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Research Article

Stability Results for a Class of Differential Equation and Application in Medicine

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A *Chemostat* system incorporating hepatocellular carcinomas is discussed. The model generalizes the classical *Chemostat* model, and it assumes that the *Chemostat* is an increasing function of the concentration. The asymptotic behavior of solutions is determined. Sufficient conditions for the local and global asymptotic stability of equilibrium and numerical simulation are obtained, which is used to select the disease control tactics.

1. Introduction

As we know, the stability of ecological systems and the persistence of species within them are fundamental concerns in ecology. Mathematical models of ecological systems, reflecting these concerns, have been used to investigate the stability of a variety of systems. For example, see [1–7]. The dynamic relationship between predator and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [4], and many good achievements have been reached [1–5, 8–10]. But few works about hepatocellular carcinomas cell *Chemostat* model have been done.

According to some medical knowledge, under certain condition, the growth rate of tumor is in proportion to the volume on the time t. Let $V(t)$ denote this volume, such differential equations take the form

$$\frac{dV(t)}{dt} = (K(t) - Q(t))V(t), \tag{1.1}$$

where $K(t)$ and $Q(t)$ denote the cell growth and death rate, respectively.

Some research on tumor cell shows that the growth rate is not invariable, but in reciprocal proportion to the factor t. Therefore, it becomes

$$K'(t) = -L(t)K(t). \tag{1.2}$$

So the growth of the tumor disciplinarian can be denoted as follows, see [2],

$$\begin{aligned} V'(t) &= (K(t) - Q(t))V(t), \\ K'(t) &= -L(t)K(t). \end{aligned} \tag{1.3}$$

Following the accumulation of experimental data, it became evident that the system (1.3) requires modification. Another research shows that the growth speed of the tumor cell is continuously decreased, which can be denoted by cube root function.

In fact, different kinds of tumor cell have different growth mode which behave exponential function, cube root function, or linear equation. It has been well established in experimental literature, so it becomes

$$V'(t) = (K(t) - Q(t))V^{\alpha}(t). \tag{1.4}$$

The data presented by clinical experience shows that the liver tumor tissue is similar to entity ball. Because the volume of the liver on the time t may be regarded as invariable, it is treated as *Chemostat* model.

In this paper, we modify the modeling approach developed in some paper [2–7]. It extends the above model by assuming that $Q(t)$ is constant.

The symbol in paper [7] is used as follows: $x(t)$ and $y(t)$ denote tumor and normal cell consistency, respectively; $Q(t)$ and $P(t)$ denote the death and growth rate of tumor cell, and a typical choice for $Q(t)$ and $P(t)$ are $P(t) = \mu x/(ky + x)\delta$, $Q(t) = Q$ respectively; x^0 is the growth speed of normal cell, $1/\delta$ is the consume rate of normal cell. The system (1.3) becomes, see [6, 7]

$$\begin{aligned} x'(t) &= Q\left(x^0 - x\right) - \frac{\mu x}{(ky + x)\delta} y^{2/3}, \\ y'(t) &= y^{2/3}\left[\frac{\mu x}{ky + x} - Q\right], \\ x(0) &\geq 0, \qquad y(0) > 0, \end{aligned} \tag{1.5}$$

where realistic meaning is the same as the paper in [7, 11].

The study of the system (1.5) shows that it can be global and local asymptotic stability. Specifically, we demonstrate that stability change occurs only when the coefficient varies thus correcting the previously published results.

This paper has the following structure. The basic properties of its solutions and equilibrium are given in Section 2. The stability of the system (1.5) is discussed in Section 3. Applications of the theorem, numerical simulation, and control strategy are presented in Section 4, two examples of oscillatory coexistence are also presented.

2. Equilibrium

In this section, we will study all possible equilibriums of the system (1.5).

For simplicity, the system (1.5) is rescaled with substitutions

$$x = x^0 \overline{x}, \qquad y = x^0 \delta \overline{y}, \qquad t = \frac{1}{Q}\bar{t}, \tag{2.1}$$

$\overline{x}, \overline{y}, \bar{t}$ are still replaced by x, y, t, respectively, then the new system is written as

$$\begin{aligned} x'(t) &= 1 - x - \frac{bmx}{(ay + x)\delta} y^{2/3}, \\ y'(t) &= by^{2/3}\left[\frac{mx}{ay + x} - 1\right], \\ x(0) &\geq 0, \qquad y(0) > 0, \end{aligned} \tag{2.2}$$

where $a = k\delta$, $b = (x^0\delta)^{-1/3}$, $m = \mu/Q$.

