1,h}^2 > 0. \end{aligned} \tag{4.30}$$

Combining (4.24)–(4.28) with (4.29) yields:

$$\|e_h\|_{1,h} \leq Ch. \tag{4.31}$$

From (4.28) and (4.31), we obtain that $\|\eta_h\|_{0,\Omega} \leq Ch$. Furthermore, we finish the proof by combining triangles inequality with Lemma 4.1, (4.28), and (4.31). □

Theorem 4.3. *Let (u,p) and (u_h,p_h) be the solutions of* (2.1) *and* (3.14), *respectively, then we have*

$$\|u-u_h\|_{0,\Omega} \leq Ch^2\Big(\|Au\|_{0,\Omega} + |p|_{1,\Omega}\Big). \tag{4.32}$$

Proof. Using the duality argument for a linearized stationary Navier-Stokes problem; for some given $g \in Y$ and the solution (u,p) of (2.1), defining $(\Phi,\Psi) \in (X,M)$ by

$$-\nu\Delta\Phi + \nabla\Psi + \widetilde{B}(u,\Phi) - B(u,\Phi) = g, \quad \text{in } \Omega, \tag{4.33}$$

$$\operatorname{div}\Phi = 0 \quad \text{in } \Omega, \tag{4.34}$$

$$u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega, \tag{4.35}$$

where $\widetilde{B}(u,\Phi)$ is defined as $\langle v,\widetilde{B}(u,\Phi)\rangle_{X\times X'} = b(v,u,\Phi)$, for all $v \in X$; multiplying (4.33) and (4.34) by $v \in X$ and $q \in M$, respectively; integrating over Ω, from (2.8)–(2.11), it is easily to see that the bilinear form $a(\Phi,v) - d(v,\Psi) + d(\Phi,q)$ is continuity and $X \times M$ coercive, by using the Lax-Milgram's Lemma, (4.33)–(4.35) have a unique solution (Φ,Ψ).

Multiplying (4.33) and (4.34) by Φ and Ψ, respectively, using (2.13) and Theorem 2.1, we have

$$\begin{aligned} \nu|\Phi|_{1,\Omega}^2 - b(\Phi, u, \Phi) &\geq \nu|\Phi|_{1,\Omega}^2 - c_0\gamma_0|u|_{1,\Omega}|\Phi|_{1,\Omega}^2 \\ &\geq \nu\Big(1 - c_0\gamma_0^2\nu^{-2}\|f\|_{0,\Omega}\Big)|\Phi|_{1,\Omega}^2 > 0. \end{aligned} \tag{4.36}$$

On the other hand, estimating the right term yields

$$(\Phi, g) \leq \|\Phi\|_{0,\Omega}\|g\|_{0,\Omega} \leq \gamma_0|\Phi|_{1,\Omega}\|g\|_{0,\Omega}. \tag{4.37}$$

By using (4.36) and (4.37), we arrive at

$$|\Phi|_{1,\Omega} \leq C\|g\|_{0,\Omega}. \tag{4.38}$$

Setting $\Psi = 0$ and taking the scalar product of (4.33) with $A\Phi$ in Y yields

$$\nu\|A\Phi\|_0^2 + b(A\Phi, u, \Phi) - b(u, \Phi, A\Phi) = (g, A\Phi). \tag{4.39}$$

Using the Gagliardo-Nirenberg inequality yields

$$\begin{aligned} \|v\|_{L^4}^2 &\leq C\|v\|_{0,\Omega}|v|_{1,\Omega}, \quad \forall v \in X, \\ \|\nabla v\|_{L^4}^2 &\leq C\|Av\|_{0,\Omega}|v|_{1,\Omega}, \quad \forall v \in H^2(\Omega)^2 \cap X. \end{aligned} \tag{4.40}$$

With the help of the Agmon's inequality, we have

$$\|v\|_{L^\infty}^2 \leq C\|v\|_{0,\Omega}\|Av\|_{0,\Omega}, \quad \forall v \in H^2(\Omega)^2 \cap X. \tag{4.41}$$

