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Research Article

On Generalized Periodic-Like Rings

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Let R be a ring with center Z , Jacobson radical J , and set N of all nilpotent elements. Call R generalized periodic-like if for all $x \in R \setminus (N \cup J \cup Z)$ there exist positive integers m, n of opposite parity for which $x^m - x^n \in N \cap Z$. We identify some basic properties of such rings and prove some results on commutativity.

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1. Introduction

Let R be a ring; and let $N = N(R)$, $Z = Z(R)$ and $J = J(R)$ denote, respectively, the set of nilpotent elements, the center, and the Jacobson radical. As usual, we call R periodic if for each $x \in R$, there exist distinct positive integers m, n such that $x^m = x^n$. In [1] we defined R to be generalized periodic (g-p) if for each $x \in R \setminus (N \cup Z)$,

there exist positive integers m, n of opposite parity such that $x^m - x^n \in N \cap Z$. (*)

We now define R to be generalized periodic-like (g-p-l) if (*) holds for each $x \in R \setminus (N \cup J \cup Z)$. Clearly, the class of g-p-l rings contains all commutative rings, all nil rings, all Jacobson radical rings, all g-p rings, and some (but not all) periodic rings. It is our purpose to exhibit some general properties of g-p-l rings and to study commutativity of such rings.

2. Preliminary results

To simplify our discussion, we denote by $((m, n))$ the ordered pair of integers m, n of opposite parity. The rest of our notation and terminology is standard. For elements $x, y \in R$, the symbol $[x, y]$ denotes the commutator $xy - yx$; for subsets $X, Y \subseteq R$, $[X, Y]$ denotes

the set $\{[x, y] \mid x \in X, y \in Y\}$; and $C(R)$ denotes the commutator ideal of R . An element $x \in R$ is called regular if it is not a zero divisor; it is called periodic if there exist distinct positive integers m, n for which $x^m = x^n$; and it is called potent if there exists an integer $n > 1$ for which $x^n = x$. The set of all potent elements of R is denoted by P or $P(R)$, and the prime radical by $\mathfrak{P}(R)$. Finally, R is called reduced if $N(R) = \{0\}$.

LEMMA 2.1. *Let R be an arbitrary g - p - l ring.*

- (i) *Every epimorphic image of R is a g - p - l ring.*
- (ii) $N \subseteq J$.
- (iii) *If $[N, J] = \{0\}$, then N is an ideal.*
- (iv) $C(R) \subseteq J$.
- (v) *If e is an idempotent, the additive order of which is not a power of 2, then $e \in Z$.*

Proof. (i) is clear, once we recall that if $\sigma : R \rightarrow S$ is an epimorphism, then $\sigma(J(R)) \subseteq J(S)$.

(ii) Let $S = R/J(R)$. Then by (i), S is a g - p - l ring; and since $J(S) = \{0\}$, S is a g - p ring. It follows from [1, Theorem 1] that $N(S)$ is an ideal of S , hence $N(S) \subseteq J(S) = \{0\}$ and therefore $N(R) \subseteq J(R)$.

(iii) Since $N \subseteq J$, N is commutative and hence $(N, +)$ is an additive subgroup. Let $a \in N$ and $x \in R$. Then $ax \in J$, so $[a, ax] = 0$, that is, $a^2x = axa$. It follows that $(ax)^2 = a^2x^2$ and that $(ax)^n = a^n x^n$ for all positive integers n . Therefore, $ax \in N$.

(iv) As in (ii), $R/J(R)$ is a g - p ring; hence, by [1, Lemma 2], $C(R/J(R)) = \{0\}$. Therefore, $C(R) \subseteq J(R)$.

(v) If $e \notin Z$, then $-e \notin J \cup Z$ and there exists $((m, n))$ such that $(-e)^m - (-e)^n \in N \cap Z$. Since m, n are of opposite parity, we get $2e \in N$, so that $2^k e = 0$ for some k . □

LEMMA 2.2. *Let R be an arbitrary g - p - l ring, and let $x \in R$. Then either $x \in J \cup Z$, or there exist a positive integer q and an idempotent e such that $x^q = x^q e$.*

Proof. If $x \notin J \cup Z$, there exists $((m, n))$ such that $x^m - x^n \in N \cap Z$. Therefore, there exist a positive integer q and $g(t) \in \mathbb{Z}[t]$ such that $x^q = x^{q+1}g(x)$. It is now easy to verify that $e = (xg(x))^q$ is an idempotent with $x^q = x^q e$. □