Substituting $z = 1 - x - y$, $\lambda = a/(m-1)$ into the system (2.2), it takes the form

$$\begin{aligned} z'(t) &= -z - y - by^{2/3}, \\ y'(t) &= b(m-1)y^{2/3}\frac{1 - (1+\lambda)y - z}{1 + (a-1)y - z}, \\ z(0) &\geq 0, \qquad y(0) \geq 0. \end{aligned} \tag{2.3}$$

By straightforward computing:

$$\begin{aligned} -z - y - by^{2/3} &= 0, \\ b(m-1)y^{2/3}\frac{1 - (1+\lambda)y - z}{1 + (a-1)y - z} &= 0, \end{aligned} \tag{2.4}$$

two equilibriums of the system (2.3) are obtained, $E_1(0,0)$ and $E_2(z^*, y^*)$, where

$$z^* = (1+\lambda)y^* - 1, \qquad 1 - \lambda y^* - by^{*2/3} = 0. \tag{2.5}$$

Obviously, according to the first equation of the system (2.3), if $y = 0$, then $z(t) = z_0 e^{-t}$. Consequently, if $t \to 0$, then $z(t) \to 0$. That is, the system (2.3) W limit set of all positive solutions are in the point $E_1(0,0)$.

We assume that if the liver volume does not obviously change, we call it health equilibrium, otherwise we named it disease equilibrium. Therefore, E_1 is the former and E_2 is the latter.

A necessary condition for existence of positive equilibrium in the system (2.3) is that

$$(H_1):\ R_0 = \frac{m+a-1}{m-1} > 0. \tag{2.6}$$

Lemma 2.1. *The system* (2.3) *has positive equilibrium if and only if* (H_1) *is hold.*

3. Stability

People always expect that any disease can be cured no matter what stage it is, that is to say that this differential equation is asymptotically stable.

Following Section 2, we apply Lyapunov's stability theorem to analyse the two equilibriums of the system (1.5).

Theorem 3.1. *If* $R_0 \leq 0$, *then the system* (2.3) *has only one equilibrium, and it is locally asymptotically stable.*

Proof. If $R_0 \leq 0$, then $E_2(z^*, y^*)$ does not exist, the system has only E_1. Therefore, we obtain the *Jacobic* matrix

$$J_{E_1} = J(0,0) = \begin{pmatrix} -1, & -1 \\ 0, & 0 \end{pmatrix}. \tag{3.1}$$

Because $T = -1$, $D = 0$, E_1 is a stable crunode, and it is locally asymptotically stable. This completes the proof. □

Theorem 3.2. *If* $R_0 > 0$ *and* $m < 1$ *hold, the positive equilibrium* E_2 *of the system* (2.3) *is globally asymptotically stable.*

Proof. Substituting $\xi = y - y^*$ into the system (2.3), we get

$$\frac{d\xi}{dt} = b\left(\xi + y^*\right)^{2/3}(m-1)\frac{1-(1+\lambda)\left(\xi + y^*\right) - z}{1+(a-1)\left(\xi + y^*\right) - z}. \tag{3.2}$$

Construct *Lyapunov* function:

$$V(\xi) = \xi - y^* \ln \frac{\xi + y^*}{y^*}, \tag{3.3}$$

therefore, the derived function of $V(\xi)$ about the system (2.3) take the form

$$\frac{dV}{dt} = \frac{\xi}{\xi + y^*}\frac{d\xi}{dt} = b(m-1)\xi\left(\xi + y^*\right)^{-1/3}\frac{1-(1+\lambda)\left(\xi + y^*\right) - z}{1+(a-1)\left(\xi + y^*\right) - z}. \tag{3.4}$$

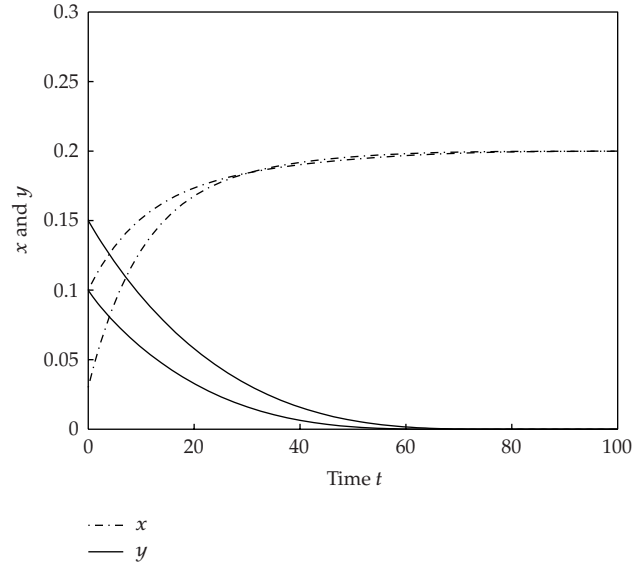

Figure 1: Solution curves for $R_0 = -0.5$ with initial value $(0.03, 0.1)$, $(0.1, 0.15)$.