Furthermore, the following estimates are hold:

$$\begin{aligned} b(A\Phi, u, \Phi) &\leq C\|A\Phi\|_{0,\Omega}\left(\|\nabla u\|_{L^4}\|\Phi\|_{L^4} + \|u\|_{L^\infty}\|\nabla\Phi\|_{0,\Omega}\right) \\ &\leq \frac{\nu}{4}\|A\Phi\|_{0,\Omega}^2 + C\nu^{-1}\|Au\|_{0,\Omega}^2\|\nabla\Phi\|_{0,\Omega}^2), \\ b(u, \Phi, A\Phi) &\leq \frac{\nu}{4}\|A\Phi\|_{0,\Omega}^2 + C\nu^{-1}\|Au\|_{0,\Omega}^2\|\nabla\Phi\|_{0,\Omega}^2), \\ (g, A\Phi) &\leq \frac{\nu}{4}\|A\Phi\|_{0,\Omega}^2 + C\nu^{-1}\|g\|_{0,\Omega}. \end{aligned} \tag{4.42}$$

Combining above inequalities with (4.38) and (4.39), we arrive at

$$\|A\Phi\|_{0,\Omega}^2 \leq c\nu^{-2}\Big(1 + \|Au\|_{0,\Omega}^2\Big)\|g\|_{0,\Omega}^2. \tag{4.43}$$

Applying the continuous Inf-Sup condition (2.8) yields

$$|\Psi|_{1,\Omega} \le c\nu\|A\Phi\|_{0,\Omega} + c\|Au\|_{0,\Omega}|\Phi|_{1,\Omega} + c\|g\|_{0,\Omega}. \tag{4.44}$$

Combining (4.43)-(4.44) with (2.18), (4.38), we arrive at

$$\|A\Phi\|_{0,\Omega} + |\Psi|_{1,\Omega} \le C\|g\|_{0,\Omega}. \tag{4.45}$$

Taking $g = R_h(u,p) - u_h$, multiplying (4.33) and (4.34) by $R_h(u,p) - u_h$ and $Q_h(u,p) - p_h$, respectively, using (4.1) yields

$$\begin{aligned}
&\|R_h(u,p) - u_h\|_{0,\Omega}^2 \\
&\quad = B\big((R_h(u,p) - u_h, Q_h(u,p) - p_h); (\Phi,\Psi)\big) + b(u, R_h(u,p) - u_h, \Phi) \\
&\qquad - \sum_{K_j} \tau_{K_j}\big(\nabla(Q_h(u,p) - p_h), \nabla\Psi\big)_{K_j} + \sum_{K_j} \langle \Psi, (R_h(u,p) - u_h)\cdot n, \rangle_j \\
&\qquad + b(R_h(u,p) - u_h, u, \Phi) - \sum_{K_j} \left\langle \frac{\partial\Phi}{\partial n}, R_h(u,p) - u_h \right\rangle_j.
\end{aligned} \tag{4.46}$$

Setting $(v_h, q_h) = (\Phi_I, \Psi_I)$ in (4.22), and using (4.46), we obtain that

$$\begin{aligned}
&\|R_h(u,p) - u_h\|_{0,\Omega}^2 \\
&= B\big((R_h(u,p) - u_h, Q_h(u,p) - p_h); (\Phi - \Phi_I, \Psi - \Psi_I)\big) \\
&- \sum_{K_j} \tau_{K_j}\big(\nabla(Q_h(u,p) - p_h), \nabla\Psi\big)_{K_j} + b(u - u_h, R_h(u,p) - u_h, \Phi) + \sum_{K_j} \left\langle \frac{\partial u}{\partial n}, v_h \right\rangle_j \\
&+ b(u - R_h(u,p), u, \Phi_I) + b(R_h(u,p) - u_h, u, \Phi - \Phi_I) + b(u_h, u - R_h(u,p), \Phi - \Phi_I) \\
&+ b(u_h, R_h(u,p) - u_h, \Phi - \Phi_I) - b(u_h, u - R_h(u,p), \Phi) + \sum_{K_j} \langle \Psi, (R_h(u,p) - u_h)\cdot n, \rangle_j \\
&- \sum_{K_j} \tau_{K_j}(f, \nabla\Psi_I)_{K_j} - \sum_{K_j} \left\langle \frac{\partial\Phi}{\partial n}, R_h(u,p) - u_h \right\rangle_j - \sum_{K_j} \langle v_h \cdot n, p\rangle_j.
\end{aligned} \tag{4.47}$$

