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Research Article

Convergence Analysis of a Fully Discrete Family of Iterated Deconvolution Methods for Turbulence Modeling with Time Relaxation

R. Ingram,[1] C. C. Manica,[2] N. Mays,[3] and I. Stanculescu[4]

[1] *Department of Mathematics, University of Pittsburgh, PA 15260, USA*
[2] *Departmento de Matemática Pura e Aplicada, Universidade Federal do Rio Grande do Sul, Porto Alegre 91509-900, RS, Brazil*
[3] *Department of Mathematics, Wheeling Jesuit University, WV 26003, USA*
[4] *Farquhar College of Arts and Sciences, Nova Southeastern University, FL 33314, USA*

Correspondence should be addressed to N. Mays, nmays@wju.edu

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We present a general theory for regularization models of the Navier-Stokes equations based on the Leray deconvolution model with a general deconvolution operator designed to fit a few important key properties. We provide examples of this type of operator, such as the (modified) Tikhonov-Lavrentiev and (modified) Iterated Tikhonov-Lavrentiev operators, and study their mathematical properties. An existence theory is derived for the family of models and a rigorous convergence theory is derived for the resulting algorithms. Our theoretical results are supported by numerical testing with the Taylor-Green vortex problem, presented for the special operator cases mentioned above.

1. Approximate Deconvolution for Turbulence Modeling

Numerical simulations of complex flows present many challenges. The Navier-Stokes (NS) equations (NSE), given by the following

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}, \qquad \nabla \cdot \mathbf{u} = 0, \text{ in } \Omega \times (0, T) \tag{1.1}$$

is an exact model for the flow of a viscous, incompressible fluid, [1]. For turbulent flows (characterized by Reynold's number $\mathrm{Re} \gg 1$), it is infeasible to properly resolve all significant scales above the Kolmogorov length scale $O(\mathrm{Re}^{-3/4})$ by direct numerical simulation. Thus, numerical simulations are often based on various regularizations of NSE, rather than NSE themselves. Accordingly, regularization methods provide a (computationally) *efficient* and (algorithmically) *simple* family of turbulence models. Several of the most commonly applied regularization methods include:

$$\text{(Leray)} \qquad \mathbf{w}_t + \overline{\mathbf{w}} \cdot \nabla \mathbf{w} - \mathrm{Re}^{-1} \Delta \mathbf{w} + \nabla p = \mathbf{f}, \qquad \nabla \cdot \mathbf{w} = 0, \tag{1.2}$$

$$\text{(NS-}\alpha\text{)} \qquad \mathbf{w}_t + \overline{\mathbf{w}} \times (\nabla \times \mathbf{w}) - \mathrm{Re}^{-1} \Delta \mathbf{w} + \nabla P = \mathbf{f}, \tag{1.3}$$

$$\text{(NS-}\overline{\omega}\text{)} \qquad \mathbf{w}_t + \mathbf{w} \times (\nabla \times \overline{\mathbf{w}}) - \mathrm{Re}^{-1} \Delta \mathbf{w} + \nabla P = \mathbf{f}, \tag{1.4}$$

$$\text{(time relaxation)} \qquad \mathbf{w}_t + \mathbf{w} \cdot \nabla \mathbf{w} - \mathrm{Re}^{-1} \Delta \mathbf{w} + \nabla p + \chi(\mathbf{w} - \overline{\mathbf{w}}) = \mathbf{f}, \tag{1.5}$$

where $\nabla \cdot \mathbf{w} = 0$ in each case and $\overline{\mathbf{w}}$ is an averaged velocity field $\mathbf{w}$, p is pressure, and P is the Bernoulli pressure. More details about these models can be found for instance in [2–5] and references therein. Although these regularization methods achieve high theoretical accuracy and perform well in select practical tests, those models do not provide a fully-developed numerical solution for decoupling the scales in a turbulent flow. In fact, results show that only time relaxation regularization truncates scales sufficiently for practical computations. Indeed, it is shown that time relaxation term $\chi(\mathbf{w} - \overline{\mathbf{w}})$ for $\chi > 0$ damps unresolved fluctuations over time [5, 6]. Note that the choice of χ is an active area of research and that solutions are very sensitive to variations in χ.

Deconvolution-based regularization is also an active area of research obtained, for example, by replacing $\overline{\mathbf{w}}$ by $D(\overline{\mathbf{w}})$ in each (1.2)–(1.5) for some deconvolution operator D. In [7], Dunca proposed the general Leray-deconvolution problem ($D(\overline{\mathbf{w}})$ instead of $\overline{\mathbf{w}}$) as a more accurate extension to Leray's model [8]. Leray used the Gaussian filter as the smoothing (averaging) filter G, denoted above by *overbar*. In [9], Germano proposed the differential filter (approximate-Gaussian) $G = (-\delta^2 \Delta + I)^{-1}$ where $\delta > 0$ is the filter length. The differential filter is easily modeled in the variational framework of the finite element (FE method (FEM)). We provide a brief overview of continuous and discrete operators (Section 3).

The deconvolution-based models have proven themselves to be very promising. However, among the very many known approximate deconvolution operators from image processing, for example, [10], so far only few have been studied for turbulence modeling, for example, the van Cittert and the modified Tikhonov-Lavrentiev deconvolution operators. Their success suggests that it is time to develop a general theory for regularization models of the NSE as a guide to development of models based on other, possibly better, deconvolution operators and refinement of existing ones.

Herein, we present a general theory for regularization models of the NSE based on the Leray deconvolution model with a general deconvolution operator. We prove energetic stability (and hence existence) and convergence of an FE (in space) and Crank-Nicolson (CN) (in time) discretization of the following family of Leray deconvolution regularization models with time relaxation: find $\mathbf{w} : \Omega \times (0, T] \to \mathbb{R}^3$ and $\pi : \Omega \times (0, T] \to \mathbb{R}$ satisfying the following

$$\mathbf{w}_t + D(\overline{\mathbf{w}}) \cdot \nabla \mathbf{w} - \mathrm{Re}^{-1} \Delta \mathbf{w} + \nabla \pi + \chi(\mathbf{w} - D(\overline{\mathbf{w}})) = \mathbf{f}, \qquad \nabla \cdot \mathbf{w} = 0, \tag{1.6}$$

for some appropriate boundary and initial conditions (the fully discrete model is presented in Problem 1 with energetic stability and well posedness proved in Theorem 4.5).

1.1. Improving Accuracy of Approximate Deconvolution Methods

The fundamental difficulties corresponding to regularization methods applied as a viable turbulence model include ensuring that:

(i) scales are appropriately truncated (model microscale = filter radius = mesh width)

(ii) smooth parts of the solution are accurately approximated ($D(\overline{\mathbf{w}}) \approx \mathbf{w}$ for smooth $\mathbf{w}$)

(iii) physical fidelity of flow is preserved.

Due to the nonlinearity in (1.6), different choices of the filter and deconvolution operator yield significant changes in the solution of the corresponding model. Implementation concerns for deconvolution methods, for example, Tikhonov-Lavrentiev regularization given by $D = (G + \alpha I)^{-1}$, include selection of deconvolution parameter $\alpha > 0$. Iterated deconvolution methods reduce approximation sensitivity relative to α-selection and, hence, allow a conservatively large α-selection for stability with updates (fixed number of iterations) used to recover higher accuracy (Section 3.3). For example we prove, under usual conditions, (Proposition 5.3),

$$\begin{aligned} &\text{Modified Tikhonov } (j = 0) \qquad \text{error}(\mathbf{w} - D(\overline{\mathbf{w}})) \le \mathcal{O}\left(\alpha\delta^2\right), \\ &\text{Iterated Modified Tikhonov } (j > 0) \qquad \text{error}(\mathbf{w} - D(\overline{\mathbf{w}})) \le \mathcal{O}\left(\left(\alpha\delta^2\right)^{j+1}\right), \end{aligned} \tag{1.7}$$

so that iterated modified Tikhonov regularization gives geometric convergence with respect to the update number j. In either case, Tikhonov-Lavrentiev or iterated Tikhonov-Lavrentiev regularization, we prove, under usual conditions, (Proposition 5.4)

$$\text{error}\left(D^h\left(\overline{\mathbf{w}}^h\right) - D(\overline{\mathbf{w}})\right) \le \mathcal{O}\left(h^k\right), \tag{1.8}$$

where D^h and $\overline{\mathbf{w}}^h$ represent the discrete deconvolution operator and discrete filter, respectively, and k is order of FE polynomial space. We propose minimal properties for a general family of deconvolution operators D and filters G (Section 3.1) satisfying (Assumptions 3.4, 3.5), for example,

(i) $\|DG\| \le 1$, forces spectrum of DG in $[0, 1]$

(ii) $\|\nabla DG\mathbf{v}\| \le d_1\|\nabla \mathbf{v}\|$, controls size of ∇DG

(iii) $\|(I - DG)\mathbf{v}\| \le c_1(\alpha, \delta)\|\Delta \mathbf{v}\| \to 0$ as $\alpha, \delta \to 0$ (for smooth $\mathbf{v}$), ensures convergence of method

(iv) $\|(DG - D^hG^h)\mathbf{v}\| \le c_2(h, \alpha, \delta) \to 0$ as $h \to 0$ (for smooth $\mathbf{v}$), ensures convergence of method.

In fact, the updates $\{\boldsymbol{\omega}_j\}_{j\geq 0}$ satisfying

$$\boldsymbol{\omega}_0 := D(\overline{\mathbf{w}}), \quad \boldsymbol{\omega}_j - \boldsymbol{\omega}_{j-1} := D\overline{\left(\mathbf{w} - \boldsymbol{\omega}_{j-1}\right)}, \tag{1.9}$$

inherit the properties assumed for the base operator D in Assumptions 3.4, 3.5 (Propositions 3.10, 3.11). These iterates represent defect correction generalization of iterated Tikhonov regularization operator [11]. We prove that the FE-CN approximation $\mathbf{w}^h$ of the general deconvolution turbulence model (Problem 1) satisfies (Theorem 4.6)

$$\operatorname{error}\left(\mathbf{w}^h - \mathbf{u}_{\mathrm{NSE}}\right) \leq C\left(h^k + \Delta t^2 + c_1(\alpha,\delta,j) + c_2(h,\alpha,\delta,j)\right) \longrightarrow 0, \quad \text{as } h,\delta,\alpha \longrightarrow 0 \tag{1.10}$$

for smooth enough solutions $\mathbf{u}_{\mathrm{NSE}}$ of the NSE (see variational formulation (2.3)–(2.5)). We show that $c_1 = C(\alpha\delta^2)^{\beta}$ (for any $0 \leq \beta \leq j+1$ and $\Delta^{\beta}\mathbf{u}_{\mathrm{NSE}} \in L^2(\Omega)$) (Proposition 5.3) and $c_2 = Ch^k$ (for $D(\overline{\mathbf{w}}) \in H^{k+1}(\Omega)$) (Proposition 5.4) for modified iterated Tikhonov-Lavrentiev regularization (see Corollary 5.5 for corresponding error estimate). We conclude with a numerical test that verifies the theoretical convergence rate predicted in Theorem 4.6 (Section 5.2).

1.2. Background and Overview

One of the most interesting approaches to generate turbulence models is via approximate deconvolution or approximate/asymptotic inverse of the filtering operator. Examples of such models include: Approximate Deconvolution Models (ADM) and Leray-Tikhonov Deconvolution Models. Layton and Rebholz compiled a comprehensive overview and detailed analysis of ADM [12] (see also references therein). Previous analysis of the ADM with and without the time-relaxation term used van Cittert deconvolution operators [5, 6]; although easily programmed, van Cittert schemes can be computationally expensive [5]. Tikhonov-Lavrentiev regularization is another popular regularization scheme [13]. Determining the appropriate value of α to ensure stability while preserving accuracy is challenging, see for example, [14–19]. Alternatively, *iterated* Tikhonov regularization is well known to decouple stability and accuracy from the selection of regularization parameter α, see for example, [11, 20–22]. Iterated Tikhonov regularization is one special case of the general deconvolution operator we propose herein.

2. Function Spaces and Approximations

Let the flow domain $\Omega \in \mathbb{R}^d$ for $d = 2,3$ be a regular and bounded polyhedral. We use standard notation for Lebesgue and Sobolev spaces and their norms. Let $\|\cdot\|$ and $(\cdot,\cdot)$ be the L^2-norm and inner product, respectively. Let $\|\cdot\|_{p,k} := \|\cdot\|_{W_p^k(\Omega)}$ represent the $W_p^k(\Omega)$-norm. We write $H^k(\Omega) := W_2^k(\Omega)$ and $\|\cdot\|_k$ for the corresponding norm. Let the context determine whether $W_p^k(\Omega)$ denotes a scalar, vector, or tensor function space. For example let $\mathbf{v} : \Omega \rightarrow \mathbb{R}^d$. Then, $\mathbf{v} \in H^1(\Omega)$ implies that $\mathbf{v} \in H^1(\Omega)^d$ and $\nabla\mathbf{v} \in H^1(\Omega)$ implies that

$\nabla \mathbf{v} \in H^1(\Omega)^{d\times d}$. Write $W_q^m(W_p^k(\Omega)) := W_q^m(0,T;W_p^k(\Omega))$ equipped with the standard norm. For example,

$$\|\mathbf{v}\|_{L^q(W_p^k)} := \begin{cases} \left(\int_0^T \|\mathbf{v}(\cdot,t)\|_{p,k}^q dt\right)^{1/q}, & \text{if } 1 \le q < \infty, \\ \operatorname{ess}\sup\limits_{0<t<T} \|\mathbf{v}(\cdot,t)\|_{p,k}, & \text{if } q = \infty. \end{cases} \tag{2.1}$$

Denote the pressure and velocity spaces by $Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$ and $X := H_0^1(\Omega) = \{\mathbf{v} \in H^1(\Omega) : \mathbf{v}|_{\partial\Omega} = 0\}$, respectively. Moreover, the dual space of X is denoted $X' := W_2^{-1}(\Omega)$ and equipped with the norm

$$\|\mathbf{f}\|_{-1} := \sup_{0\neq \mathbf{v}\in X} \frac{\langle \mathbf{f}, \mathbf{v}\rangle_{X'\times X}}{|\mathbf{v}|_1}. \tag{2.2}$$

Fix $\mathbf{f} \in X'$ and $\mathrm{Re} > 0$. In this setting, we consider strong NS solutions: find $\mathbf{u} \in L^2(X) \cap L^\infty(L^2(\Omega))$ and $p \in W^{-1,\infty}(Q)$ satisfying

$$(\mathbf{u}_t, \mathbf{v}) + (\mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{v}) + \mathrm{Re}^{-1}(\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \text{a.e. } t \in (0,T],\ \forall \mathbf{v} \in X, \tag{2.3}$$

$$(q, \nabla\cdot\mathbf{u}) = 0, \quad \text{a.e. } t \in (0,T],\ \forall q \in Q, \tag{2.4}$$

$$\mathbf{u}(\cdot,0) = \mathbf{u}_0 \quad \text{in } L^2(\Omega). \tag{2.5}$$

Let $V := \{\mathbf{v} \in X : \nabla\cdot\mathbf{v} = 0\}$. Restricting test functions $\mathbf{v} \in V$ reduces (2.3)–(2.5) to find $\mathbf{u} : (0,T] \to V$ satisfying

$$(\mathbf{u}_t, \mathbf{v}) + (\mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{v}) + \mathrm{Re}^{-1}(\nabla\mathbf{u}, \nabla\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \text{a.e. } t \in (0,T],\ \forall \mathbf{v} \in V \tag{2.6}$$

and (2.5). For smooth enough solutions, solving the problem associated with (2.6), (2.5) is equivalent to (2.3)–(2.5).