LEMMA 2.3. *Let R be a g - p - l ring and σ an epimorphism from R to S . Then $N(S) \subseteq \sigma(J(R)) \cup Z(S)$.*

Proof. Let $s \in N(S)$ with $s^k = 0$ and let $d \in R$ such that $\sigma(d) = s$. If $d \in J(R) \cup Z(R)$, then obviously $s \in \sigma(J(R)) \cup Z(S)$; hence we may suppose that there exists $((m, n))$ with $n > m$ such that $d^m - d^n \in N(R) \cap Z(R)$. It is easy to show that $d - d^h \in N$, where $h = n - m + 1$; thus

$$d - d^{k+1} d^{k(h-2)} = d - d^h + d^{h-1}(d - d^h) + \dots + (d^{h-1})^{k-1}(d - d^h) \tag{2.1}$$

is a sum of commuting nilpotent elements, hence it is in $N(R)$ and therefore in $J(R)$. Consequently, $s - s^{k+1} s^{k(h-2)} \in \sigma(J(R))$; and since $s^{k+1} = 0$, $s \in \sigma(J(R))$. □

We finish this section by stating two known results on periodic elements.

LEMMA 2.4. *Let R be an arbitrary ring, and let $N^* = \{x \in R \mid x^2 = 0\}$.*

- (i) [2, Lemma 1] *If $x \in R$ is periodic, then $x \in P + N$.*
- (ii) [3, Theorem 2] *If N^* is commutative and N is multiplicatively closed, then $PN \subseteq N$.*

3. Commutativity results

THEOREM 3.1. *If R is a g - p - l ring with $J \subseteq Z$, then R is commutative.*

Proof. Suppose $x \notin Z$. Then by Lemma 2.1(ii), we have $((m, n))$ with $n > m$ such that $x^m - x^n \in N \cap Z$. Consequently $x^{n-m+1} - x \in N$; and since $N \subseteq Z$, commutativity of R follows by a well-known theorem of Herstein [4]. □

THEOREM 3.2. *If R is any g - p - l ring with 1, then R is commutative.*

Proof. We show that if R is g - p - l with 1, then $J \subseteq Z$. Suppose that $x \in J \setminus Z$. Then $-1 + x \notin J \cup Z$, so there exists $((m, n))$ such that $(-1 + x)^m - (-1 + x)^n \in N \cap Z$; and we may assume that m is even and n is odd. Since $N \subseteq J$, it follows that $2 \in J$; thus for every integer $m, 2m \in J$, and hence $2m + 1$ is invertible.

Now consider $((m_1, n_1))$ such that $(1 + x)^{m_1} - (1 + x)^{n_1} \in N \cap Z$. Then $(m_1 - n_1)x + x^2 p(x) \in N \cap Z$ for some $p(t) \in \mathbb{Z}[t]$; and since $m_1 - n_1$ is central and invertible, we get $x + x^2 w \in N \cap Z$ for some w in R with $[x, w] = 0$. Thus, we have a positive integer q and an element y in R such that $[x, y] = 0$ and $x^q = x^{q+1}y$. It follows that $e = (xy)^q$ is an idempotent such that $x^q = x^q e$; and since J contains no nonzero idempotents, x is in N .

Let α be the smallest positive integer for which $x^k \in Z$ for all $k \geq \alpha$, and note that, since $x \notin Z$, $\alpha \geq 2$. But $1 + x^{\alpha-1} \notin J \cup Z$, so there exists $((m_2, n_2))$ such that $(1 + x^{\alpha-1})^{m_2} - (1 + x^{\alpha-1})^{n_2} \in N \cap Z$; hence $(m_2 - n_2)x^{\alpha-1} \in Z$. But since $m_2 - n_2$ is invertible and central, we conclude that $x^{\alpha-1} \in Z$ —a contradiction. □

THEOREM 3.3. *If R is a reduced g - p - l ring with $R \neq J$, then R is commutative.*

Proof. If $R = J \cup Z$, then $R = Z$ and we are finished. Otherwise, if $x \in R \setminus (J \cup Z)$, there exists $((m, n))$ such that $x^m - x^n \in N \cap Z = \{0\}$; hence x is periodic, and by Lemma 2.4(i), $x \in P$. Thus, $R = P \cup J \cup Z$; and to complete the proof we need only to show that $P \subseteq Z$.