We now estimate the right terms of (4.47). Thanks to (3.3); Theorems 2.1, 3.4-3.5, and 4.2; inverse inequality; Lemma 4.1; we know that

$$
\begin{aligned}
&B((R_h(u,p)-u_h, Q_h(u,p)-p_h);(\Phi-\Phi_I,\Psi-\Psi_I))\\
&\le C\left(\|R_h(u,p)-u_h\|_{1,h}+\|Q_h(u,p)-p_h\|_{0,\Omega}\right)(\|\Phi-\Phi_I\|_{1,h}+\|\Psi-\Psi_I\|_{0,\Omega}+h|\Psi|_{1,\Omega})\\
&\le Ch^2\left(\|Au\|_{0,\Omega}+|p|_{1,\Omega}\right)(\|A\Phi\|_{0,\Omega}+|\Psi|_{1,\Omega}),\\
&b(R_h(u,p)-u_h,u,\Phi-\Phi_I)+b(u_h,R_h(u,p)-u_h,\Phi-\Phi_I)\\
&\le 2c_0(|u|_{1,\Omega}+\|u_h\|_{1,h})\|R_h(u,p)-u_h\|_{1,h}\|\Phi-\Phi_I\|_{1,h}\\
&\le Ch^2\left(\|Au\|_{0,\Omega}+|p|_{1,\Omega}\right)\|A\Phi\|_{0,\Omega},\\
&\sum_{K_j}\tau_{K_j}(\nabla(Q_h(u,p)-p_h),\nabla\Psi)_{K_j}\le\sum_{K_j}\widetilde{C}h_K^2\|\nabla(Q_h(u,p)-p_h)\|_{0,K_j}\|\nabla\Psi\|_{0,K_j}\\
&\qquad\le Ch^2(\|Au\|_{0,\Omega}+|p|_1)|\Psi|_{1,\Omega},\\
&b(u-R_h(u,p),u,\Phi_I)\le C\|u-R_h(u,p)\|_{0,\Omega}\|Au\|_{0,\Omega}|\Phi_I|_{1,h}\\
&\qquad\le Ch^2\left(\|Au\|_{0,\Omega}+|p|_{1,\Omega}\right)|\Phi|_{1,\Omega},\\
&b(u-u_h,R_h(u,p)-u_h,\Phi)\le c_0\|u-u_h\|_{1,h}\|R_h(u,p)-u_h\|_{1,h}|\Phi|_{1,\Omega}\\
&\qquad\le Ch^2(\|Au\|_{0,\Omega}+|p|_1)|\Phi|_{1,\Omega},\\
&b(u_h,u-R_h(u,p),\Phi)\le C\|u_h\|_{1,h}\|u-R_h(u,p)\|_{0,\Omega}\|A\Phi\|_{0,\Omega},\\
&\sum_{K_j}\tau_{K_j}(f,\nabla\Psi_I)_{K_j}\le\sum_{K_j}\widetilde{C}h_{K_j}^2\|f\|_{0,K_j}\|\nabla\Psi_I\|_{0,K_j}\le Ch^2\|f\|_{0,\Omega}|\Psi|_{1,\Omega},\\
&b(u_h,u-R_h(u,p),\Phi-\Phi_I)\le c_0\|u_h\|_{1,h}\|u-R_h(u,p)\|_{1,h}\|\Phi-\Phi_I\|_{1,h}\\
&\qquad\le Ch^2\left(\|Au\|_{0,\Omega}+|p|_{1,\Omega}\right)\|A\Phi\|_{0,\Omega}.
\end{aligned}
\tag{4.48}
$$