Control of the nonlinear term is essential for establishing a priori estimates and convergence estimates. We state a selection of inequalities here that will be utilized later:

$$|\mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{w}| \le C(\Omega) \begin{cases} \|\mathbf{u}\|_1 |\mathbf{v}|_1 \|\mathbf{w}\|_1 & \\ \|\mathbf{u}\| \|\mathbf{v}\|_2 \|\mathbf{w}\|_1 & \forall \mathbf{v} \in H^2(\Omega) \\ \|\mathbf{u}\|_2 \|\mathbf{v}\| \|\mathbf{w}\|_1 & \forall \mathbf{u} \in H^2(\Omega) \cap V. \end{cases} \tag{2.7}$$

2.1. Discrete Function Setting

Fix $h > 0$. Let $\mathcal{T}^h$ be a family of subdivisions (e.g., triangulation) of $\overline{\Omega} \subset \mathbb{R}^d$ satisfying $\overline{\Omega} = \bigcup_{E\in\mathcal{T}^h} E$ so that $diameter(E) \le h$ and any two closed elements $E_1, E_2 \in \mathcal{T}^h$ are either disjoint or share exactly one face, side, or vertex. See Chapter II, Appendix A in [23] for more on this subject in context of Stokes problem and [24] for a more general treatment. For example, $\mathcal{T}^h$ consists of triangles for $d = 2$ or tetrahedra for $d = 3$ that are nondegenerate as $h \to 0$.

Let $X^h \subset X$ and $Q^h \subset Q$ be a conforming velocity-pressure mixed FE space. For example, let X^h and Q^h be continuous, piecewise (on each $E \in \mathcal{T}^h$) polynomial spaces. The discretely divergence-free space is given by

$$V^h = \left\{ \mathbf{v}^h \in X^h : \left(q^h, \nabla \cdot \mathbf{v}^h \right) = 0 \ \forall q^h \in Q^h \right\}. \tag{2.8}$$

Note that in general $V^h \not\subset V$ (e.g., for Taylor-Hood elements). In order to avoid stability issues arising when FE solutions are not exactly divergence free (i.e., when $V^h \not\subset V$), we introduce the explicitly skew-symmetric convective term

$$b^h(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}((\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - (\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v})), \tag{2.9}$$

so that

$$b^h(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0. \tag{2.10}$$

Note that $b^h(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w})$ when $\mathbf{u} \in V$. Moreover, the trilinear from $b^h(\cdot, \cdot, \cdot)$ is continuous and skew-symmetric on $X \times X \times X$.

Lemma 2.1. *If* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X$,

$$\left| b^h(\mathbf{u}, \mathbf{v}, \mathbf{w}) \right| \leq C(\Omega) \begin{cases} (\|\mathbf{u}\| \|\mathbf{u}\|_1)^{1/2} \|\mathbf{v}\|_1 \|\mathbf{w}\|_1 & \\ \|\mathbf{u}\| \|\mathbf{v}\|_2 \|\mathbf{w}\|_1 & \forall \mathbf{v} \in H^2(\Omega). \end{cases} \tag{2.11}$$

Proof. The proof of the first inequality can be found in [25]. The second follows from Hölder's and Poincaré's inequalities. □

For the time discretization, let $0 = t_0 < t_1 < \cdots < t_{M-1} = T < \infty$ be a discretization of the time interval $[0, T]$ for a constant time step $\Delta t = t_{n+1} - t_n$. Write $t_{n+1/2} := (t_{n+1} + t_n)/2$, $z_n = z(t_n)$ and, if $z \in C^0([t_n, t_{n+1}])$, $z_{n+1/2} = (z_{n+1} + z_n)/2$. Define

$$\|\mathbf{u}\|_{l^q(m_1, m_2; W_p^k(\Omega))} := \begin{cases} \left(\Delta t \sum_{n=m_1}^{m_2} \|\mathbf{u}_n\|_{k,p}^q \right)^{1/q}, & q \in [1, \infty), \\ \max_{m_1 \leq n \leq m_2} \|\mathbf{u}_n\|_{k,p}, & q = \infty, \end{cases} \tag{2.12}$$

for any $0 \leq n = m_1,\ m_1 + 1, \ldots, m_2 \leq M$. Write $\|\mathbf{u}\|_{l^q(W_p^k(\Omega))} = \|\mathbf{u}\|_{l^q(0, M; W_p^k(\Omega))}$. We say that $\mathbf{u} \in l^q(m_1, m_2; W_p^k(\Omega))$ if the associated norm defined above stays finite as $\Delta t \to 0$.

The discrete Gronwall inequality is essential to the convergence analysis in Section 4.2.

Lemma 2.2. *Let* $D \geq 0$ *and* κ_n, A_n, B_n, $C_n \geq 0$ *for any integer* $n \geq 0$ *and satisfy*

$$A_M + \Delta t \sum_{n=0}^{M} B_n \leq \Delta t \sum_{n=0}^{M} \kappa_n A_n + \Delta t \sum_{n=0}^{M} C_n + D, \quad \forall M \geq 0. \tag{2.13}$$

Suppose that for all n, $\Delta t\kappa_n < 1$ and set $g_n = (1 - \Delta t\kappa_n)^{-1}$. Then,

$$A_M + \Delta t\sum_{n=0}^{M} B_n \leq \exp\left(\Delta t\sum_{n=0}^{M} g_n\kappa_n\right)\left[\Delta t\sum_{n=0}^{M} C_n + D\right], \quad \forall M \geq 0. \tag{2.14}$$

Proof. The proof follows from [26]. □

2.2. Approximation Theory

Let $C > 0$ be a generic constant independent of $h \to 0^+$. Preserving an abstract framework for the FE spaces, we assume that $X^h \times Q^h$ inherit several fundamental approximation properties.

Assumption 2.3. The FE spaces $X^h \times Q^h$ satisfy:

Uniform inf-sup (LBB) condition

$$\inf_{q^h\in Q^h}\sup_{\mathbf{v}^h\in X^h}\frac{(q^h, \nabla\cdot\mathbf{v}^h)}{|\mathbf{v}^h|_1\|q\|} \geq C > 0. \tag{2.15}$$

FE-approximation

$$\inf_{\mathbf{v}^h\in X^h}\left|\mathbf{u} - \mathbf{v}^h\right|_1 \leq Ch^k\|\mathbf{u}\|_{k+1}, \quad \inf_{q^h\in Q^h}\left\|p - q^h\right\| \leq Ch^{s+1}\|p\|_{s+1} \quad \text{for } k \geq 0,\ s \geq -1$$
$$\text{when } \mathbf{u} \in H^{k+1}(\Omega)\cap X,\ p \in H^{s+1}(\Omega)\cap Q. \tag{2.16}$$

Inverse-estimate

$$\left|\mathbf{v}^h\right|_1 \leq Ch^{-1}\left\|\mathbf{v}^h\right\|, \quad \forall \mathbf{v}^h \in X^h. \tag{2.17}$$

The well-known Taylor-Hood mixed FE is one such example satisfying Assumption 2.3.

Estimates in (2.18)–(2.20) stated below are used in proving error estimates for time-dependent problems: for any $n = 0, 1, \ldots, M-1$,

$$\left\|\frac{\boldsymbol{\theta}_{n+1} - \boldsymbol{\theta}_n}{\Delta t}\right\|_k^2 \leq C\Delta t^{-1}\int_{t_n}^{t_{n+1}}\|\boldsymbol{\theta}_t(t)\|_k^2 dt, \tag{2.18}$$

$$\|\boldsymbol{\theta}_{n+1/2} - \boldsymbol{\theta}(t_{n+1/2})\|_k^2 \leq C\Delta t^3\int_{t_n}^{t_{n+1}}\|\boldsymbol{\theta}_{tt}(t)\|_k^2 dt, \tag{2.19}$$

$$\left\|\frac{1}{\Delta t}(\boldsymbol{\theta}_{n+1} - \boldsymbol{\theta}_n) - (\boldsymbol{\theta}_t)_{n+1/2}\right\|_k^2 \leq C\Delta t^3\int_{t_n}^{t_{n+1}}\|\boldsymbol{\theta}_{ttt}(t)\|_k^2 dt, \tag{2.20}$$

where $\boldsymbol{\theta} \in H^1(H^k(\Omega))$, $\boldsymbol{\theta} \in H^2(H^k(\Omega))$, and $\boldsymbol{\theta} \in H^3(H^k(\Omega))$ is required, respectively, for some $k \geq -1$. Each estimate (2.18)–(2.20) is a result of a Taylor expansion with integral remainder.

These higher-order spatial ($k \geq 2$ or $s \geq 1$) and temporal estimates (2.18)–(2.20) require that the nonlocal compatibility condition addressed by Heywood and Rannacher in [26, 27] (and more recently, for example, by He in [28, 29] and He and Li in [30, 31]) is satisfied. Suppose, for example, that p_0 is the solution of the (well-posed) Neumann problem

$$\begin{aligned} \Delta p_0 &= \nabla \cdot (\mathbf{f}_0 - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0), \quad \text{in } \Omega, \\ \nabla p_0 \cdot \widehat{\mathbf{n}}\big|_{\partial\Omega} &= (\Delta \mathbf{u}_0 + \mathbf{f}_0 - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0) \cdot \widehat{\mathbf{n}}|_{\partial\Omega}. \end{aligned} \tag{2.21}$$

In order to avoid the accompanying factor $\min\{t^{-1}, 1\}$ in the error estimates contained herein, the following compatibility condition is necessarily required (e.g., see [27, Corollary 2.1]):

$$\nabla p_0\big|_{\partial\Omega} = (\Delta \mathbf{u}_0 + \mathbf{f}_0 - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0)|_{\partial\Omega}. \tag{2.22}$$

Replacing (2.21) with (2.21)(a), (2.22) defines an overdetermined Neumann-type problem. Condition (2.22) is a nonlocal condition relating $\mathbf{u}_0$ and $\mathbf{f}_0$. Condition (2.22) is satisfied for several practical applications including start up from rest with zero force, $\mathbf{u}_0 = 0$, $\mathbf{f}_0 = 0$. In general, however, condition (2.22) cannot be verified. In this case, it is shown that, for example, $|\mathbf{u}_\mathbf{t}(\cdot, t)|_1, \|\mathbf{u}(\cdot, t)\|_3 \rightarrow \infty$ as $t \rightarrow 0^+$.

We finish with an approximation property of the L^2-projection. Indeed, Assumption 2.4 holds for smooth enough Ω.

Assumption 2.4. Fix $\mathbf{w} \in V$ and let $\mathbf{w}^h \in V^h$ be the unique solution satisfying $(\mathbf{w} - \mathbf{w}^h, \mathbf{v}^h)$ for all $\mathbf{v}^h \in V^h$. Then

$$\left\|\mathbf{w} - \mathbf{w}^h\right\|_{-m} \leq C h^{m+1} \inf_{\mathbf{v}^h \in X^h} \left|\mathbf{w} - \mathbf{v}^h\right|_1 \tag{2.23}$$

for $m = -1, 0, 1$.

Note that the infimum in (2.23) is over all X^h (see intermediate estimate (1.16) of Theorem II.1.1 in [23] for the corresponding estimate relating the spaces V^h and X^h).

3. Filters and Deconvolution

We prescribe the essential properties our filter G and deconvolution operator D in this section.

Definition 3.1. Let Y be a Hilbert space and $T : Y \rightarrow Y$. Write $T \geq 0$ if T is self-adjoint $T = T'$ and $(T\mathbf{v}, \mathbf{v})_Y \geq 0$ for all $\mathbf{v} \in Y$ and call T symmetric nonnegative (snn). Write $T > 0$ if T is self-adjoint $T = T'$ and $(T\mathbf{v}, \mathbf{v})_Y > 0$ for all $0 \neq \mathbf{v} \in Y$ and call T symmetric positive definite (spd).

Let $G = G(\delta) > 0$ be a linear, bounded, compact operator on X representing a generic *smoothing filter* with filter radius $\delta > 0$:

$$G : L^2(\Omega) \longrightarrow L^2(\Omega), \qquad \overline{\phi} := G\phi. \tag{3.1}$$

One example of this operator is the continuous differential filter $G = A^{-1} = (-\delta^2\Delta + I)^{-1}$ (Definition 3.2), which is used, together with its discrete counterpart $(A^h)^{-1}$ (Definition 3.3), for implementation of our numerical scheme (Section 5.2).

Definition 3.2 (continuous differential filter). Fix $\phi \in L^2(\Omega)$. Then $\overline{\phi} \in X$ is the unique solution of $-\delta^2\Delta\overline{\phi} + \overline{\phi} = \phi$ with corresponding weak formulation

$$\delta^2\left(\nabla\overline{\phi}, \nabla\mathbf{v}\right) + \left(\overline{\phi}, \mathbf{v}\right) = (\phi, \mathbf{v}), \quad \forall\mathbf{v} \in X. \tag{3.2}$$

Set $A = -\delta^2\Delta + I$ so that $A^{-1} : L^2(\Omega) \to X$, defined by $\overline{\phi} := A^{-1}\phi$, is well defined.

Definition 3.3 (discrete differential filter). Fix $\phi \in L^2(\Omega)$. Then $\overline{\phi}^h \in X^h$ is the unique solution of the following

$$\delta^2\left(\nabla\overline{\phi}^h, \nabla\mathbf{v}^h\right) + \left(\overline{\phi}^h, \mathbf{v}^h\right) = \left(\phi, \mathbf{v}^h\right), \quad \forall\mathbf{v}^h \in X^h. \tag{3.3}$$

Set $A^h = -\delta^2\Delta^h + \Pi^h$ so that $(A^h)^{-1} : L^2(\Omega) \to X^h$, defined by $\overline{\phi}^h := (A^h)^{-1}\phi$, is well defined. Here, $\Pi^h : L^2(\Omega) \to X^h$ is the L^2 projection and $\Delta^h : X \to X^h$ the discrete Laplace operator satisfying the following

$$\left(\Pi^h\phi - \phi, \mathbf{v}^h\right) = 0, \qquad \left(\Delta^h\phi, \mathbf{v}^h\right) = -\left(\nabla\phi, \nabla\mathbf{v}^h\right) \quad \forall\mathbf{v}^h \in X^h. \tag{3.4}$$

It is well known that A^{-1} and $(A^h)^{-1}$ are each linear and bounded, A^{-1} is compact, and the spectrum of A and A^h (on X and X^h, resp.) is contained in $[1, \infty)$ and spectrum of A^{-1} and $(A^h)^{-1}$ (on X and X^h, resp.) is contained in $(0, 1]$ so that

$$A^{-1} > 0 \quad \text{on } X, \qquad \left(A^h\right)^{-1} > 0 \quad \text{on } X^h. \tag{3.5}$$

For more detailed exposition on these operators, see [13].

3.1. A Family of Deconvolution Operators

We analyze (1.6) for stable, accurate deconvolution D of the smoothing filter G introduced in Section 3 so that $DG\mathbf{u}$ accurately approximates the smooth parts of $\mathbf{u}$.