Let $y \in P$, and let $k > 1$ be such that $y^k = y$. Then $e = y^{k-1}$ is an idempotent for which $y = ye$, and $e \in Z$ since $N = \{0\}$. Now eR is an ideal of R , so that $J(eR) = eR \cap J(R)$; hence eR is a g - p - l ring with 1, which is commutative by Theorem 3.2. Therefore, $[ey, ew] = 0$ for all $w \in R$; and since $ey = y$ and $e \in Z$, we conclude that $[y, w] = 0$ for all $w \in R$, that is, $y \in Z$. □

THEOREM 3.4. *If R is a g - p - l ring in which J is commutative and all idempotents are central, then R is commutative.*

Proof. We may express R as a subdirect product of subdirectly irreducible rings, each of which is an epimorphic image of R . Let R_α be such a subdirectly irreducible ring, and let $\sigma : R \rightarrow R_\alpha$ be an epimorphism. Let $x_\alpha \in R_\alpha$ and let $x \in R$ such that $\sigma(x) = x_\alpha$. By Lemma 2.2, $x \in J(R) \cup Z(R)$ or there exist an idempotent $e \in R$ and a positive integer q such that $x^q = x^q e$. Thus, either $x_\alpha \in \sigma(J(R)) \cup Z(R_\alpha)$ or $x_\alpha^q = x_\alpha^q e_\alpha$, where $e_\alpha = \sigma(e)$ is a central idempotent of R_α . But R_α is subdirectly irreducible, hence if R_α has a nonzero central idempotent, then R_α has 1 and is commutative by Theorem 3.2.

To complete the proof, we need only consider the case that for each $x_\alpha \in R_\alpha$, $x_\alpha \in \sigma(J(R)) \cup Z(R_\alpha) \cup N(R_\alpha)$. Now by Lemma 2.3, $N(R_\alpha) \subseteq \sigma(J(R)) \cup Z(R_\alpha)$; hence $R_\alpha = \sigma(J(R)) \cup Z(R_\alpha)$, which is clearly commutative. Therefore, R is commutative. \square

Theorem 3.4 has two corollaries, the first of which is immediate when we recall Lemma 2.1(v).

COROLLARY 3.5. *If R is a 2-torsion-free g - p - l ring with J commutative, then R is commutative.*

COROLLARY 3.6. *Let R be a g - p - l ring containing a regular central element c . If J is commutative, then R is commutative.*

Proof. It suffices to show that $N \subseteq Z$ since this condition implies that idempotents are central. Consider first the case $c \in J$. Then $cJ \subseteq J^2$, which is central since J is commutative. Since c is regular and central, it is immediate that $J \subseteq Z$, so certainly $N \subseteq Z$.

Now assume that $c \notin J$, and suppose that $a \in N \setminus Z$. Then $c + a \notin J \cup Z$, and there exists $((m, n))$ such that $(c + a)^m - (c + a)^n \in N \cap Z$. It follows that $c^m - c^n$ is a sum of commuting nilpotent elements, hence $c^m - c^n \in N$ and there exists q such that $c^q = c^{q+1}p(c)$ for some $p(t) \in \mathbb{Z}[t]$. As before, we get an idempotent e such that $c^q = c^q e$ and $[c, e] = 0$. Now e cannot be a zero divisor, since that would force c to be a zero divisor; therefore, R has a regular idempotent, that is, R has 1. We have contradicted Theorem 3.2, so $N \subseteq Z$ as claimed. \square

4. Nil-commutator-ideal theorems

THEOREM 4.1. *Let R be a g - p - l ring. If $R \neq J$ and N is an ideal, then $C(R)$ is nil.*

Proof. We may assume $R \neq J \cup Z$, since otherwise R is commutative. Let $\bar{R} = R/N$, and let the element $x + N$ of \bar{R} be denoted by \bar{x} . We need to show that \bar{R} is commutative—a conclusion that follows from Theorem 3.3 once we show that $J(\bar{R}) \neq \bar{R}$.