Applying the argument used for (4.18)-(4.21), (4.28) gives

$$
\begin{aligned}
&\sum_{K_j}\left\langle\frac{\partial\Phi}{\partial n},R_h(u,p)-u_h\right\rangle_j+\sum_{K_j}\langle\Psi,(R_h(u,p)-u_h)\cdot n,\rangle_j\\
&\qquad\le Ch^2\left(\|Au\|_{0,\Omega}+|p|_{1,\Omega}\right)(\|A\Phi\|_{0,\Omega}+|\Psi|_{1,\Omega}),\\
&\sum_{K_j}\left\langle\frac{\partial u}{\partial n},\Phi_I\right\rangle+\sum_{K_j}\langle\Phi_I\cdot n,p\rangle_j\le Ch^2\left(\|Au\|_{0,\Omega}+\|p\|_{1,\Omega}\right)\|\Phi_I\|_{1,h}\\
&\qquad\le Ch^2\left(\|Au\|_{0,\Omega}+\|p\|_{1,\Omega}\right)\|\Phi_I\|_{1,\Omega}.
\end{aligned}
\tag{4.49}
$$

Combining above inequalities, Theorems 2.1, 3.5, (4.46), and (4.51), we get that

$$\begin{aligned}
&\left\|R_h(u,p)-u_h\right\|_{0,\Omega}^2 \\
&\quad\le Ch^2\left(\|Au\|_{0,\Omega}+\left|p\right|_{1,\Omega}\right)\left(\|A\Phi\|_{0,\Omega}+|\Psi|_{1,\Omega}\right)+C\|u_h\|_{1,h}\left\|u-R_h(u,p)\right\|_{0,\Omega}\|A\Phi\|_{0,\Omega} \\
&\quad\le Ch^2\left(\|Au\|_{0,\Omega}+\left|p\right|_{1,\Omega}\right)\left\|R_h(u,p)-u_h\right\|_{1,\Omega}+C\frac{\gamma_0+\nu^{1/2}}{\nu}\|f\|_{0,\Omega}\left\|u-R_h(u,p)\right\|_{0,\Omega}^2.
\end{aligned} \tag{4.50}$$

Choosing the appropriate ν,Ω and f such that $1-C((\gamma_0+\nu^{1/2})/\nu)\|f\|_0>0$, then we have

$$\left\|R_h(u,p)-u_h\right\|_{0,\Omega}\le Ch^2\left(\|Au\|_{0,\Omega}+\left\|p\right\|_{1,\Omega}\right). \tag{4.51}$$

By applying the triangles inequality, Lemma 4.1 and (4.51), we finish the proof. □

Lemma 4.4. *Under the assumptions of Theorem 4.2, the following estimate about $u-u_h$ in mesh-dependent norm*

$$|||u-u_h|||_h\le Ch, \tag{4.52}$$

holds, where u and u_h are the solution of problem (2.1) *and* (3.14), *respectively.*

Proof. According to the definition of $|||\cdot|||_h$, $u_h\in X_h$, with Theorems 2.1 and 4.2, inverse and local trace inequalities (3.21), we have

$$\begin{aligned}
|||u-u_h|||_h^2&=\nu|||u-u_h|||_{1,h}^2+\sum_{E\in\Gamma_h}\tau_E\|\nu\partial_n(u-u_h)\|_{0,E}^2 \\
&=\nu|||u-u_h|||_{1,h}^2+\sum_{E\in\Gamma_h}\frac{h_e}{12\nu}\|\nu\partial_n(u-u_h)\|_{0,E}^2 \\
&\le Ch^2+C\sum_{E\in\Gamma_h}h_e\left[h_{K_j}^{-1}\|\nabla(u-u_h)\cdot n\|_{0,K_j}^2+h_{K_j}|\nabla(u-u_h)\cdot n|_{1,K_j}^2\right] \\
&\le Ch^2.
\end{aligned} \tag{4.53}$$

□

5. Numerical Validations

In this section, we provide two numerical examples to illustrate the theoretical analysis of the method (3.14). In all experiments, we consider the domain Ω to be the square $[0,1]\times[0,1]$. The mesh consists of triangular elements that are obtained by dividing Ω into subsquares of equal size and then drawing the diagonal in each sub-square, see Figure 1. The software Freefem++, developed by Hecht et al. [23], is used in our experiments.