Assumption 3.4 (continuous deconvolution operator). Suppose that $D : X \to X$ is linear, bounded, spd, and commutes with G so that

$$\|DG\| \leq 1, \qquad |D\overline{\mathbf{v}}|_1 \leq d_1|\mathbf{v}|_1 \quad \forall\mathbf{v} \in X, \tag{3.6}$$

for some constant $d_1 > 0$. Moreover, suppose that D is parametrized by $\alpha > 0$, $\delta > 0$ so that

$$\|(I - DG)\mathbf{v}\| \leq c_1(\delta, \alpha)\|\Delta \mathbf{v}\| \longrightarrow 0, \quad \text{as } \alpha, \delta \longrightarrow 0 \tag{3.7}$$

for smooth enough $\mathbf{v} \in X$.

Note that the first estimate in (3.6) is required so that the spectral radius satisfies $\rho(DG) \leq 1$. The second estimate in (3.6) (which controls the H^1-seminorm of DG) is required for the convergence analysis in Section 4.2.

Assumption 3.5 prescribes properties of the discrete analogue $D^h : X^h \to X^h$ corresponding to the continuous deconvolution operator $D : X \to X$ (Assumptions 3.4).

Assumption 3.5 (discrete deconvolution operator). Let D satisfy Assumption 3.4. Let $G^h : X^h \to X^h$ be a discrete analogue of G that is linear, bounded, spd. Suppose that $D^h : X^h \to X^h$ is linear, bounded, spd, and commutes with G^h such that

$$\left\|D^h G^h\right\| \leq 1, \qquad \left|D^h \overline{\mathbf{v}}^h\right|_1 \leq d_1 |\mathbf{v}|_1 \quad \forall \mathbf{v} \in X, \tag{3.8}$$

for some constant $d_1 > 0$. Moreover, suppose that D^h is parametrized by $\alpha > 0$, $\delta > 0$, h such that $D = D(h, \delta, \alpha)$ and

$$\left\|\left(DG - D^h G^h\right)\mathbf{v}\right\| \leq c_2(h, \delta, \alpha)\|\mathbf{v}\|_{k+1} \longrightarrow 0, \quad \text{as } h, \delta, \alpha \longrightarrow 0, \tag{3.9}$$

for all $\mathbf{v} \in X \cap H^{k+1}(\Omega)$ for some $k \geq 0$.

The estimates in (3.8) are motivated by the continuous case of (3.6). The approximation (3.9) is required for the convergence analysis in Section 4.2 (see Theorem 4.6, Corollary 4.7).

Remark 3.6. If $D = f(G)$ for some continuous map $f : \mathbb{R} \to \mathbb{R}$, then commutativity is satisfied $DG = GD$. Tikhonov-Lavrentiev (modified) regularization with $G = A^{-1}$, $G^h = (A^h)^{-1}$ given by $D = ((1-\alpha)A^{-1} + \alpha I)^{-1}$, $D^h = ((1-\alpha)(A^h)^{-1} + \alpha \Pi^h)^{-1}$ is one such example with $f(x) = ((1-\alpha)x + \alpha)^{-1}$ and $d_1 = 1$, $c_1 = \alpha\delta^2$, $c_2 = \alpha\delta^2 h^k + h^{k+1}$, see [13].

Remark 3.7. Letting $\lambda_k(\cdot)$ denote the kth (ordered) eigenvalue of a given operator, commutativity of D and G provides $\lambda_k(DG) = \lambda_k(D)\lambda_k(G)$ and similarly for the discrete operator $D^h G^h$.

We next derive several important consequences of D and D^h under Assumptions 3.4, 3.5 required in the forthcoming analysis.

Lemma 3.8. *Suppose that G, G^h, D, D^h satisfy Assumptions 3.4, 3.5. Then,*

$$\|D\overline{\mathbf{v}}\| \leq \|\mathbf{v}\|, \qquad \left\|D^h \overline{\mathbf{v}}^h\right\| \leq \|\mathbf{v}\| \quad \forall \mathbf{v} \in L^2(\Omega). \tag{3.10}$$

Proof. For the continuous operator,

$$\|D\overline{\mathbf{v}}\| \leq \|DG\|\|\mathbf{v}\|. \tag{3.11}$$

Then, (3.10)(a) follows from Assumption 3.4, and (3.10)(b) is derived similarly applying Assumption 3.5 instead. □

Lemma 3.9. *Suppose that G, G^h, D, D^h satisfy Assumptions 3.4, 3.5. Then, the spectrum of both DG and D^hG^h are contained in* $[0,1]$ *so that*

$$\|I - DG\| \leq 1, \qquad \left\|I - D^hG^h\right\| \leq 1. \tag{3.12}$$

As a consequence,

$$\left.\begin{aligned} \|\mathbf{v}\|_{\star}^2 &:= (\mathbf{v} - D\overline{\mathbf{v}}, \mathbf{v}) \\ \|\mathbf{v}\|_{\star h}^2 &:= \left(\mathbf{v} - D^h\overline{\mathbf{v}}^h, \mathbf{v}\right) \end{aligned}\right\} \geq 0, \quad \forall \mathbf{v} \in L^2(\Omega). \tag{3.13}$$

Proof. Assumptions 3.4, 3.5 guarantee that the spectral radius $\rho(DG) \leq 1$ and $\rho(D^hG^h) \leq 1$. Also, $D > 0$ and $G > 0$ and commute so that $DG \geq 0$. Similarly, $D^hG^h \geq 0$. Therefore, the spectrum of DG, $D^hG^h \geq 0$ are each contained in $[0,1]$. So, $I - DG$, $I - D^hG^h \geq 0$ have spectrum contained in $[0,1]$ which ensures the non-negativity of both $\|\cdot\|_{\star}$ and $\|\cdot\|_{\star h}$. □

3.2. *Iterated Deconvolution*

One can show, by eliminating intermediate steps in the definition of the iterated regularization operator D_j in (1.9) with base operator D satisfying Assumption 3.4, that

$$D_j = D\sum_{i=0}^{j}(FD)^i, \qquad F := D^{-1} - G. \tag{3.14}$$

Similarly, the discrete iterated regularization operator D_j^h with discrete base operator D^h satisfying Assumption 3.5, is given by the following

$$D_j^h = D^h\sum_{i=0}^{j}\left(F^hD^h\right)^i, \qquad F^h := \left(D^h\right)^{-1} - G^h. \tag{3.15}$$

We next show that D_j and D_j^h for $j > 0$ inherit several important properties from D and D^h, respectively, via Assumption 3.5.

Proposition 3.10. *Fix $j \in \mathbb{N}$. Then $D_j : X \to X$ defined by* (3.14) *satisfies Assumption 3.4. In particular, $D_j > 0$ is linear, bounded, commutes with G and satisfies* (3.6)*(a). Estimate* (3.6)*(b) is replaced by the following*

$$\left|D_j\overline{\mathbf{v}}\right|_1 \leq d_{1,j}|\mathbf{v}|_1 \quad \forall \mathbf{v} \in X \tag{3.16}$$

for some constant $d_{1,j} > 0$. Estimate (3.7) *is replaced by the following*

$$\left\|(I - D_j G)\mathbf{v}\right\| \leq c_{1,j}(\delta,\alpha)\|\Delta \mathbf{v}\|^2 \longrightarrow 0, \quad \text{as } \alpha,\delta \longrightarrow 0. \tag{3.17}$$

Moreover, $d_{1,j} \leq \sum_{i=0}^{j} d_1^i$ and $c_{1,j} \leq \sum_{i=0}^{j} c_1^i$.

Proof. First notice that D_j is linear and bounded since it is a linear combination of linear and bounded operators $D(FD)^i = D(I-DG)^i$, for $i = 0,1,\ldots,j$. Moreover, since G commutes with D, it follows that G commutes with $D(I-DG)^i$ and hence with D_j. Next, D_j is a sum of spd and snn operators $D > 0$, $D(I-DG)^i \geq 0$. Hence, $D_j > 0$. Next, notice that

$$D_j G = \left(\sum_{i=0}^{j}(I - DG)^i\right)DG = \left(I + (I - DG) + \cdots + (I - DG)^j\right)DG. \tag{3.18}$$

Letting $\lambda_k(\cdot)$ denote the kth (ordered) eigenvalue of a given operator, we can characterize the spectrum of D_j by summing the resulting finite geometric series (3.18) to get

$$\lambda_k\left(D_j G\right) = \lambda_k(D)\lambda_k(G)\sum_{i=0}^{j}(1 - \lambda_k(DG))^i = \left(1 - (1 - \lambda_k(DG))^{j+1}\right). \tag{3.19}$$

Then under Assumption 3.4, Lemma 3.9 with (3.19) implies that $0 \leq \lambda_k(D_j G) \leq \|D_j G\| \leq 1$. Hence, D_j satisfies (3.6)(a). Expanding the terms in (3.18) as powers of DG, we see that (3.18) can be written as a polynomial (with coefficients a_i) in DG, so that

$$\nabla D_j \overline{\mathbf{v}} = \sum_{i=0}^{j} a_i \nabla (DG)^i \mathbf{v}, \qquad \left|D_j \overline{\mathbf{v}}\right|_1 \leq \sum_{i=0}^{j} d_1^i |\mathbf{v}|_1, \tag{3.20}$$

since $|D\overline{\mathbf{v}}|_1 \leq d_1|\mathbf{v}|_1$ can be applied successfully. Therefore (3.16) follows with $d_{1,j} = \sum_{i=0}^{j} d_1^i$. Next, start with (3.18) to get

$$\begin{aligned}\left\|(I - D_j G)\mathbf{v}\right\| &= \left\|\left((I - DG)\mathbf{v} + DG(I - DG)\mathbf{v} + \cdots + DG(I - DG)^j \mathbf{v}\right)\right\| \\ &\leq \|(I - DG)\mathbf{v}\| + \|DG\|\|(I - DG)\mathbf{v}\| + \cdots + \|DG\|\|I - DG\|^{j-1}\|(I - DG)\mathbf{v}\|.\end{aligned} \tag{3.21}$$

Estimate (3.17) follows by noting $\|DG\| \leq 1$, $\|I - DG\| \leq 1$, and by Assumption 3.5, $\|(I - DG)\mathbf{v}\| \leq c_1\|\Delta \mathbf{v}\|$. □

Proposition 3.11. *Fix $j \in \mathbb{N}$. Then $D_j^h : X^h \to X^h$ defined by* (3.15) *satisfies Assumption 3.5. In particular, $D_j^h > 0$ is linear, bounded, commutes with G^h and satisfies* (3.8)*(a). Estimate* (3.8)*(b) is replaced by the following*

$$\left|D_j^h \overline{\mathbf{v}}^h\right|_1 \leq d_{1,j}|\mathbf{v}|_1 \quad \forall \mathbf{v} \in X \tag{3.22}$$

for some constant $d_{1,j} > 0$. *Estimate* (3.9) *is replaced by the following*

$$\left\| \left(D_j G - D_j^h G^h \right) \mathbf{v} \right\| \leq c_{2,j}(h, \delta, \alpha) \|\mathbf{v}\|_{k+1} \longrightarrow 0, \quad \text{as } h, \delta, \alpha \longrightarrow 0 \tag{3.23}$$

for any $\mathbf{v} \in X \cap H^{k+1}(\Omega)$ *for some* $k \geq 0$. *Moreover,* $c_{2,j} \leq \beta(j) c_2$ *for some constant* $\beta = \beta(j) > 0$.

Proof. The first two assertions follow similarly as in the previous proof of Proposition 3.10. To prove (3.23), we start by writing

$$D_j^h G^h = \left(\sum_{i=0}^{j} \left(I - D^h G^h \right)^i \right) D^h G^h, \tag{3.24}$$

and then subtract (3.24) from (3.18) to get

$$D_j G - D_j^h G^h = \Lambda_j \left(DG - D^h G^h \right) + \left(\Lambda_j - \Lambda_j^h \right) D^h G^h, \tag{3.25}$$

where

$$\Lambda_j = \sum_{i=0}^{j} (I - DG)^i, \qquad \Lambda_j^h = \sum_{i=0}^{j} \left(I - D^h G^h \right)^i. \tag{3.26}$$

Then taking norms across (3.25), we get

$$\left\| \left(D_j G - D_j^h G^h \right) \mathbf{v} \right\| = \|\Lambda_j\| \left\| \left(DG - D^h G^h \right) \mathbf{v} \right\| + \left\| D^h G^h \right\| \left\| \left(\Lambda_j - \Lambda_j^h \right) \mathbf{v} \right\|. \tag{3.27}$$

Notice that $\|I - DG\| \leq 1$ so that $\|\Lambda_j\| \leq j + 1$. Moreover, $\|(DG - D^h G^h)\mathbf{v}\| \leq c_2 \|\mathbf{v}\|_{k+1}$ via Assumption 3.5. Next, using the binomial theorem and factoring, we get

$$\begin{aligned} \left\| \left(\Lambda_j - \Lambda_j^h \right) \mathbf{v} \right\| &= \left\| \sum_{i=0}^{j} \frac{j!}{i!(j-i)!} (-1)^i \left[(DG)^i - \left(D^h G^h \right)^i \mathbf{v} \right] \right\| \\ &= \left\| \sum_{i=0}^{j} \frac{j!}{i!(j-i)!} (-1)^i \left[\sum_{n=0}^{i} (DG)^n \left(D^h G^h \right)_{n-i} \right] \left(DG - D^h G^h \right) \mathbf{v} \right\|. \end{aligned} \tag{3.28}$$

Then, applying $\|DG\| \leq 1$, $\|D^h G^h\| \leq 1$ to (3.28) provides

$$\left\| \left(\Lambda_j - \Lambda_j^h \right) \mathbf{v} \right\| = \left(\sum_{i=0}^{j} \frac{j! i}{(i)!(j-i)!} \right) \left\| \left(DG - D^h G^h \right) \mathbf{v} \right\|. \tag{3.29}$$

Again, $\|(DG - D^h G^h)\| \leq c_2 \|\mathbf{v}\|_{k+1}$ via Assumption 3.5. So, we combine these above results to conclude (3.23) with $\beta(j) = \sum_{i=0}^{j} j! i / (i)!(j-i)!$. □

3.3. Tikhonov-Lavrentiev Regularization

We provide two examples of discrete deconvolution operators D^h to make the abstract formulation in the previous section more concrete. The Tikhonov-Lavrentiev and modified Tikhonov-Lavrentiev operator (for linear, compact $G > 0$) is given by the following

$$\begin{aligned} \text{Tikhonov-Lavrentiev} &\Longrightarrow D_{\alpha,0} = (G + \alpha I)^{-1} \\ \text{modified Tikhonov-Lavrentiev} &\Longrightarrow D_{\alpha,0} = ((1-\alpha)G + \alpha I)^{-1}. \end{aligned} \tag{3.30}$$

Definition 3.12 ((weak) modified Tikhonov-Lavrentiev deconvolution). Fix $\alpha > 0$. Let $G = A^{-1}$. For any $\mathbf{w} \in X$, let $\boldsymbol{\omega}_0 := D_{\alpha,0}\overline{\mathbf{w}} \in X$ be the unique solution of

$$\alpha\delta^2(\nabla\boldsymbol{\omega}_0, \nabla\mathbf{v}) + (\boldsymbol{\omega}_0, \mathbf{v}) = (\mathbf{w}, \mathbf{v}), \quad \forall \mathbf{v} \in X. \tag{3.31}$$