According to the formal conclusion: $m < 1$, $b > 0$, $\xi(\xi + y^*)^{-1/3} > 0$, we obtain

$$\frac{dV}{dt} < 0. \tag{3.5}$$

Therefore, if $R_0 > 0$ and $m < 1$ hold, E_2 is globally asymptotically stable. This completes the proof. □

Theorem 3.3. *If $R_0 > 0$ and $m < 1$ hold, the positive equilibrium E_0 of the system* (2.2) *is globally asymptotically stable.*

4. Numerical Simulation and Control Policy

In this section, we will select proper parameter and present numerical computer simulation to obtain control policy.

4.1. When $R_0 \leq 0$

Then $(m + a - 1)(m - 1) < 0$. To illustrate the result of this subsection numerically, we fix the initialization points $(0.03, 0.1)$, $(0.1, 0.15)$ and $(0.01, 0.2)$, $(0.5, 1)$, respectively and present computer simulations of the system (1.5), which based on the experiment and clinic date. The critical parameters δ, k, μ, Q, x^0 are as follows: $\delta = 10$, $k = 0.03$, $\mu = 0.08$, $Q = 0.1$, $x^0 = 0.2$, where $R_0 = -0.5$; $\delta = 10$, $k = -0.03$, $\mu = 0.12$, $Q = 0.1$, $x^0 = 0.2$, where $R_0 = -0.5$, respectively. Figures 1 and 2 show that it is asymptotically stable.

Therefore, the system has only one equilibrium in which the volume of liver does not obviously change. This probably is in the delitescence or no disease in which the liver has changed at functionality at most, but not organic. If proper measure is taken which will control the growth of the tumor, the development of tumor will be stagnancy.

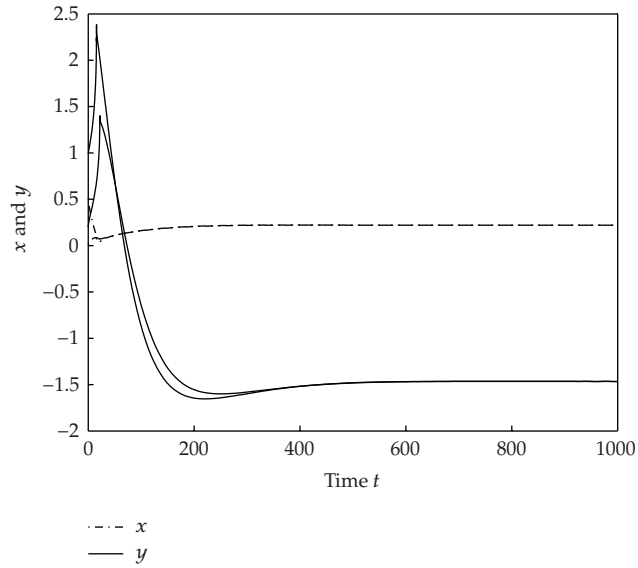

Figure 2: Solution curves for $R_0 = -0.5$ with initial value $(0.01, 0.2)$, $(0.5, 1)$.

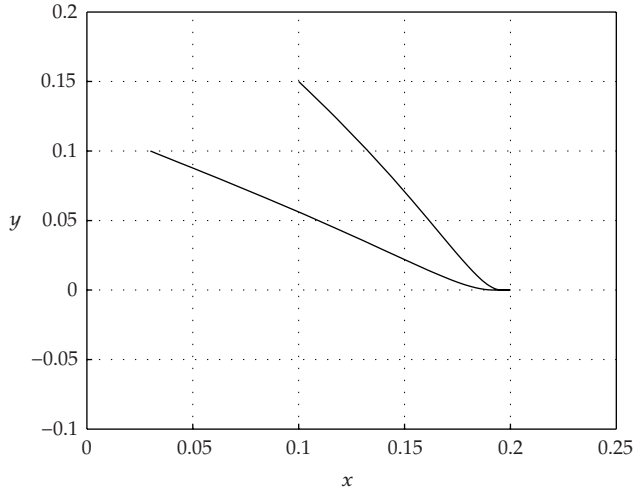

Figure 3: Solution curves for $R_0 = 0.25$ with initial value $(0.03, 0.1)$, $(0.1, 0.15)$.