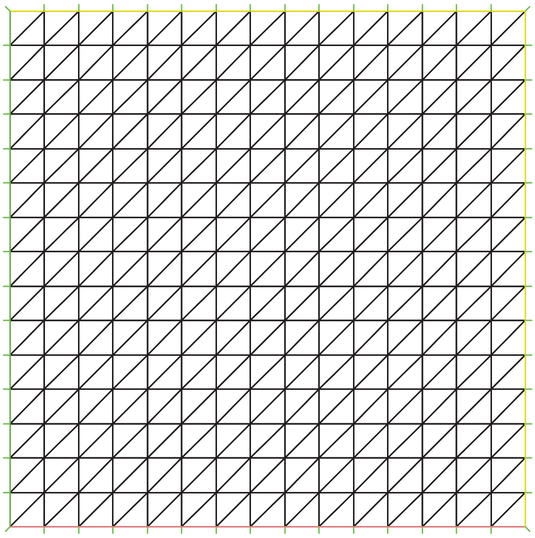


Figure 1: Uniform triangulation of Ω into triangulars.

Table 1: Numerical results for enriched multiscale method with NCP_1-P_1 element.

$1/h$	$\|p_h - \tilde{p}\|_0/\|\tilde{p}\|_0$	$\|u_h - \tilde{u}\|_0/\|\tilde{u}\|_0$	$\|u_h - \tilde{u}\|_1/\|\tilde{u}\|_1$	p_{L^2} rate	u_{L^2} rate	u_{H^1} rate
20	0.160667	0.15627	0.816235			
25	0.126534	0.10132	0.665410	1.0703	1.9418	0.9155
30	0.105785	0.0708881	0.556179	0.9823	1.9591	0.9835
35	0.083427	0.0523941	0.478261	1.5403	1.9611	0.9791
40	0.0667358	0.0405399	0.420987	1.6717	1.9209	0.9552

Table 2: Numerical results for PPM in [6] with NCP_1-P_1 element.

$1/h$	$\|p_h - p\|_0/\|\tilde{p}\|_0$	$\|u_h - u\|_0/\|u\|_0$	$\|u_h - u\|_1/\|u\|_1$	p_{L^2} rate	u_{L^2} rate	u_{H^1} rate
20	0.208238	0.0182227	0.12718	1.8797	1.9629	0.9963
25	0.136834	0.0117122	0.101772	1.8818	1.9809	0.9988
30	0.0972068	0.0081486	0.0848213	1.8754	1.9898	0.9993
35	0.0727087	0.00599297	0.0727009	1.8837	1.9932	1.0002
40	0.056478	0.00459177	0.0636082	1.8918	1.9945	1.0006

5.1. An Analytical Solution: Convergence Validation

For this test, our purpose is to verify the theoretical analysis which has been established in the previous section by setting the viscosity coefficient $\nu = 1$ and f is given by the exact solution

$$\begin{aligned} u_1 &= 10x^2(x-1)^2 y(y-1)(2y-1), \\ u_2 &= -10x(x-1)(2x-1)y^2(y-1)^2, \\ p &= 10(2x-1)(2y-1). \end{aligned} \tag{5.1}$$

In our numerical validation, The experimental rates of convergence with respect to the mesh size h are calculated by the formula $(\log(E_i/E_{i+1}))/(\log(h_i/h_{i+1}))$, where E_i and E_{i+1} are the relative errors corresponding to the meshes of sizes h_i and h_{i+1}.

Table 3: Numerical results for SGM with P_1b-P_1 element.

$1/h$	$\|p_h - p\|_0/\|p\|_0$	$\|u_h - u\|_0/\|u\|_0$	$\|u_h - u\|_1/\|u\|_1$	p_{L^2} rate	u_{L^2} rate	u_{H^1} rate
20	0.194855	0.0167771	0.120396	1.9352	2.0150	1.0040
25	0.127314	0.010703	0.0962298	1.9073	2.0144	1.0041
30	0.0903239	0.00741503	0.080144	1.8827	2.0130	1.0033
35	0.0675341	0.00543371	0.068793	1.8863	2.0168	0.9907
40	0.0527341	0.00416312	0.059691	1.8532	1.9947	1.0628

Table 4: Numerical results for LGIM in [4, 7, 19] with NCP_1-P_1 element.