Definition 3.13 ((discrete) modified Tikhonov-Lavrentiev deconvolution). Fix $\alpha > 0$. Let $G^h = (A^h)^{-1}$ and $D^h_{\alpha,0} = ((1-\alpha)(A^h)^{-1} + \alpha\Pi^h)^{-1}$. For any $\mathbf{w} \in X$, let $\boldsymbol{\omega}^h_0 := D^h_{\alpha,0}\overline{\mathbf{w}}^h \in X^h$ be the unique solution of

$$\alpha\delta^2\left(\nabla\boldsymbol{\omega}^h_0, \nabla\mathbf{v}^h\right) + \left(\boldsymbol{\omega}^h_0, \mathbf{v}^h\right) = \left(\mathbf{w}, \mathbf{v}^h\right), \quad \forall \mathbf{v}^h \in X^h. \tag{3.32}$$

The iterated modified Tikhonov-Lavrentiev operator (for linear, compact $G > 0$) is obtained from the Tikhonov-Lavrentiev operator with updates via (1.9):

$$\begin{aligned} \text{Iterated Tikhonov-Lavrentiev} &\Longrightarrow D_{\alpha,j} = D_{\alpha,0}\sum_{i=0}^{j}(\alpha D_{\alpha,0})^i, \\ \text{Iterated modified Tikhonov-Lavrentiev} &\Longrightarrow D_{\alpha,j} = D_{\alpha,0}\sum_{i=0}^{j}(\alpha(I-G)D_{\alpha,0})^i. \end{aligned} \tag{3.33}$$

Definition 3.14 (iterated modified Tikhonov-Lavrentiev deconvolution (weak)). Fix $\alpha > 0$ and $J \in \mathbb{N}$. Let $G = A^{-1}$. Define $\boldsymbol{\omega}_{-1} = 0$, then for any $\mathbf{w} \in X$ and $j = 0, 1, \ldots, J$, let $\boldsymbol{\omega}_j := D_{\alpha,j}\overline{\mathbf{w}} \in X$ be the unique solution of

$$\alpha\delta^2(\nabla\boldsymbol{\omega}_j, \nabla\mathbf{v}) + (\boldsymbol{\omega}_j, \mathbf{v}) = (\mathbf{w}, \mathbf{v}) + \alpha\delta^2(\nabla\boldsymbol{\omega}_{j-1}, \nabla\mathbf{v}), \quad \forall \mathbf{v} \in X. \tag{3.34}$$

Definition 3.15 (iterated modified Tikhonov-Lavrentiev deconvolution (discrete)). Fix $\alpha > 0$ and $J \in \mathbb{N}$. Let $G^h = (A^h)^{-1}$, and $D^h_{\alpha,j} = D^h_{\alpha,0}\sum_{i=0}^{j}(\alpha(\Pi^h - (A^h)^{-1})D^h_{\alpha,0})^i$. Define $\boldsymbol{\omega}^h_{-1} = 0$, then for any $\mathbf{w} \in X$ and $j = 0, 1, \ldots, J$, let $\boldsymbol{\omega}^h_j := D^h_{\alpha,j}\overline{\mathbf{w}}^h \in X^h$ be the unique solution of

$$\alpha\delta^2\left(\nabla\boldsymbol{\omega}^h_j, \nabla\mathbf{v}^h\right) + \left(\boldsymbol{\omega}^h_j, \mathbf{v}^h\right) = \left(\mathbf{w}, \mathbf{v}^h\right) + \alpha\delta^2\left(\nabla\boldsymbol{\omega}^h_{j-1}, \nabla\mathbf{v}^h\right), \quad \forall \mathbf{v}^h \in X^h. \tag{3.35}$$

4. Well Posedness of the Fully Discrete Model

We now state the proposed algorithm.

Problem 1 (CNFE for Leray-deconvolution). Let $(\mathbf{w}_0, \pi_0) \in (X^h, Q^h)$. Then, for each $n = 0, 1, \ldots, M-1$, find $(\mathbf{w}_{n+1}^h, \pi_{n+1}^h) \in (X^h, Q^h)$ satisfying

$$\begin{aligned}\frac{1}{\Delta t}\left(\mathbf{w}_{n+1}^h - \mathbf{w}_n^h, \mathbf{v}^h\right) + b^h\left(\varphi^h\left(\mathbf{w}_{n+1/2}^h\right), \mathbf{w}_{n+1/2}^h, \mathbf{v}^h\right) - \left(\pi_{n+1/2}^h, \nabla \cdot \mathbf{v}^h\right) \\ + \mathrm{Re}^{-1}\left(\nabla \mathbf{w}_{n+1/2}^h, \nabla \mathbf{v}^h\right) + \chi\left(\mathbf{w}_{n+1/2}^h - \varphi^h\left(\mathbf{w}_{n+1/2}^h\right), \mathbf{v}^h\right) = \left(\mathbf{f}_{n+1/2}, \mathbf{v}^h\right), \quad \forall \mathbf{v}^h \in X^h,\end{aligned} \tag{4.1}$$

$$\left(\nabla \cdot \mathbf{w}_{n+1}^h, q^h\right) = 0, \quad \forall q^h \in Q^h, \tag{4.2}$$

where $\varphi^h(\mathbf{w}_{n+1/2}^h) = D^h \overline{\mathbf{w}_{n+1/2}^h}^h$.

Notice that $(q_{n+1/2}^h, \nabla \cdot \mathbf{v}^h) = 0$ when $\mathbf{v}^h \in \mathbf{V}^h$ so that the problem of finding $\mathbf{w}_{n+1}^h \in V^h$ satisfying

$$\begin{aligned}\frac{1}{\Delta t}\left(\mathbf{w}_{n+1}^h - \mathbf{w}_n^h, \mathbf{v}^h\right) + b^h\left(\varphi^h\left(\mathbf{w}_{n+1/2}^h\right), \mathbf{w}_{n+1/2}^h, \mathbf{v}^h\right) + \mathrm{Re}^{-1}\left(\nabla \mathbf{w}_{n+1/2}^h, \nabla \mathbf{v}^h\right) \\ + \chi\left(\mathbf{w}_{n+1/2}^h - \varphi^h\left(\mathbf{w}_{n+1/2}^h\right), \mathbf{v}^h\right) = \left(\mathbf{f}_{n+1/2}, \mathbf{v}^h\right), \quad \forall \mathbf{v}^h \in V^h.\end{aligned} \tag{4.3}$$

4.1. Well Posedness

We establish existence of $\mathbf{w}$ at each time step of (4.3) by Leray-Schauder's fixed-point theorem.

Lemma 4.1. *Let*

$$\begin{aligned} a\left(\boldsymbol{\theta}^h, \mathbf{v}^h\right) &= \frac{\Delta t}{2\,\mathrm{Re}}\left(\nabla \boldsymbol{\theta}^h, \nabla \mathbf{v}^h\right) + \frac{\chi \Delta t}{2}\left(\boldsymbol{\theta}^h - \varphi^h\left(\boldsymbol{\theta}^h\right), \mathbf{v}^h\right), \\ l_y\left(\mathbf{v}^h\right) &= \left(\mathbf{y}, \mathbf{v}^h\right). \end{aligned} \tag{4.4}$$

for any $\mathbf{y} \in X'$ *and* $\boldsymbol{\theta}^h$, $\mathbf{v}^h \in V^h$. *Suppose that* D^h *satisfies Assumption 3.5. Then* $a(\cdot,\cdot) : V^h \times V^h \to \mathbb{R}$ *is a continuous and coercive bilinear form and* $l_y(\cdot) : V^h \to \mathbb{R}$ *is a linear, continuous functional.*

Proof. Linearity for $l_y(\cdot)$ is obvious, and continuity follows from an application of Hölder's inequality. Continuity for $a(\cdot,\cdot)$ also follows from Hölder's inequality and Assumption 3.5. Coercivity is proven by application of (3.13). □

Lemma 4.2. *Let* $T : X' \to V^h$ *be such that, for any* $\mathbf{y} \in X'$, $\boldsymbol{\theta}^h := T(\mathbf{y})$ *solves*

$$a\left(\boldsymbol{\theta}^h, \mathbf{v}^h\right) = l_y\left(\mathbf{v}^h\right), \quad \forall \mathbf{v}^h \in V^h. \tag{4.5}$$

Then T *is a well-defined, linear, bounded operator.*

Proof. Linearity is clear. The results of Lemma 4.1, and the Lax-Milgram theorem prove the rest. □

Lemma 4.3. *Fix $n = 0, 1, \ldots, M-1$. Let $\mathbf{w}_n^h$ be a solution of Problem 1 and let $N : V^h \to X'$ satisfy, for any $\boldsymbol{\theta}^h \in V^h$,*

$$\begin{aligned}\left(N\left(\boldsymbol{\theta}^h\right), \mathbf{v}^h\right) &= -\left(\boldsymbol{\theta}^h - 2\mathbf{w}_n^h, \mathbf{v}^h\right) - \frac{\Delta t}{4} b^h\left(\varphi^h\left(\boldsymbol{\theta}^h\right), \boldsymbol{\theta}^h, \mathbf{v}^h\right) + \Delta t\left(\mathbf{f}_{n+1/2}, \mathbf{v}^h\right) \\ &=: c\left(\boldsymbol{\theta}^h, \mathbf{v}^h\right), \quad \forall \mathbf{v}^h \in V^h.\end{aligned} \tag{4.6}$$

Then $N(\boldsymbol{\theta}^h)$ is well-defined, bounded, and continuous.

Proof. For each $\boldsymbol{\theta}^h \in V^h$, the map $\mathbf{v}^h \in V^h \mapsto c(\boldsymbol{\theta}^h, \mathbf{v}^h)$ is a bounded, linear functional (apply Hölder's inequality and (2.11)). Since V^h is a Hilbert space, we conclude that $N(\boldsymbol{\theta}^h)$ is well defined, by the Riesz-Representation theorem. Moreover, $N(\boldsymbol{\theta}^h)$ is bounded on V^h and since the underlying function space is finite dimensional, continuity follows. □

Lemma 4.4. *Fix $n \in \mathbb{N}$. Let $F : V^h \to V^h$ be defined such that $F(\boldsymbol{\theta}^h) = (T \circ N)(\boldsymbol{\theta}^h)$. Then, F is a compact operator.*

Proof. $N(\cdot)$ is a compact operator (continuous on a finite dimensional function space). Thus, F is a continuous composition of a compact operator and hence compact itself. □

Theorem 4.5 (well posedness). *Fix $n = 0, 1, 2, \ldots, M-1 < \infty$. There exists $(\mathbf{w}_n^h, \pi_n^h) \in X^h \times Q^h$ satisfying Problem 1. Moreover,*

$$\left\|\mathbf{w}_m^h\right\|^2 + \frac{1}{2\,\mathrm{Re}}\Delta t \sum_{n=0}^{m-1}\left|\mathbf{w}_{n+1/2}^h\right|_1^2 + \chi\Delta t\sum_{n=0}^{m-1}\left\|\mathbf{w}_{n+1/2}^h\right\|_{\star h}^2 \le \left\|\mathbf{w}_0^h\right\|^2 + \frac{\Delta t\,\mathrm{Re}}{2}\sum_{n=0}^{m-1}\|\mathbf{f}_{n+1/2}\|_{-1}^2, \tag{4.7}$$

for all integers $1 \le m \le M$, independent of $\Delta t > 0$.

Proof. First, assume that $(\mathbf{w}_{n+1}^h, q_{n+1}^h)$ is a solution to (4.1), (4.2). Set $\mathbf{v}^h = \mathbf{w}_{n+1/2}^h$ in (4.1) so that skew-symmetry of the nonlinear term provides

$$\begin{aligned}&\frac{1}{2\Delta t}\left(\left\|\mathbf{w}_{n+1}^h\right\|^2 - \left\|\mathbf{w}_m^h\right\|^2\right) + \mathrm{Re}^{-1}\left|\mathbf{w}_{n+1/2}^h\right|_1^2 \\ &\quad + \chi\left(\mathbf{w}_{n+1/2}^h - \varphi^h\left(\mathbf{w}_{n+1/2}^h\right), \mathbf{w}_{n+1/2}^h\right) = \left(\mathbf{f}_{n+1/2}, \mathbf{w}_{n+1/2}^h\right).\end{aligned} \tag{4.8}$$

Duality of $X' \times X$ with Young's inequality implies

$$\left(\mathbf{f}_{n+1/2}, \mathbf{w}_{n+1/2}^h\right) \le \frac{\mathrm{Re}}{2}\|\mathbf{f}_{n+1/2}\|_{-1}^2 + \frac{\mathrm{Re}}{2}\left|\mathbf{w}_{n+1/2}^h\right|_1^2. \tag{4.9}$$

From (3.13), we have

$$\left\|\mathbf{w}^h_{n+1/2}\right\|^2_{\star h} = \left(\mathbf{w}^h_{n+1/2} - \varphi^h\left(\mathbf{w}^h_{n+1/2}\right), \mathbf{w}^h_{n+1/2}\right) \geq 0. \tag{4.10}$$

Then applying (4.9), (4.10) to (4.8), combining-like terms and simplifying provides

$$\frac{1}{2\Delta t}\left(\left\|\mathbf{w}^h_{n+1}\right\|^2 - \left\|\mathbf{w}^h_{n}\right\|^2\right) + \frac{\mathrm{Re}}{2}\left|\mathbf{w}^h_{n+1/2}\right|^2_1 + \chi\left\|\mathbf{w}^h_{n+1/2}\right\|^2_{\star h} \leq \frac{\mathrm{Re}}{2}\|\mathbf{f}_{n+1/2}\|^2_{-1}. \tag{4.11}$$

Summing from $n = 0$ to $m - 1$, we get the desired bound.