We suppose that if the volume of liver does not obviously change, disease will not exacerbate, not to mention proper measure is adopted; therefore, hepatocellular carcinomas early diagnosis is very important which can control the development of the disease. Therefore we advocate early detection, early diagnosis, and early treatment.

4.2. When $R_0 > 0$

The system has one equilibrium on which the volume of liver has obviously changed, that is to say it is in serious condition.

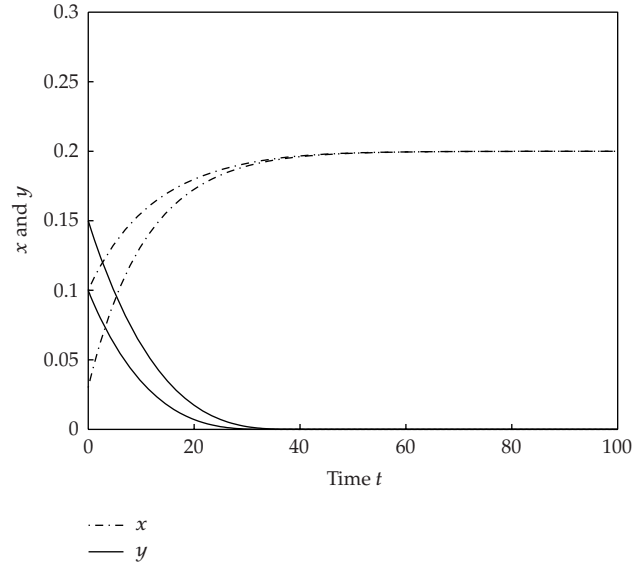

Figure 4: Solution curves for $R_0 = 0.25$ with initial value $(0.03, 0.1)$, $(0.1, 0.15)$.

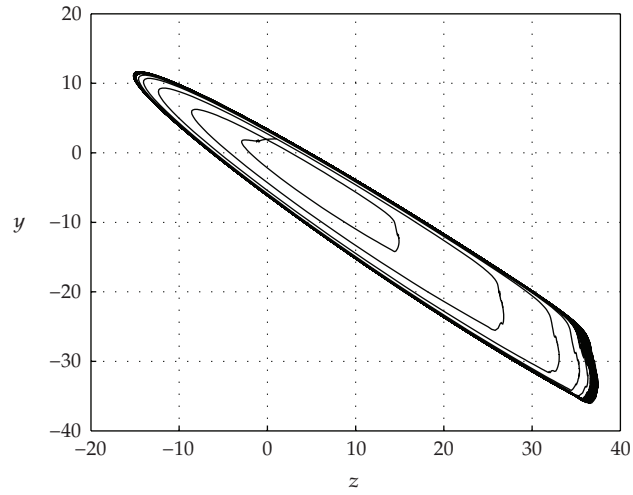

Figure 5: Solution curves for $R_0 = 0.25$ with initial value $(1, 2)$.

Therefore, $m + a - 1 > 0$, $m < 1$. Based on the experiment and clinic date, we fix the initialization points of the system (1.5): $(0.03, 0.1)$, $(0.1, 0.15)$, the critical parameters are as follows: $\delta = 10$, $k = 0.03$, $\mu = 0.06$, $Q = 0.1$, $x^0 = 0.2$, where $R_0 = 0.25$. It is shown in Figures 3 and 4.

Otherwise, we fix the initialization points of the system (2.3): $(1, 2)$, the critical parameters are as follows: $m = 1.2$, $a = -0.3$, $\lambda = -1.5$, $b = 2^{-1/3}$, where $R_0 = 0.25$. Figure 5 shows that such system may exhibit a stable periodic solution.

We suppose that when the volume of liver has obviously changed, if proper measure such as chemotherapy or radiology is adopted, disease will not exacerbate at least, then the development of the tumor is in the phase of stagnation; therefore, hepatocellular carcinomas

diagnosis is very important as it can control the development of disease. So we prefer medication treatment.

5. Concluding Remarks

In this paper we have considered a *Chemostat* system incorporating hepatocellular carcinomas. We obtained a more realistic model by incorporating hepatocellular carcinomas in system (1.5). We have proved that local asymptotic stability of the positive equilibrium implies that it is global asymptotic stability. All the results indicated that medication treatment had a stabilizing effect on hepatocellular carcinomas development. We have given a numerical simulation to verify some of the key results we have obtained.

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