$1/h$	$\|p_h - p\|_0/\|p\|_0$	$\|u_h - u\|_0/\|u\|_0$	$\|u_h - u\|_1/\|u\|_1$	p_{L^2} rate	u_{L^2} rate	u_{H^1} rate
20	0.263248	0.0186335	0.127546	1.7919	2.0088	1.0010
25	0.176286	0.0118944	0.102010	1.7970	2.0117	1.0012
30	0.128021	0.00824161	0.0849864	1.7547	2.0122	1.0014
35	0.0978302	0.00604517	0.0728229	1.7448	2.0106	1.0020
40	0.0776455	0.00462309	0.0637016	1.7305	2.0085	1.0022

Table 5: CPU time for solving the steady Navier-Stokes equations with different mesh sizes.

$1/h$	20	25	30	35	40
Our method with NCP_1-P_1	1.6324	3.286	5.975	8.792	14.088
PPM in [6] with NCP_1-P_1	1.875	3.594	6.406	10.891	17.453
LGIM in [10] with NCP_1-P_1	1.844	3.547	6.391	10.593	17.047
SGM with P_1b-P_1	3.031	6.078	11.375	21.106	35.020

In order to show the efficiency of the enriched multiscale method, we compare the numerical results obtained by using different methods, which are shown in Tables 1, 2, 3, and 4. The compared methods include the pressure projection method (PPM) in [6], the local Gauss integration method (LGIM) in [7, 10, 19], and the standard Galerkin method (SGM) with MINI element (see [1]), respectively. From these tables, we can see that the stabilized multiscale method has good precision for pressure, and the precision of velocity, worse than other methods. Table 5 explains the CPU times that needed for solving the steady Navier-Stokes equations in different mesh sizes. From these data, we know that our method takes less time than other methods. Furthermore, from Tables 1–4, we can see that the numerical results reproduce the established theoretical analysis and show an $\mathcal{O}(h)$ order of convergence for $|u - u_h|_{1,\Omega}$ and $\|p - p_h\|_{0,\Omega}$, and an $\mathcal{O}(h^2)$ convergence for $\|u - u_h\|_{0,\Omega}$.

5.2. Lid-Driven Cavity Problem

In this test, we consider the incompressible lid-driven cavity flow problem defined on the unit square. Setting $f = 0$ and the boundary condition $u = 0$ on $[\{0\} \times (0,1)] \cup [(0,1) \times \{0\}] \cup [\{1\} \times (0,1)]$ and $u = (1,0)^T$ on $(0,1) \times \{1\}$, see Figure 2. The mesh consists of triangular element and the mesh size $h = 1/50$.

Figure 3 shows the pressure contours at different Reylond numbers, where the stopping criterion $\|u_h^{n+1} - u_h^n\|_{0,\Omega}/\|u_h^{n+1}\|_{0,\Omega} \leq 10^{-6}$ is employed, where u_h^{n+1} is the approximation of u_h at the $n+1$ Newton iterative. From Figure 3, we can see that the oscillations are absented for the pressure isovalues by the NCP_1-P_1 approximations, and compared with the results given in [24], we can see that our method has the effect to stabilized flow field (see Figure 4).

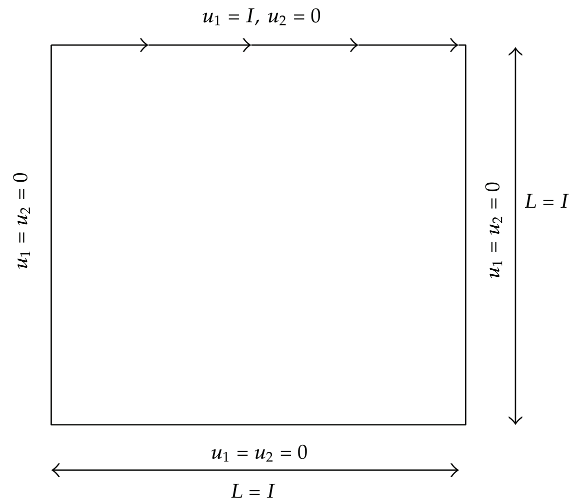


Figure 2: Lid-driven cavity flow.