Next, let $\mathbf{W}^h_n = \mathbf{w}^h_{n+1} + \mathbf{w}^h_n$. Showing that $\mathbf{W}^h_n = F(\mathbf{W}^h_n)$ has a fixed point will ensure existence of solutions to (4.3). Indeed, if we can show that $\mathbf{W}^h_0 = F(\mathbf{W}^h_0)$, then since $\mathbf{w}^h_0$ is given initial data, existence of $\mathbf{w}^h_1$ is immediate. Induction can be applied to prove existence of $(\mathbf{w}^h_n)_{1\leq n\leq M}$. To this end, since F is compact, it is enough to show (via Leray Schauder) that any solution $\mathbf{W}^h_{n,\lambda}$ of the fixed-point problem $\mathbf{W}^h_{n,\lambda} = \lambda F(\mathbf{W}^h_{n,\lambda})$ is uniformly bounded with respect to $0 \leq \lambda \leq 1$. Hence, we consider

$$a\left(\mathbf{W}^h_{n,\lambda}, \mathbf{v}^h\right) = \lambda\left(N\left(\mathbf{W}^h_{n,\lambda}\right), \mathbf{v}^h\right). \tag{4.12}$$

Test with $\mathbf{v}^h = \mathbf{W}^h_{n,\lambda}$, use skew-symmetry of the trilinear form and properties of D^h given in Assumption 3.5 and (3.13) to get

$$\lambda\left\|\mathbf{W}^h_{n,\lambda}\right\|^2 + \frac{\Delta t}{2\,\mathrm{Re}}\left|\mathbf{W}^h_{n,\lambda}\right|^2_1 + \frac{\chi\Delta t}{2}\left\|\mathbf{W}^h_{n,\lambda}\right\|^2_{\star h} \leq 2\lambda\left(\mathbf{w}^h_n, \mathbf{W}^h_{n,\lambda}\right) + \lambda\Delta t\left(\mathbf{f}_{n+1/2}, \mathbf{W}^h_{n,\lambda}\right). \tag{4.13}$$

Duality of $X' \times X$ followed by Young's inequality implies

$$\lambda\Delta t\left(\mathbf{f}_{n+1/2}, \mathbf{W}^h_{n,\lambda}\right) \leq \Delta t\,\mathrm{Re}\,\|\mathbf{f}_{n+1/2}\|^2_{-1} + \frac{\Delta t}{4\,\mathrm{Re}}\left|\mathbf{W}^h_{n,\lambda}\right|^2_1. \tag{4.14}$$

Since $\mathbf{w}^h_n \in L^2(\Omega)$ from the a priori estimate (4.7), we apply Hölder's and Young's inequalities to get

$$2\lambda\left(\mathbf{w}^h_n, \mathbf{W}^h_{n,\lambda}\right) \leq 2\left\|\mathbf{w}^h_n\right\|^2 + \frac{\lambda}{2}\left\|\mathbf{W}^h_{n,\lambda}\right\|^2. \tag{4.15}$$

Applying estimates (4.14), (4.15) to (4.13) we get that $|\mathbf{W}^h_{n,\lambda}|_1 \leq C < \infty$ independent of λ. By the Leray-Schauder fixed point theorem, given $\mathbf{w}^h_n$, there exists a solution to the fixed-point theorem $\mathbf{W}^h_n = F(\mathbf{W}^h_n)$. By the induction argument noted above, there exists a solution $\mathbf{w}^h_n$ for each $n = 0, 1, 2, \ldots, M-1$ to (4.3). Existence of an associated discrete pressure follows by a classical argument, since the pair (X^h, Q^h) satisfies the discrete inf-sup condition (2.15). □

4.2. Convergence Analysis

Under usual regularity assumptions, we summarize the main convergence estimate in Theorem 4.6. Suppose that D represents deconvolution with J-updates.

Theorem 4.6. *Suppose that* $(\mathbf{u}, p)$ *are strong solutions to* (2.3), (2.4), (2.5) *and that* G, G^h, D, D^h *satisfy Assumptions 3.4, 3.5. Suppose further that* $\mathbf{u} \in l^2(H^2(\Omega) \cap V) \cap l^\infty(X)$, $\mathbf{u}_t \in l^2(X') \cap L^2(X')$, $\Delta^{\beta+1}\mathbf{u} \in L^2(\Omega)$ *for some* $0 \leq \beta \leq J+1$, $p \in l^2(Q)$. *If*

$$C \operatorname{Re} \Delta t \|\mathbf{u}_n\|_2^2 < 1, \quad \forall n = 0, 1, \ldots, M \tag{4.16}$$

then,

$$\begin{aligned} &\left\|\mathbf{u}_M - \mathbf{w}_M^h\right\|^2 + \operatorname{Re}^{-1}\Delta t \sum_{n=0}^{M-1} \left|\mathbf{u}_{n+1/2} - \mathbf{w}_{n+1/2}^h\right|_1^2 \\ &\quad \leq C\Bigg(\left\|\mathbf{u}_0 - \mathbf{w}_0^h\right\|^2 + E + \|p\|^2_{l^2(L^2(\Omega))} + \left(\|\nabla \mathbf{u}\|^2_{l^\infty(L^2(\Omega))} + \chi^2\right)\|\mathbf{u}\|^2_{l^2(H^1(\Omega))} \\ &\qquad + \|\mathbf{u}\|^2_{l^\infty(L^2(\Omega))} + \|\mathbf{u}\|^2_{l^2(L^2(\Omega))} + \left(\chi^2 + \|\mathbf{u}\|^2_{l^\infty(H^2(\Omega))}\right)c_{1,J}^2\left\|\Delta^\beta \mathbf{u}\right\|^2 + c_{2,j}^2 \|\nabla \mathbf{u}\|^2_{l^2(L^2(\Omega))}\Bigg), \end{aligned} \tag{4.17}$$

where $E > 0$ *is given in* (4.52).

Corollary 4.7 (convergence estimate). *Under the assumptions of Theorem 4.6, suppose further that* $(\mathbf{u}, p)$ *satisfy the assumptions for* (2.16) *for some* $k \geq 1$ *and* $s \geq 0$, $\mathbf{u} \in l^\infty(H^k(\Omega)) \cap l^2(H^{k+1}(\Omega))$, $\mathbf{u}_t \in L^2(H^{k-1}(\Omega) \cap H^1(\Omega)) \cap l^\infty(H^1(\Omega))$, $\mathbf{u}_{tt} \in L^2(L^2(\Omega))$, $\mathbf{u}_{ttt} \in L^2(X')$, *and* $p \in l^2(H^{s+1}(\Omega))$. *If* $\|\mathbf{u}_0 - \mathbf{w}_0^h\| \leq C(h^k + (\alpha\delta^2)^\beta)$ *then*

$$\begin{aligned} &\left\|\mathbf{u}_M - \mathbf{w}_M^h\right\|^2 + \operatorname{Re}^{-1}\Delta t \sum_{n=0}^{M-1} \left|\mathbf{u}_{n+1/2} - \mathbf{w}_{n+1/2}^h\right|_1^2 \\ &\quad \leq C\left(h^{2k} + h^{2s+2} + \Delta t^4 + c_{1,J}(\delta,\alpha)^2 + c_{2,J}(h,\delta,\alpha)^2\right). \end{aligned} \tag{4.18}$$

Proof of Theorem 4.6, Corollary 4.7. Suppose that $\mathbf{u}$, p satisfying (2.3), (2.4) also satisfy $\mathbf{u} \in C^0(V)$, $\mathbf{u}_t \in C^0(X')$, and $p \in C^0(Q)$ so that, for each $n = n_0, n_0+1, \ldots, M-1$,

$$\begin{aligned} &\left((\mathbf{u}_t)_{n+1/2}, \mathbf{v}\right) + \frac{1}{2}(\mathbf{u}_{n+1} \cdot \nabla \mathbf{u}_{n+1}, \mathbf{v}) + \frac{1}{2}(\mathbf{u}_n \cdot \nabla \mathbf{u}_n, \mathbf{v}) + \operatorname{Re}^{-1}(\nabla \mathbf{u}_{n+1/2}, \nabla \mathbf{v}) - \left(p_{n+1/2}, \nabla \cdot \mathbf{v}\right) \\ &\quad = (\mathbf{f}_{n+1/2}, \mathbf{v}), \quad \forall \mathbf{v} \in X, \\ &\qquad \nabla \cdot \mathbf{u}_{n+1} = 0. \end{aligned} \tag{4.19}$$

The consistency error for the time-discretization $\tau_n^{(1)}(\mathbf{u}, p; \mathbf{v}^h)$ and regularization/time-relaxation error $\tau_n^{(2)}(\mathbf{u}, p; \mathbf{v}^h)$ are given by, for $n = 0, 1, \ldots, M-1$,

$$\begin{aligned}
\tau_n^{(1)}\left(\mathbf{u}, p; \mathbf{v}^h\right) &:= \left(\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - (\mathbf{u}_t)_{n+1/2}, \mathbf{v}^h\right) + \left(\mathbf{u}_{n+1/2} \cdot \nabla \mathbf{u}_{n+1/2}, \mathbf{v}^h\right) \\
&\qquad - \frac{1}{2}\left(\mathbf{u}_{n+1} \cdot \nabla \mathbf{u}_{n+1}, \mathbf{v}^h\right) - \frac{1}{2}(\mathbf{u}_n \cdot \nabla \mathbf{u}_n, \mathbf{v}), \\
\tau_n^{(2)}\left(\mathbf{u}, p; \mathbf{v}^h\right) &:= -b^h\left(\mathbf{u}_{n+1/2} - \varphi^h(\mathbf{u}_{n+1/2}), \mathbf{u}_{n+1/2}, \mathbf{v}^h\right) + \chi\left(\mathbf{u}_{n+1/2} - \varphi^h(\mathbf{u}_{n+1/2}), \mathbf{v}^h\right),
\end{aligned} \tag{4.20}$$

where $\mathbf{v}^h \in X^h$. Write $\tau_n := \tau_n^{(1)} + \tau_n^{(2)}$. Using (4.20), rewrite (4.19) in a form conducive to analyzing the error between the continuous and discrete models:

$$\begin{aligned}
&\left(\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t}, \mathbf{v}^h\right) + b^h\left(\varphi^h(\mathbf{u}_{n+1/2}), \mathbf{u}_{n+1/2}, \mathbf{v}^h\right) + \mathrm{Re}^{-1}\left(\nabla \mathbf{u}_{n+1/2}, \nabla \mathbf{v}^h\right) \\
&\quad - \left(p_{n+1/2}, \nabla \cdot \mathbf{v}^h\right) + \chi\left(\mathbf{u}_{n+1/2} - \varphi^h(\mathbf{u}_{n+1/2}), \mathbf{v}^h\right) = \left(\mathbf{f}_{n+1/2}, \mathbf{v}^h\right) + \tau_n\left(\mathbf{u}, p; \mathbf{v}^h\right).
\end{aligned} \tag{4.21}$$

Let $\tilde{\mathbf{v}}^h = \mathbf{U}_n^h$ be the L^2-projection of $\mathbf{u}(\cdot, t_n)$ so that $(\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n, \phi_{n+1/2}^h) = 0$. Decompose the velocity error

$$\mathbf{e}_n = \mathbf{w}_n^h - \mathbf{u}_n = \phi_n^h - \boldsymbol{\eta}_n, \qquad \phi_n^h = \mathbf{w}_n^h - \mathbf{U}_n^h, \qquad \boldsymbol{\eta}_n = \mathbf{u}_n - \mathbf{U}_n^h, \tag{4.22}$$

where $\mathbf{U}_n^h \in V^h$. Fix $\tilde{q}_{n+1/2}^h \in Q^h$. Note that $(\tilde{q}_{n+1/2}^h, \nabla \cdot \mathbf{v}^h) = 0$ for any $\mathbf{v}^h \in V^h$. Subtract (4.21) from (4.3), apply (3.13)(b), and test with $\mathbf{v}^h = \phi_{n+1/2}^h$ to get

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\left\|\phi_{n+1}^h\right\|^2 - \left\|\phi_n^h\right\|^2\right) + \mathrm{Re}^{-1}\left|\phi_{n+1/2}^h\right|_1^2 + \chi\left\|\phi_{n+1/2}^h\right\|_{\star h}^2 \\
&\quad = -\left(\tilde{q}_{n+1/2}^h - p_{n+1/2}, \nabla \cdot \phi_{n+1/2}^h\right) - b^h\left(\varphi^h\left(\phi_{n+1/2}^h\right), \mathbf{u}_{n+1/2}, \phi_{n+1/2}^h\right) \\
&\qquad + b^h\left(\varphi^h\left(\boldsymbol{\eta}_{n+1/2}\right), \mathbf{u}_{n+1/2}, \phi_{n+1/2}^h\right) + b^h\left(\varphi^h\left(\mathbf{w}_{n+1/2}^h\right), \boldsymbol{\eta}_{n+1/2}, \phi_{n+1/2}^h\right) \\
&\qquad + \mathrm{Re}^{-1}\left(\nabla \boldsymbol{\eta}_{n+1/2}, \nabla \phi_{n+1/2}^h\right) + \chi\left(\boldsymbol{\eta}_{n+1/2} - \varphi^h\left(\boldsymbol{\eta}_{n+1/2}\right), \phi_{n+1/2}^h\right) - \tau_n\left(\mathbf{u}, p; \phi_{n+1/2}^h\right).
\end{aligned} \tag{4.23}$$

(Spatial discretization error): Fix $\varepsilon > 0$. First, apply Hölder's and Young's inequalities to get

$$\begin{aligned}
&\left|\mathrm{Re}^{-1}\left(\nabla \boldsymbol{\eta}_{n+1/2}, \nabla \phi_{n+1/2}^h\right) + \left(p_{n+1/2} - \tilde{q}_{n+1/2}^h, \nabla \cdot \phi_{n+1/2}^h\right)\right| \\
&\quad \le C\mathrm{Re}^{-1}\left|\boldsymbol{\eta}_{n+1/2}\right|_1^2 + C\,\mathrm{Re}\left\|p_{n+1/2} - \tilde{q}_{n+1/2}^h\right\|^2 + \frac{1}{\varepsilon\,\mathrm{Re}}\left|\phi_{n+1/2}^h\right|_1^2.
\end{aligned} \tag{4.24}$$

Apply (3.12) and duality estimate on $X \times X'$ to get

$$\left|\chi\left(\boldsymbol{\eta}_{n+1/2} - \varphi^h\left(\boldsymbol{\eta}_{n+1/2}\right), \phi^h_{n+1/2}\right)\right| \le C\chi^2 \operatorname{Re} \left\|\boldsymbol{\eta}_{n+1/2}\right\|_{-1}^2 + \frac{1}{\varepsilon \operatorname{Re}}\left|\phi^h_{n+1/2}\right|_1. \tag{4.25}$$

We bound the convective terms next. First, $\mathbf{u} \in H^2(\Omega)$ and estimate (2.11)(b) give

$$\left|b^h\left(\varphi^h\left(\phi^h_{n+1/2}\right), \mathbf{u}_{n+1/2}, \phi^h_{n+1/2}\right)\right| \le C \operatorname{Re} \left\|\varphi^h\left(\phi^h_{n+1/2}\right)\right\|^2 \|\mathbf{u}_{n+1/2}\|_2^2 + \frac{1}{\varepsilon \operatorname{Re}}\left|\phi^h_{n+1/2}\right|_1^2 \tag{4.26}$$

and $\mathbf{u} \in L^\infty(X)$ with (2.11)(a) give

$$\left|b^h\left(\varphi^h\left(\boldsymbol{\eta}_{n+1/2}\right), \mathbf{u}_{n+1/2}, \phi^h_{n+1/2}\right)\right| \le C \operatorname{Re} \|\nabla \mathbf{u}\|^2_{L^\infty(L^2(\Omega))}\left|\varphi^h\left(\boldsymbol{\eta}_{n+1/2}\right)\right|_1^2 + \frac{1}{\varepsilon \operatorname{Re}}\left|\phi^h_{n+1/2}\right|_1^2. \tag{4.27}$$

Next, rewrite the remaining nonlinear term

$$\begin{aligned} b^h\left(\varphi^h\left(\mathbf{w}^h_{n+1/2}\right), \boldsymbol{\eta}_{n+1/2}, \phi^h_{n+1/2}\right) = {} & b^h\left(\varphi^h(\mathbf{u}_{n+1/2}), \boldsymbol{\eta}_{n+1/2}, \phi^h_{n+1/2}\right) \\ & - b^h\left(\varphi^h\left(\boldsymbol{\eta}_{n+1/2}\right), \boldsymbol{\eta}_{n+1/2}, \phi^h_{n+1/2}\right) \\ & + b^h\left(\varphi^h\left(\phi^h_{n+1/2}\right), \boldsymbol{\eta}_{n+1/2}, \phi^h_{n+1/2}\right). \end{aligned} \tag{4.28}$$