Suppose that $J(\bar{R}) = \bar{R}$, and let $x \in R \setminus (J \cup Z)$. By Lemma 2.2, there exists a positive integer q and an idempotent $e \in R$ such that $x^q = x^q e$; and it follows that \bar{e} is an idempotent of \bar{R} such that $\bar{x}^q = \bar{x}^q \bar{e}$. But $\bar{R} = J(\bar{R})$ contains no nonzero idempotents, so that $\bar{x}^q = 0 = \bar{x}$ and hence $x \in N(R)$. This contradicts the fact that $x \notin J \cup Z$, hence $\bar{R} \neq J(\bar{R})$ as required. \square

THEOREM 4.2. *If R is a g - p - l ring and J is commutative, then $C(R)$ is nil.*

Proof. If $R = J$, then R is commutative. If $R \neq J$, N is an ideal by Lemma 2.1(iii) and $C(R)$ is nil by Theorem 4.1. \square

In fact, we can improve this result as follows.

THEOREM 4.3. *Let R be a g - p - l ring with $R \neq J$. If N is commutative, then $C(R)$ is nil.*

This result follows from Theorem 4.1, once we prove our final theorem.

THEOREM 4.4. *Let R be a g - p - l ring with $R \neq J$. If N is commutative, then N is an ideal.*

Proof. Again we may assume that $R \neq J \cup Z$. Since N is commutative, N is an additive subgroup of R . To show that $RN \subseteq N$, it is convenient to work with the ring $\bar{R} = R/\mathfrak{P}(R)$. As in the proof of Theorem 4.1, we have $J(\bar{R}) \neq \bar{R}$; and if $\bar{R} = Z(\bar{R})$, then $C(R) \subseteq \mathfrak{P}(R) \subseteq N$. Therefore, we assume that $\bar{R} \neq J(\bar{R}) \cup Z(\bar{R})$. We note that if $x + N = \bar{x} \in N(\bar{R})$, then $x \in N(R)$; consequently, $N(\bar{R})$ is commutative and hence is an additive subgroup of \bar{R} .

Now \bar{R} is semiprime and therefore $N(\bar{R}) \cap Z(\bar{R}) = \{0\}$. It follows that if $\bar{x} \in \bar{R} \setminus (J(\bar{R}) \cup Z(\bar{R}))$, there exists $((m, n))$ such that $\bar{x}^m = \bar{x}^n$, that is, \bar{x} is periodic. Thus $\bar{x} \in P(\bar{R}) + N(\bar{R})$ by Lemma 2.4(i); and by commutativity of $N(\bar{R})$ and Lemma 2.4(ii) we get $\bar{x}N(\bar{R}) \subseteq N(\bar{R})$. Moreover, if $\bar{y} \in Z(\bar{R})$, $\bar{y}N(\bar{R}) \subseteq N(\bar{R})$. Now let $\bar{y} \in J(\bar{R}) \setminus Z(\bar{R})$, and let $\bar{x} \in \bar{R} \setminus (J(\bar{R}) \cup Z(\bar{R}))$. Then $\bar{x} + \bar{y} \notin J(\bar{R})$, hence it is in $\bar{R} \setminus (J(\bar{R}) \cup Z(\bar{R}))$ or in $Z(\bar{R})$; and in either case $(\bar{x} + \bar{y})N(\bar{R})$ and $\bar{x}N(\bar{R})$ are in $N(\bar{R})$, so that $\bar{y}N(\bar{R}) \subseteq N(\bar{R})$. We have shown that $N(\bar{R})$ is an ideal of \bar{R} ; therefore, if $x \in R$ and $a \in N(R)$, $\bar{x}\bar{a} \in N(\bar{R})$ and hence $xa \in N(R)$. Thus, $N(R)$ is an ideal of R . □

Remark 4.5. There exist noncommutative g - p - l rings with J commutative. An accessible example is

$$\left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in GF(2) \right\}. \tag{4.1}$$

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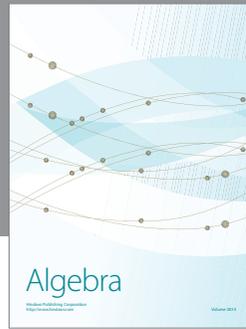
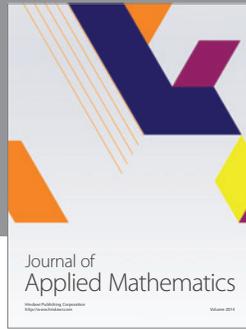
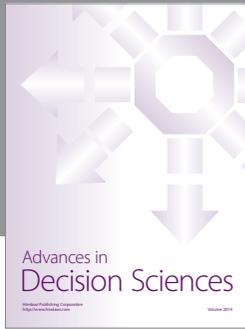
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