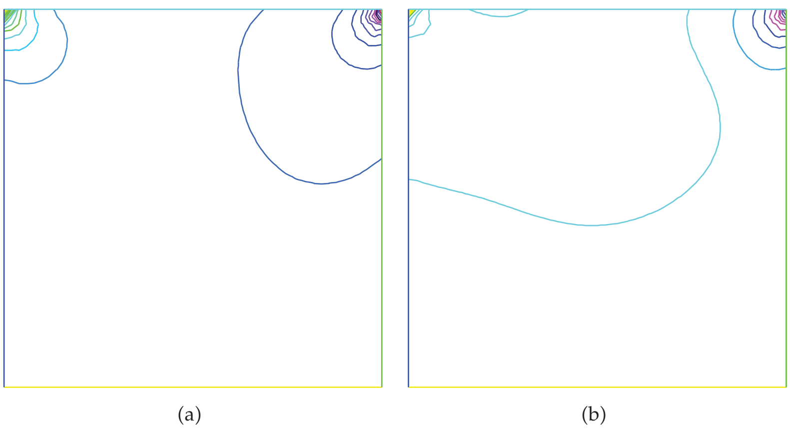
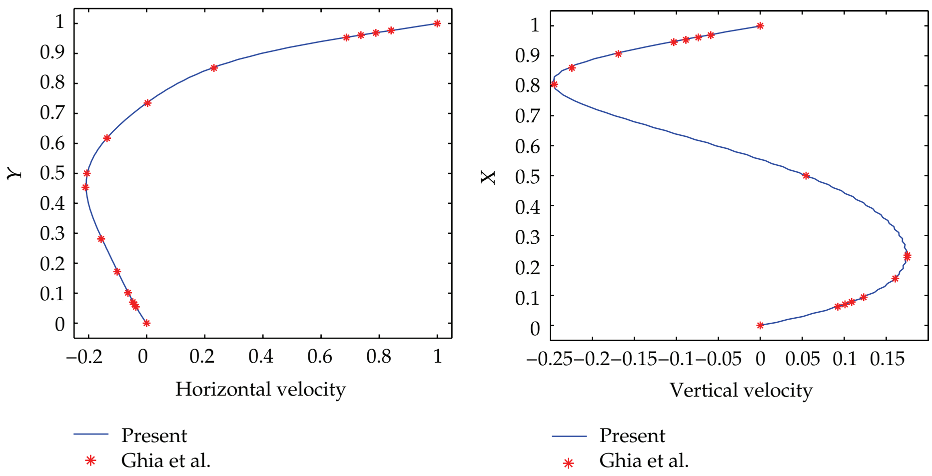


Figure 3: Pressure contours of the driven cavity by using the stabilized multiscale method with NCP_1-P_1 at different Reynolds numbers. (a) Re = 1, (b) Re = 100.

In this sense, we say that the stabilized multiscale nonconforming finite element method is effective for the stationary Navier-Stokes problem.

6. Conclusion

In this paper we have derived a theoretical analysis of enriched multiscale nonconforming finite element method for the steady incompressible Navier-Stokes equations. The analysis has extended the work in [16] from the linear problem to the nonlinear problem. The discretization uses nonconforming and conforming piecewise linear finite elements for velocity and pressure over triangles elements, respectively. Numerical tests show that this stabilized method is computationally efficient, and it can be performed locally at the element



(a) (b)

Figure 4: The computed velocity profiles through the geometric center at Re = 100. (a) Horizontal velocity, (b) vertical velocity.

level with minimal additional cost; at the same time, our numerical results obtained are in good agreement with the established theoretical results.

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