Once again, $\mathbf{u} \in L^\infty(X)$ and (2.11)(a) give

$$\begin{aligned} &\left|b^h\left(\varphi^h(\mathbf{u}_{n+1/2}), \boldsymbol{\eta}_{n+1/2}, \phi^h_{n+1/2}\right) + b^h\left(\varphi^h\left(\boldsymbol{\eta}_{n+1/2}\right), \boldsymbol{\eta}_{n+1/2}, \phi^h_{n+1/2}\right)\right| \\ &\quad \le C \operatorname{Re} \left\|\nabla \varphi^h(\mathbf{u})\right\|_{L^\infty(L^2(\Omega))}\left\|\varphi^h(\mathbf{u})\right\|_{L^\infty(L^2(\Omega))}\left|\boldsymbol{\eta}_{n+1/2}\right|_1^2 \\ &\qquad + C \operatorname{Re}\left|\varphi^h\left(\boldsymbol{\eta}_{n+1/2}\right)\right|_1\left|\varphi^h\left(\boldsymbol{\eta}_{n+1/2}\right)\right|_1\left|\boldsymbol{\eta}_{n+1/2}\right|_1^2 + \frac{1}{\varepsilon \operatorname{Re}}\left|\phi^h_{n+1/2}\right|_1^2. \end{aligned} \tag{4.29}$$

Lastly, estimate (2.11)(a) and inverse inequality (2.17) give

$$\begin{aligned} &\left|b^h\left(\varphi^h\left(\phi^h_{n+1/2}\right), \boldsymbol{\eta}_{n+1/2}, \phi^h_{n+1/2}\right)\right| \\ &\quad \le C \operatorname{Re} h^{-1}\left\|\varphi^h\left(\phi^h_{n+1/2}\right)\right\|^2\left|\boldsymbol{\eta}_{n+1/2}\right|_1^2 + \frac{1}{\varepsilon \operatorname{Re}}\left|\phi^h_{n+1/2}\right|_1^2. \end{aligned} \tag{4.30}$$

Apply estimates $\|\varphi^h(\mathbf{v})\| \leq \|\mathbf{v}\|$ and $|\varphi^h(\mathbf{v})|_1 \leq d_1|\mathbf{v}|_1$ from Assumption 3.5 along with estimates (4.26)–(4.30) to get

$$\begin{aligned}
&\Big|b^h\Big(\varphi^h\Big(\phi^h_{n+1/2}\Big), \mathbf{u}_{n+1/2}, \phi^h_{n+1/2}\Big) + b^h\Big(\varphi^h\Big(\boldsymbol{\eta}_{n+1/2}\Big), \mathbf{u}_{n+1/2}, \phi^h_{n+1/2}\Big) \\
&+b^h\Big(\varphi^h\Big(\mathbf{w}^h_{n+1/2}\Big), \boldsymbol{\eta}_{n+1/2}, \phi^h_{n+1/2}\Big)\Big| \\
&\quad \leq \frac{4}{\varepsilon \operatorname{Re}}\Big|\phi^h_{n+1/2}\Big|_1^2 + C\operatorname{Re}\Big(\|\mathbf{u}_{n+1/2}\|_2^2 + h^{-1}\Big|\boldsymbol{\eta}_{n+1/2}\Big|_1^2\Big)\Big\|\phi^h_{n+1/2}\Big\|^2 \\
&\qquad + Cd_1\operatorname{Re}\Big(\|\nabla\mathbf{u}\|^2_{L^\infty(L^2(\Omega))} + \Big|\boldsymbol{\eta}_{n+1/2}\Big|_1^2\Big)\Big|\boldsymbol{\eta}_{n+1/2}\Big|_1^2.
\end{aligned} \tag{4.31}$$

(Time discretization error): First, apply duality estimate on $X \times X'$ to get

$$\left|\left(\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - (\mathbf{u}_t)_{n+1/2}, \phi^h_{n+1/2}\right)\right| \leq C\operatorname{Re}\left\|\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - (\mathbf{u}_t)_{n+1/2}\right\|_{-1}^2 + \frac{1}{\varepsilon\operatorname{Re}}\left|\phi^h_{n+1/2}\right|_1^2. \tag{4.32}$$

Taylor-expansion about $t_{n+1/2}$ with integral remainder gives

$$\begin{aligned}
&\frac{1}{2}(\mathbf{u}_{n+1} \cdot \nabla\mathbf{u}_{n+1}, \mathbf{v}) + \frac{1}{2}(\mathbf{u}_n \cdot \nabla\mathbf{u}_n, \mathbf{v}) \\
&\quad = (\mathbf{u}(\cdot, t_{n+1/2}) \cdot \nabla\mathbf{u}(\cdot, t_{n+1/2}), \mathbf{v}) \\
&\qquad + \frac{1}{2}\int_{t_{n+1/2}}^{t^{n+1}} (t_{n+1} - t)\frac{d^2}{dt^2}(\mathbf{u}(\cdot, t) \cdot \nabla\mathbf{u}(\cdot, t), \mathbf{v})dt \\
&\qquad + \frac{1}{2}\int_{t_n}^{t_{n+1/2}} (t - t_n)\frac{d^2}{dt^2}(\mathbf{u}(\cdot, t) \cdot \nabla\mathbf{u}(\cdot, t), \mathbf{v})dt.
\end{aligned} \tag{4.33}$$

Add/subtract $(\mathbf{u}_{n+1/2} \cdot \nabla\mathbf{u}(\cdot, t_{n+1/2}), \mathbf{v})$ and apply (4.33) to get

$$\begin{aligned}
&(\mathbf{u}_{n+1/2} \cdot \nabla\mathbf{u}_{n+1/2}, \mathbf{v}) - \frac{1}{2}(\mathbf{u}_{n+1} \cdot \nabla\mathbf{u}_{n+1}, \mathbf{v}) - \frac{1}{2}(\mathbf{u}_n \cdot \nabla\mathbf{u}_n, \mathbf{v}) \\
&\quad = (\mathbf{u}_{n+1/2} \cdot \nabla(\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})), \mathbf{v}) + ((\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})) \cdot \nabla\mathbf{u}(\cdot, t_{n+1/2}), \mathbf{v}) \\
&\qquad - \frac{1}{2}\int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t)\int(\mathbf{u}_{tt} \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u}_{tt} + 2\mathbf{u}_t \cdot \nabla\mathbf{u}_t) \cdot \mathbf{v}dt \\
&\qquad - \frac{1}{2}\int_{t_n}^{t_{n+1/2}} (t - t_n)\int(\mathbf{u}_{tt} \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u}_{tt} + 2\mathbf{u}_t \cdot \nabla\mathbf{u}_t) \cdot \mathbf{v}dt.
\end{aligned} \tag{4.34}$$

Majorize either directly or with (2.7)(a) to get

$$\begin{aligned} &\left|(\mathbf{u}_{n+1/2}\cdot\nabla\mathbf{u}_{n+1/2},\mathbf{v})-\frac{1}{2}(\mathbf{u}_{n+1}\cdot\nabla\mathbf{u}_{n+1},\mathbf{v})-\frac{1}{2}(\mathbf{u}_{n}\cdot\nabla\mathbf{u}_{n},\mathbf{v})\right| \\ &\qquad\le C\|\nabla\mathbf{u}\|_{l^\infty(L^2(\Omega))}(|\mathbf{u}_{n+1}|_1+|\mathbf{u}_n|_1)\left|\mathbf{v}^h\right|_1, \end{aligned} \tag{4.35}$$

or with (2.7)(b), (2.7)(c) and Hölder's inequality (in time) applied to (4.34) to get

$$\begin{aligned} &\left|(\mathbf{u}_{n+1/2}\cdot\nabla\mathbf{u}_{n+1/2},\mathbf{v})-\frac{1}{2}(\mathbf{u}_{n+1}\cdot\nabla\mathbf{u}_{n+1},\mathbf{v})-\frac{1}{2}(\mathbf{u}_{n}\cdot\nabla\mathbf{u}_{n},\mathbf{v})\right| \\ &\quad\le C\|\mathbf{u}\|_{l^\infty(H^2(\Omega))}\|\mathbf{u}_{n+1/2}-\mathbf{u}(\cdot,t_{n+1/2})\|\left|\mathbf{v}^h\right|_1 \\ &\qquad+\frac{C\Delta t^{3/2}}{\sqrt{t^{n+1/2}}}\|\mathbf{u}\|_{l^\infty(n,n+1;H^2(\Omega))}\left(\int_{t_n}^{t_{n+1}} t\;\|\mathbf{u}_{tt}(\cdot,t)\|^2dt\right)^{1/2}\left|\mathbf{v}^h\right|_1 \\ &\qquad+\frac{C\Delta t^{3/2}}{\sqrt{t^{n+1/2}}}\|\mathbf{u}_t\|_{l^\infty(n,n+1;L^2)}\left(\int_{t_n}^{t_{n+1}} t\|\mathbf{u}_{t}(\cdot,t)\|_2^2\;dt\right)^{1/2}\left|\mathbf{v}^h\right|_1. \end{aligned} \tag{4.36}$$

Then, to prove Corollary 4.7, apply (4.32), (4.36) with Young's inequality give

$$\begin{aligned} \left|\tau_n^{(1)}\left(\mathbf{u},p;\phi_{n+1/2}^h\right)\right| &\le \frac{2}{\varepsilon\,\mathrm{Re}}\left|\phi_{n+1/2}^h\right|_1^2+C\,\mathrm{Re}\left\|\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t}-(\mathbf{u}_t)_{n+1/2}\right\|_{-1}^2 \\ &+C\,\mathrm{Re}\,\Delta t^3\|\mathbf{u}\|_{l^\infty(H^2)}^2\|\mathbf{u}_{tt}\|_{L^2(t^n,t^{n+1};L^2(\Omega))}^2+C\,\mathrm{Re}\,\Delta t^3\|\mathbf{u}_t\|_{l^\infty(L^2(\Omega))}^2\|\mathbf{u}_t\|_{L^2(t^n,t^{n+1};H^2(\Omega))}^2. \end{aligned} \tag{4.37}$$

We apply (4.35) instead of (4.36) to prove Theorem 4.6:

$$\begin{aligned} \left|\tau_n^{(1)}\left(\mathbf{u},p;\phi_{n+1/2}^h\right)\right| &\le \frac{2}{\varepsilon\,\mathrm{Re}}\left|\phi_{n+1/2}^h\right|_1^2+C\,\mathrm{Re}\left\|\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t}-(\mathbf{u}_t)_{n+1/2}\right\|_{-1}^2 \\ &\qquad+C\,\mathrm{Re}\,\|\nabla\mathbf{u}\|_{l^\infty(L^2(\Omega))}^2\left(|\mathbf{u}_{n+1}|_1^2+|\mathbf{u}_n|_1^2\right). \end{aligned} \tag{4.38}$$

(Deconvolution error): Next, add/subtract $\varphi(\mathbf{u}_{n+1/2})$ we write

$$\begin{aligned} \tau_n^{(2)}\left(\mathbf{u},p;\mathbf{v}^h\right) = &-b^h\left(\mathbf{u}_{n+1/2}-\varphi(\mathbf{u}_{n+1/2}),\mathbf{u}_{n+1/2},\phi_{n+1/2}^h\right) \\ &-b^h\left(\left(\varphi(\mathbf{u}_{n+1/2})-\varphi^h(\mathbf{u}_{n+1/2})\right),\mathbf{u}_{n+1/2},\phi_{n+1/2}^h\right) \\ &+\chi\left(\mathbf{u}_{n+1/2}-\varphi(\mathbf{u}_{n+1/2}),\phi_{n+1/2}^h\right)+\chi\left(\varphi(\mathbf{u}_{n+1/2})-\varphi^h(\mathbf{u}_{n+1/2}),\phi_{n+1/2}^h\right). \end{aligned} \tag{4.39}$$

Then duality on $X \times X'$ and Young's inequalities give

$$
\begin{aligned}
&\left|\chi\left(\mathbf{u}_{n+1/2}-\varphi(\mathbf{u}_{n+1/2}),\phi^h_{n+1/2}\right)+\chi\left(\varphi(\mathbf{u}_{n+1/2})-\varphi^h(\mathbf{u}_{n+1/2}),\phi^h_{n+1/2}\right)\right|\\
&\quad\le C\chi^2\operatorname{Re}\left\|\mathbf{u}_{n+1/2}-\varphi(\mathbf{u}_{n+1/2})\right\|_{-1}^2\\
&\qquad+C\chi^2\operatorname{Re}\left\|\varphi(\mathbf{u}_{n+1/2})-\varphi^h(\mathbf{u}_{n+1/2})\right\|_{-1}^2+\frac{1}{\varepsilon\operatorname{Re}}\left|\phi^h_{n+1/2}\right|_1^2,
\end{aligned}
\tag{4.40}
$$

and $\mathbf{u}\in H^2(\Omega)$ along with (2.11)(b) give

$$
\begin{aligned}
&\left|b^h\left(\mathbf{u}_{n+1/2}-\varphi(\mathbf{u}_{n+1/2}),\mathbf{u}_{n+1/2},\phi^h_{n+1/2}\right)+b^h\left(\varphi(\mathbf{u}_{n+1/2})-\varphi^h(\mathbf{u}_{n+1/2}),\mathbf{u}_{n+1/2},\phi^h_{n+1/2}\right)\right|\\
&\quad\le C\operatorname{Re}\|\mathbf{u}_{n+1/2}\|_2^2\left(\left\|\mathbf{u}_{n+1/2}-\varphi(\mathbf{u}_{n+1/2})\right\|^2+\left\|\varphi(\mathbf{u}_{n+1/2})-\varphi^h(\mathbf{u}_{n+1/2})\right\|^2\right)+\frac{1}{\varepsilon\operatorname{Re}}\left|\phi^h_{n+1/2}\right|_1^2.
\end{aligned}
\tag{4.41}
$$

The estimates (4.37), (4.40), (4.41) with identity (4.39) give

$$
\left|\tau_n^{(1)}\left(\mathbf{u},p;\phi^h_{n+1/2}\right)+\tau_n^{(2)}\left(\mathbf{u},p;\phi^h_{n+1/2}\right)\right|\le\frac{4}{\varepsilon\operatorname{Re}}\left|\phi^h_{n+1/2}\right|_1^2+C\operatorname{Re}E_n.
\tag{4.42}
$$

Then estimates (3.17) and (3.23) give

$$
\begin{aligned}
E_n:=&\left(\chi^2+\|\mathbf{u}\|^2_{l^\infty(H^2(\Omega))}\right)\left(c_{1,j}(\alpha,\delta)^2\left\|\Delta^{j+1}\mathbf{u}_{n+1/2}\right\|^2+c_{2,j}(h,\alpha,\delta)^2\|\mathbf{u}_{n+1/2}\|^2_{k+1}\right)\\
&+\left\|\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t}-(\mathbf{u}_t)_{n+1/2}\right\|_{-1}^2+\|\mathbf{u}\|^2_{l^\infty(H^2)}\|\mathbf{u}_{n+1/2}-\mathbf{u}(\cdot,t_{n+1/2})\|^2\\
&+\Delta t^3\|\mathbf{u}\|^2_{l^\infty(H^2)}\|\mathbf{u}_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))}+\Delta t^3\|\mathbf{u}_t\|^2_{l^\infty(L^2(\Omega))}\|\mathbf{u}_t\|^2_{L^2(t^n,t^{n+1};H^2(\Omega))}.
\end{aligned}
\tag{4.43}
$$

Apply estimates from (4.24), (4.25), (4.31), (4.37), (4.42) to (4.23). Set $\varepsilon=20$ and absorb all terms including $|\phi^h_{n+1/2}|_1$ from the right into left-hand-side of (4.23). Sum the resulting inequality on both sides from $n=0$ to $n=M-1$ to get

$$
\begin{aligned}
&\left\|\phi^h_M\right\|^2+\operatorname{Re}^{-1}\Delta t\sum_{n=0}^{M-1}\left|\phi^h_{n+1/2}\right|_1^2+\chi\Delta t\sum_{n=0}^{M-1}\left\|\phi^h_{n+1/2}\right\|_{\star h}^2\\
&\quad\le\left\|\phi^h_0\right\|^2+C\operatorname{Re}\Delta t\sum_{n=0}^{M-1}E_n+C\operatorname{Re}\Delta t\sum_{n=0}^{M-1}\left\|p_{n+1/2}-\tilde{q}^h_{n+1/2}\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ C\,\mathrm{Re}\, d_1^2 \Delta t \sum_{n=0}^{M-1} \left(\|\nabla \mathbf{u}\|^2_{L^\infty(L^2(\Omega))} + \left|\boldsymbol{\eta}_{n+1/2}\right|_1^2 \right) \left|\boldsymbol{\eta}_{n+1/2}\right|_1^2 \\
&+ C\,\mathrm{Re}\, \Delta t \sum_{n=0}^{M-1} \left(\|\mathbf{u}_{n+1/2}\|_2^2 + h^{-1} \left|\boldsymbol{\eta}_{n+1/2}\right|_1^2 \right) \left\|\phi^h_{n+1/2}\right\|^2 \\
&+ C\chi^2\,\mathrm{Re}\, \Delta t \sum_{n=0}^{M-1} \left\|\boldsymbol{\eta}_{n+1/2}\right\|_{-1}^2 + C\mathrm{Re}^{-1} \Delta t \sum_{n=0}^{M-1} \left|\boldsymbol{\eta}_{n+1/2}\right|_1^2 .
\end{aligned}
\tag{4.44}
$$

Estimates (2.23), (2.16)(a) imply

$$
\sup_n \left|\boldsymbol{\eta}_n\right|_1 \le C \|\nabla \mathbf{u}\|_{l^\infty(L^2(\Omega))}, \qquad \left|\boldsymbol{\eta}_n\right|_1 \le Ch\|\mathbf{u}_n\|_2. \tag{4.45}
$$

These estimates applied to (4.44) give

$$
\begin{aligned}
&\left\|\phi^h_M\right\|^2 + \mathrm{Re}^{-1} \Delta t \sum_{n=0}^{M-1} \left|\phi^h_{n+1/2}\right|_1^2 + \chi \Delta t \sum_{n=0}^{M-1} \left\|\phi^h_{n+1/2}\right\|_{\star h}^2 \\
&\quad \le \left\|\phi^h_0\right\|^2 + C\,\mathrm{Re}\, \Delta t \sum_{n=0}^{M-1} E_n + C\,\mathrm{Re}\, \Delta t \sum_{n=0}^{M-1} \left\| p_{n+1/2} - \tilde{q}^h_{n+1/2} \right\|^2 \\
&\quad + C\,\mathrm{Re}\, \Delta t \sum_{n=0}^{M-1} \left(\left(d_1^2 \|\nabla \mathbf{u}\|^2_{L^\infty(L^2(\Omega))} + \mathrm{Re}^{-2} \right) \left|\boldsymbol{\eta}_{n+1/2}\right|_1^2 + \|\mathbf{u}_{n+1/2}\|_2^2 \left\|\phi^h_{n+1/2}\right\|^2 \right) \\
&\quad + C\chi^2\,\mathrm{Re}\, \Delta t \sum_{n=0}^{M-1} \left\|\boldsymbol{\eta}_{n+1/2}\right\|_{-1}^2 .
\end{aligned}
\tag{4.46}
$$

Suppose that the Δt-restriction (4.16) is satisfied. Then the discrete Gronwall Lemma 2.2 applies to (4.46) and gives

$$
\begin{aligned}
&\left\|\phi^h_M\right\|^2 + \mathrm{Re}^{-1} \Delta t \sum_{n=0}^{M-1} \left|\phi^h_{n+1/2}\right|_1^2 + \chi \Delta t \sum_{n=0}^{M-1} \left\|\phi^h_{n+1/2}\right\|_{\star h}^2 \\
&\quad \le G_M \left\|\phi^h_0\right\|^2 + G_M\,\mathrm{Re}\, \Delta t \sum_{n=0}^{M-1} E_n \\
&\quad + G_M\,\mathrm{Re}\, \Delta t \sum_{n=0}^{M-1} \left(\left\| p_{n+1/2} - \tilde{q}^h_{n+1/2} \right\|^2 + \left(d_1^2 \|\nabla \mathbf{u}\|^2_{L^\infty(L^2(\Omega))} + \mathrm{Re}^{-2} \right) \left|\boldsymbol{\eta}_{n+1/2}\right|_1^2 \right) \\
&\quad + G_M \chi^2\,\mathrm{Re}\, \Delta t \sum_{n=0}^{M-1} \left\|\boldsymbol{\eta}_{n+1/2}\right\|_{-1}^2 ,
\end{aligned}
\tag{4.47}
$$

where

$$G_M = C \exp\left(\operatorname{Re} \Delta t \sum_{n=0}^{M} g_n \|\mathbf{u}_n\|_2^2 \right), \qquad g_n = \left(1 - C \operatorname{Re} \Delta t \|\mathbf{u}_n\|_2^2\right)^{-1}. \tag{4.48}$$

Lastly, the triangle inequality and approximation theory estimates (2.23), (2.16) along with (3.23) applied to (4.47) give

$$\begin{aligned} &\left\|\mathbf{u}_M - \mathbf{w}_M^h\right\|^2 + \operatorname{Re}^{-1} \Delta t \sum_{n=0}^{M-1} \left|\mathbf{u}_{n+1/2} - \mathbf{w}_{n+1/2}^h\right|_1^2 \\ &\quad \le G_M \left\|\mathbf{e}_0^h\right\|^2 + Ch^{2k} \left(G_M \|\mathbf{u}_0\|_k + \|\mathbf{u}\|_{l^\infty(H^k(\Omega))}^2 + \operatorname{Re}^{-1} \|\mathbf{u}\|_{l^2(H^{k+1}(\Omega))}^2 \right) \\ &\qquad + G_M \operatorname{Re} \Delta t \sum_{n=0}^{M-1} E_n + G_M \operatorname{Re} h^{2s+2} \|p\|_{l^2(H^{s+1}(\Omega))}^2 \\ &\qquad + G_M \operatorname{Re}\left(d_1^2 \|\nabla \mathbf{u}\|_{L^\infty(L^2(\Omega))}^2 + \operatorname{Re}^{-2} + \chi^2 h^4 \right) h^{2k} \|\mathbf{u}\|_{l^2(H^{k+1}(\Omega))}^2. \end{aligned} \tag{4.49}$$

It remains to bound $\Delta t \sum_n E_n$.

(Theorem 4.6): Suppose that $\partial_t \mathbf{u} \in L^2(X') \cap l^2(X')$. The triangle inequality and (2.18) gives

$$\Delta t \sum_{n=n_0}^{M-1} \left\| \frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - (\mathbf{u}_t)_{n+1/2} \right\|_{-1}^2 \le C\left(\|\mathbf{u}_t\|_{L^2(X')}^2 + \|\mathbf{u}_t\|_{l^2(X')}^2 \right). \tag{4.50}$$

Apply (4.38), instead of (4.37), to derive E_n in (4.43). Then

$$\operatorname{Re} \Delta t \sum_{n=0}^{M-1} E_n \le E, \tag{4.51}$$

where

$$\begin{aligned} E &:= C\operatorname{Re}\left(\chi^2 + \|\mathbf{u}\|_{l^\infty(H^2(\Omega))}^2 \right) \left(c_{1,j}(\alpha,\delta)^2 \left\| \Delta^{j+1} \mathbf{u}_{n+1/2} \right\|^2 + c_{2,j}(h,\alpha,\delta)^2 \|\mathbf{u}_{n+1/2}\|_{k+1}^2 \right) \\ &\quad + C\operatorname{Re}\left(\|\mathbf{u}_t\|_{L^2(X')}^2 + \|\mathbf{u}_t\|_{l^2(X')}^2 + \|\nabla \mathbf{u}\|_{l^\infty(L^2(\Omega))}^2 \|\nabla \mathbf{u}\|_{l^2(L^2(\Omega))}^2 \right). \end{aligned} \tag{4.52}$$

(Corollary 4.7): Suppose that $\mathbf{u} \in l^{\infty}(H^2(\Omega))$, $\mathbf{u}_t \in l^{\infty}(H^1(\Omega)) \cap L^2(H^1(\Omega))$, $\mathbf{u}_{tt} \in L^2(L^2(\Omega))$, and $\mathbf{u}_{ttt} \in L^2(X')$. Write

$$E := C\,\mathrm{Re}\Big(\chi^2 + \|\mathbf{u}\|^2_{l^{\infty}(H^2(\Omega))}\Big)\Big(c_{1,j}(\alpha,\delta)^2\Big\|\Delta^{j+1}\mathbf{u}_{n+1/2}\Big\|^2 + c_{2,j}(h,\alpha,\delta)^2\|\mathbf{u}_{n+1/2}\|^2_{k+1}\Big)$$
$$:= C\,\mathrm{Re}\Big(\|\mathbf{u}_{ttt}\|^2_{L^2(X')} + \|\mathbf{u}\|^2_{l^{\infty}(H^2(\Omega))}\|\mathbf{u}_{tt}\|^2_{L^2(L^2(\Omega))} + \cdots + \|\mathbf{u}_t\|^2_{l^{\infty}(L^2(\Omega))}\|\mathbf{u}_t\|^2_{L^2(H^2(\Omega))}\Big). \tag{4.53}$$

Then apply (2.19), (2.20) to bound E_n given in (4.43):

$$\mathrm{Re}\,\Delta t\sum_{n=0}^{M-1}E_n \le E\Delta t^4. \tag{4.54}$$

□

5. Applications

We show that the iterated (modified) Tikhonov regularization operator satisfied Assumption 3.4, 3.5 in Section 5.1 and verify the theoretical convergence rate predicted by Theorem 4.6, Corollary 4.7 in Section 5.2.

5.1. Iterated (Modified) Tikhonov-Lavrentiev Regularization

We will prove that $D_{\alpha,J}$, $D^h_{\alpha,J}$ (Definitions 3.14, 3.15) with the differential filter $G = A^{-1}$ satisfies Assumptions 3.4, 3.5. Proposition 3.10 implies that it is enough to show that $D_{\alpha,0}$ satisfies Assumption 3.4. Additionally, we provide sharpened estimates for $d_{1,j}$, $c_{1,j}$, $c_{2,j}$. The key is that $A^{-1} > 0$ is a continuous function of the Laplace operator $-\Delta \ge 0$ and hence they commute (on X). Moreover, $D_{\alpha,0} > 0$ is a continuous function of A^{-1} so that $D_{\alpha,0}$ commutes with A^{-1} and $-\Delta$ (on X).

We first characterize the spectrum of $D_{\alpha,0}$, $D^h_{\alpha,0}$.

Lemma 5.1. *Fix $0 < \alpha \le 1$. Define $f : (0,1] \to \mathbb{R}$ and $g : (0,1] \to \mathbb{R}$ by*

$$f(x) := \frac{1}{(1-\alpha)x+\alpha}, \qquad g(x) := \frac{x}{(1-\alpha)x+\alpha}. \tag{5.1}$$

The maps f and g are continuous and $f((0,1]) = [1,\alpha^{-1})$ and $g((0,1]) = (0,1]$.

Proof. The functions f, g are clearly continuous with f *decreasing* and g increasing on $(0,1]$. Hence, the range of f is $[1,\alpha^{-1})$ and range of g is $(0,1]$. □

The next result shows that $D_{\alpha,0}$ and $D^h_{\alpha,0}$ satisfy part of Assumptions 3.4, 3.5.

Proposition 5.2. *$D_{\alpha,0}$ and $D^h_{\alpha,0}$ (on X and X^h, resp.) are linear, bounded, spd, and commute with A^{-1}, $(A^h)^{-1}$ (resp.). Moreover,*

$$\begin{aligned} \left\| D_{\alpha,0} A^{-1} \right\| &\leq 1, \qquad |D_{\alpha,0}\overline{\mathbf{u}}|_1 \leq |\mathbf{u}|_1 \quad \forall \mathbf{u} \in X, \\ \left\| D^h_{\alpha,0} \left(A^h\right)^{-1} \right\| &\leq 1, \qquad \left| D^h_{\alpha,0} \overline{\mathbf{u}^h}^h \right|_1 \leq \left| \mathbf{u}^h \right|_1 \quad \forall \mathbf{u}^h \in X^h. \end{aligned} \tag{5.2}$$

Hence $d_1 = 1$ in Assumptions 3.4, 3.5.

Proof. It is immediately clear that $D_{\alpha,0}$, $D^h_{\alpha,0}$ are linear. As a consequence, since $A^{-1} > 0$ with spectrum in $(0,1]$, then $D_{\alpha,0} = f(A^{-1})$ with spectrum contained in $[1, \alpha^{-1})$ so that $D > 0$. Therefore, $D_{\alpha,0}A^{-1} = g(A^{-1})$ with spectrum contained in $(0,1]$. A similar argument shows that $(A^h)^{-1}$ has spectrum in $(0,1]$, $D^h_{\alpha,0}$ has spectrum in $[1, \alpha^{-1})$, and $D^h_{\alpha,0}(A^h)^{-1}$ has spectrum in $(0,1]$. Thus $D_{\alpha,0} > 0$, $D^h_{\alpha,0} > 0$ and $\|D_{\alpha,0}A^{-1}\| \leq 1$ and $\|D^h_{\alpha,0}(A^h)^{-1}\| \leq 1$. Therefore, $D_{\alpha,0}$ and $D^h_{\alpha,0}$ are bounded and commute with A^{-1} and $(A^h)^{-1}$, respectively, as discussed above.

The second set of inequalities on each line can be proved with an appropriate choice of $\mathbf{v}$ and $\mathbf{v}^h$ in Definitions 3.2 and 3.3. Starting with Definition 3.2, take $\phi = \mathbf{u}$ and choose $\mathbf{v} = \Delta D_{\alpha,0}\overline{\mathbf{u}}$. Then integration by parts and the Cauchy-Schwartz inequality give the result. The discrete form is proved using Definition 3.3 and choosing $\phi = \mathbf{u}^h$ and $\mathbf{v} = \Delta D^h_{\alpha,0}\overline{\mathbf{u}^h}^h$. □

It remains to provide estimates for c_1 and c_2, and sharpened estimates for $c_{1,j}$ and $c_{2,j}$. Indeed, as a direct consequence of Propositions 5.3, 5.4, we have, for each $j = 0, 1, \ldots, J$,

$$\begin{aligned} c_{1,j} &= \left(\alpha\delta^2\right)^{j+1} \left\| \Delta^{j+1}\mathbf{v} \right\|, \quad \forall \mathbf{v} \in H^{j+1}(\Omega), \\ c_{2,j} &= C\left(h + \left(2^j \alpha\delta^2\right)^{1/2}\right) h^k \max_{0\leq n\leq j} \|D_{\alpha,n}\overline{\mathbf{v}}\|_{k+1}, \quad \forall \mathbf{v} \in H^{k+1}(\Omega). \end{aligned} \tag{5.3}$$

Proposition 5.3. *Let $j = 0, 1, \ldots, J$. Then, for some $0 \leq \beta \leq j+1$,*

$$\|\mathbf{v} - D_{\alpha,j}\overline{\mathbf{v}}\| \leq \left(\alpha\delta^2\right)^{\beta} \left\| \Delta^{\beta}\mathbf{v} \right\|, \quad \forall \mathbf{v} \in H^{\beta}(\Omega). \tag{5.4}$$

Proof. Using (1.9), we have

$$D^{-1}_{\alpha,0}\left(D_{\alpha,J}\overline{\mathbf{v}} - D_{\alpha,J-1}\overline{\mathbf{v}}\right) = \overline{\mathbf{v}} - A^{-1}D_{\alpha,J-1}\overline{\mathbf{v}}. \tag{5.5}$$

Subtracting (5.5) from the identity

$$D^{-1}_{\alpha,0}(\mathbf{v} - \mathbf{v}) = \overline{\mathbf{v}} - \overline{\mathbf{v}}, \tag{5.6}$$

gives us

$$D^{-1}_{\alpha,0}\left(\left(I - D_{\alpha,J}A^{-1}\right)\mathbf{v} - \left(I - D_{\alpha,J-1}A^{-1}\right)\mathbf{v}\right) = -A^{-1}\left(I - D_{\alpha,J-1}A^{-1}\right)\mathbf{v}. \tag{5.7}$$

Multiplying by $D_{\alpha,0}$, rearranging, simplifying, and using $A - I = -\delta^2\Delta$ (Definition 3.2) gives

$$\begin{aligned}
\left(I - D_{\alpha,J}A^{-1}\right)\mathbf{v} &= \left[-D_{\alpha,0}A^{-1}\left(I - D_{\alpha,J-1}A^{-1}\right) + \left(I - D_{\alpha,J-1}A^{-1}\right)\right]\mathbf{v} \\
&= \left(I - D_{\alpha,J-1}A^{-1}\right)\left(I - D_{\alpha,0}A^{-1}\right)\mathbf{v} \\
&= \left(I - D_{\alpha,J-1}A^{-1}\right)D_{\alpha,0}A^{-1}\left(D_{\alpha,0}^{-1}A - I\right)\mathbf{v} \\
&= \left(I - D_{\alpha,J-1}A^{-1}\right)D_{\alpha,0}A^{-1}\left(\left((1-\alpha)A^{-1} + \alpha I\right)A - I\right)\mathbf{v} \\
&= \left(I - D_{\alpha,J-1}A^{-1}\right)D_{\alpha,0}A^{-1}\alpha(A - I)\mathbf{v} \\
&= -\alpha\delta^2\Delta D_{\alpha,0}A^{-1}\left(I - D_{\alpha,J-1}A^{-1}\right)\mathbf{v}.
\end{aligned} \tag{5.8}$$

Applying recursion, we obtain, for any $0 \le \beta < J$,

$$\left(I - D_{\alpha,J}A^{-1}\right)\mathbf{v} = \left(-\alpha\delta^2\right)^{\beta}\left(D_{\alpha,J-\beta}A^{-1}\right)^{\beta}\Delta^{\beta}\mathbf{v}. \tag{5.9}$$

Thus, taking norms and applying $\|D_{\alpha,J-\beta}A^{-1}\| \le 1$, we get (5.4). □

Proposition 5.4. *Let $j = 0, 1, \ldots, J$. Then*

$$\left\|D_{\alpha,j}\overline{\mathbf{w}} - D_{\alpha,j}^h\overline{\mathbf{w}}^h\right\|^2 \le C\left(h^2 + 2^{j+1}\alpha\delta^2\right)h^{2k}\max_{0\le n\le j}\|D_{\alpha,n}\overline{\mathbf{w}}\|_{k+1}^2, \quad \forall \mathbf{w} \in H^{k+1}(\Omega). \tag{5.10}$$

Proof. Let $\tilde{\mathbf{v}}_j^h \in X^h$ be the L^2-projection of $D_{\alpha,j}\overline{\mathbf{w}}$. Take $\mathbf{v} = \mathbf{v}^h$ in (3.34). For $j = 1, \ldots, J$, let $\mathrm{e}_j = D_{\alpha,j}\overline{\mathbf{w}} - D_{\alpha,j}^h\overline{\mathbf{w}}^h := \boldsymbol{\eta}_j - \phi_j^h$, where $\boldsymbol{\eta}_j := D_{\alpha,j}\overline{\mathbf{w}} - \tilde{\mathbf{v}}_j^h$, and $\phi_j^h := D_{\alpha,j}^h\overline{\mathbf{w}}^h - \tilde{\mathbf{v}}_j^h$. Subtract (3.34) and (3.35) to get

$$\alpha\delta^2\left(\nabla\phi_j^h, \nabla\mathbf{v}^h\right) + \left(\phi_j^h, \mathbf{v}^h\right) = \alpha\delta^2\left(\nabla\boldsymbol{\eta}_j, \nabla\mathbf{v}^h\right) + \alpha\delta^2\left(\nabla\mathbf{e}_{j-1}, \nabla\mathbf{v}^h\right). \tag{5.11}$$

Take $\mathbf{v}^h = \phi_j^h$ in (5.11) to get

$$\alpha\delta^2\left|\phi_j^h\right|_1^2 + \left\|\phi_j^h\right\|^2 = \alpha\delta^2\left(\nabla\boldsymbol{\eta}_j, \nabla\phi_j^h\right) + \alpha\delta^2\left(\nabla\mathbf{e}_{j-1}, \nabla\phi_j^h\right). \tag{5.12}$$

Fix $\varepsilon > 0$. Apply Hölder's and Young's inequalities to (5.12) to get

$$\alpha\delta^2\left|\phi_j^h\right|_1^2 + \left\|\phi_j^h\right\|^2 \le \alpha\delta^2\left|\boldsymbol{\eta}_j\right|_1^2 + \varepsilon\alpha\delta^2\left|\mathbf{e}_{j-1}\right|_1^2 + \frac{1}{\varepsilon}\alpha\delta^2\left|\phi_j^h\right|_1^2. \tag{5.13}$$

Taking $\varepsilon = 1$ and $\varepsilon = 2$ in (5.13) gives

$$\left\|\phi_j^h\right\|^2 \leq \alpha\delta^2\left|\boldsymbol{\eta}_j\right|_1^2 + \alpha\delta^2\left|\mathbf{e}_{j-1}\right|_1^2. \tag{5.14}$$

$$\alpha\delta^2\left|\phi_j^h\right|_1^2 + 2\left\|\phi_j^h\right\|^2 \leq 2\alpha\delta^2\left|\boldsymbol{\eta}_j\right|_1^2 + 4\alpha\delta^2\left|\mathbf{e}_{j-1}\right|_1^2. \tag{5.15}$$

The triangle inequality and estimate (5.14) give

$$\left\|\mathbf{e}_j\right\|^2 \leq \left\|\boldsymbol{\eta}_j\right\|^2 + \alpha\delta^2\left|\boldsymbol{\eta}_j\right|_1^2 + \alpha\delta^2\left|\mathbf{e}_{j-1}\right|_1^2. \tag{5.16}$$

Backward induction, estimate (5.15), and (2.23) give

$$\left\|\mathbf{e}_j\right\|^2 \leq \left|\mathbf{e}_0\right|_1^2 + \left(h^2 + \alpha\delta^2\left(1 + \sum_{i=0}^{j} 2^i\right)\right)\max_{0\leq n\leq j}\inf_{\mathbf{v}^h\in X^h}\left|D_{\alpha,n}\overline{\mathbf{w}} - \mathbf{v}^h\right|_1^2. \tag{5.17}$$

It has been shown (Estimate (2.36) in the proof of Lemma 2.7 [25]) that

$$\left|\mathbf{e}_0\right|_1^2 \leq C\left(h^2 + \alpha\delta^2\right)h^{2k}\left|D_0\overline{\mathbf{w}}\right|_{k+1}^2. \tag{5.18}$$

Note that $\sum_{i=0}^{j} 2^i = 2^{j+1} - 1$. Then, along with application of (2.16), we prove (5.10). □

Corollary 5.5 (convergence estimate). *Under the assumptions of Corollary 4.7, suppose further that, for some $J = 0, 1, \ldots$, that $G = A^{-1}$, $G^h = (A^h)^{-1}$, $D = D_{\alpha,J}$, $D^h = D_{\alpha,J}^h$. If $\Delta^\beta \mathbf{u} \in L^2(\Omega)$ for some $0 \leq \beta \leq J+1$, then*

$$\begin{aligned}
&\left\|\mathbf{u}_M - \mathbf{w}_M^h\right\|^2 + \mathrm{Re}^{-1}\Delta t\sum_{n=0}^{M-1}\left|\mathbf{u}_{n+1/2} - \mathbf{w}_{n+1/2}^h\right|_1^2 \\
&\quad\leq C\left(\left(1 + h^2 + 2^J\alpha\delta^2\right)h^{2k} + h^{2s+2} + \Delta t^4 + \left(\alpha\delta^2\right)^{2\beta}\right).
\end{aligned} \tag{5.19}$$

Proof. Apply estimates for $c_{1,j}$, $c_{2,j}$ from (5.3), resulting from Propositions 5.3, 5.4. □

5.2. Numerical Testing

This section presents the calculation of a flow with an exact solution to verify the convergence rates of the algorithm. FreeFEM++ [32] was used to run the simulations. The convergence

Table 1: Error and convergence rates for Leray-deconvolution with $J = 0$ for the Taylor-Green vortex with $\mathrm{Re} = 10,000$, $\alpha = \sqrt{h}$, and $\delta = \sqrt[4]{h}$. Note the convergence rate is approaching 1 as predicted by (5.21).

m (=1/h)	$\|u - w^h\|_{\infty,0}$	Rate	$\|\nabla(u - w^h)\|_{2,0}$	Rate
20	0.038975		1.651230	
40	0.024334	0.680	1.468510	0.169
60	0.017751	0.778	1.159840	0.582
80	0.013854	0.862	0.935247	0.748
100	0.011255	0.931	0.774285	0.846

Table 2: Error and convergence rates for Leray-deconvolution with $J = 1$ for the Taylor-Green vortex with $\mathrm{Re} = 10,000$, $\alpha = \sqrt{h}$, and $\delta = \sqrt[4]{h}$. Note the convergence rate is approaching 2 as predicted by (5.21).

m (=1/h)	$\|u - w^h\|_{\infty,0}$	Rate	$\|\nabla(u - w^h)\|_{2,0}$	Rate
20	0.023384		1.070400	
40	0.009739	1.264	0.640360	0.741
60	0.004997	1.646	0.357779	1.436
80	0.002899	1.892	0.212560	1.810
100	0.001915	1.858	0.136724	1.977

rates are tested against the Taylor-Green vortex problem [13, 33–35]. We use a domain of $\Omega = (0,1) \times (0,1)$ and take $\mathbf{u} = (u_1, u_2)$, where

$$\begin{aligned} u_1(x,y,t) &= -\cos(n\pi x)\sin(n\pi y)e^{-2n^2\pi^2 t/\tau}, \\ u_2(x,y,t) &= \sin(n\pi x)\cos(n\pi y)e^{-2n^2\pi^2 t/\tau}, \\ p(x,y,t) &= -\frac{1}{4}(\cos(n\pi x) + \cos(n\pi y))e^{-2n^2\pi^2 t/\tau}. \end{aligned} \tag{5.20}$$

The pair $(\mathbf{u}, p)$ is a solution the two-dimensional NSE when $\tau = \mathrm{Re}$ and $\mathbf{f} = 0$.

We used CN discretization in time and P2-P1 elements in space according to Problem 1. That is, we used continuous piecewise quadratic elements for the velocity and continuous piecewise linear elements for the pressure. We chose the spatial discretization elements and parameters $n = 1$, $T = 0.5$, $\chi = 0.1$ and $\mathrm{Re} = 10,000$ as a illustrative example. We chose $h = 1/m$, $dt = (1/4)h$, $\delta = \sqrt[4]{h}$ and $\alpha = \sqrt{h}$, where m is the number of mesh divisions per side of $[0,1]$. These were chosen so that the result of Corollary 5.5 reduces to

$$\left\|\mathbf{u}_M - \mathbf{w}_M^h\right\| + \left[\mathrm{Re}^{-1}\Delta t\sum_{n=0}^{M-1}\left|\mathbf{u}_{n+1/2} - \mathbf{w}_{n+1/2}^h\right|_1^2\right]^{1/2} \le C\left(h^2 + h^{J+1}\right). \tag{5.21}$$

We summarize the results in Tables 1 and 2. Table 1 displays error estimates corresponding to no iterations; that is, $J = 0$ in Definition 3.14. For the particular choice of α and δ, the computed errors $\|u - w^h\|_{\infty,0}$ and $\|\nabla(u - w^h)\|_{2,0}$ tend to the predicted convergence rate $\mathcal{O}(h)$. Table 2 displays error estimates corresponding to one update; that is, when $J = 1$ in Definition 3.14. Again, for the particular choice of α and δ,the computed errors $\|u - w^h\|_{\infty,0}$ and $\|\nabla(u - w^h)\|_{2,0}$ tend to the predicted convergence rate $\mathcal{O}(h^2)$.

6. Conclusion

It is infeasible to resolve all persistent and energetically significant scales down to the Kolmogorov microscale of $\mathcal{O}(\mathrm{Re}^{-3/4})$ for turbulent flows in complex domains using direct numerical simulations in a given time constraint. Regularization methods are used to find approximations to the solution. The modification of iterated Tikhonov-Lavrentiev to the modified iterated Tikhonov-Lavrentiev deconvolution in Definition 3.14 is a highly accurate method of solving the deconvolution problem in the Leray-deconvolution model, with errors $\mathbf{u} - D_{\alpha,0}\overline{\mathbf{u}} = \mathcal{O}((\alpha\delta^2)^{J+1})$ when applied to the differential filter. We use this result to show that under a regularity assumption, the error between the solutions to the NSE and to the Leray deconvolution model with time relaxation using the modified iterated Tikhonov-Lavrentiev deconvolution and discretized with CN in time and FE's in space are $\mathcal{O}(h^k(h + \sqrt{\alpha\delta^2}) + h^{s+1} + \Delta t^2 + (\alpha\delta^2)^{J+1})$.

We also examined the Taylor-Green vortex problem using Problem 1 with the deconvolution in Definition 3.14. We use this problem because it has an exact analytic solution to the NSE. The regularization parameters α and δ were chosen so that the convergence of the approximate solution to the error would be $\mathcal{O}(h^{J+1})$ for $J = 0$ and $J = 1$. The convergence rates calculated correspond to those predicted, that is $\mathcal{O}(h^1)$ for $J = 0$ and $\mathcal{O}(h^2)$ for $J = 1$.

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