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Research Article

Stability of Rotation Pairs of Cycles for the Interval Maps

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Let $C^0(I)$ be the set of all continuous self-maps of the closed interval I, and $\mathbf{P}(u,v) = \{f \in C^0(I) : f$ has a cycle with rotation pair $(u,v)\}$ for any positive integer $v > u$. In this paper, we prove that if $(2^m ns, 2^m nt) \dashv (\gamma, \lambda)$, then $\mathbf{P}(2^m ns, 2^m nt) \subset \operatorname{int} \mathbf{P}(\gamma, \lambda)$, where $m \geq 0$ is integer, $n \geq 1$ odd, $1 \leq s < t$ with s, t coprime, and $1 \leq \gamma < \lambda$.

1. Introduction

Let $C^0(I)$ be the set of all continuous self-maps of the closed interval I. For any $f, g \in C^0(I)$, we define the distance between f and g by

$$d(f,g) = \sup_{x \in I} |f(x) - g(x)|. \tag{1.1}$$

Then $(C^0(I), d)$ becomes a metric space. For any subset M of $C^0(I)$, we use $\operatorname{int} M$ to denote the interior of M. A point $x \in I$ is called a *periodic point* of f with period n if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leq i \leq n-1$, and $\{f^i(x) : 0 \leq i \leq n-1\}$ is called a *cycle* with period n. Write $F(f) = \{x : f(x) = x\}$, which is called the set of fixed points of f. For any subset $A \subset I$, we use $\#A$ and $[A]$ to denote the cardinal number of A and the smallest closed subinterval of I containing A, respectively. Write $[A] = [a; b]$ if $A = \{a, b\}$. For any positive integer n, write $\mathbf{P}(n) = \{f \in C^0(I) : f$ has a cycle with period $n\}$.

One of the remarkable results in one-dimensional dynamics is the *Sharkovskii theorem*. To state it, let us first introduce the *Sharkovskii ordering* for positive integers:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \cdots \rhd 2^k\cdot 3 \rhd 2^k\cdot 5 \rhd 2^k\cdot 7 \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1. \tag{1.2}$$

Theorem A (see [1]). *For any positive integers m and n, $\mathbf{P}(n) \subset \mathbf{P}(m)$ if $n \rhd m$.*

Block [2] studied stability of cycles in the theorem of Sarkovskii and obtained the following theorem.

Theorem B (see [2]). *For any positive integers m and n, $\mathbf{P}(n) \subset \operatorname{int}\mathbf{P}(m)$ if $n \rhd m$.*

Blokh [3] introduced the following ordering among all pairs of positive integers (k,l) with $k<l$.

(1) *If $u/v \neq 1/2$ and $k/l \in [1/2, u/v)$ or $k/l \in (u/v, 1/2]$, then $(u,v) \dashv (k,l)$.*

(2) *If $u/v = k/l = m/n$, where m,n are coprime, then $(u,v) \dashv (k,l)$ if and only if $u/m \rhd k/m$.*

He also defined the rotation pair and the rotation number of cycles with period $n>1$ for the interval maps.

Definition 1.1 (see [3]). Let $f \in C^0(I)$, P be a cycle of f with period $n>1$, and $m = \#\{y \in P : f(y) < y\}$. Then (m,n) is called the *rotation pair* of P and m/n the *rotation number* of P.

For any positive integer $v > u$, write $\mathbf{P}(u,v) = \{f \in C^0(I) : f$ has a cycle with rotation pair $(u,v)\}$.

Theorem C (see [3]). *For any positive integers $v>u$ and $l>k$, $\mathbf{P}(u,v) \subset \mathbf{P}(k,l)$ if $(u,v) \dashv (k,l)$.*

In this paper, we will study stability of rotation pairs of cycles for the interval maps. Our main result is the following theorem.

Theorem 1.2. *If $(2^m ns, 2^m nt) \dashv (\gamma,\lambda)$, then*

$$\mathbf{P}(2^m ns, 2^m nt) \subset \operatorname{int}\mathbf{P}(\gamma,\lambda), \tag{1.3}$$

where $m \geq 0$ is integer, $n \geq 1$ odd, $1 \leq s < t$ with s, t coprime, and $1 \leq \gamma < \lambda$.

2. Some Lemmas

In this section, we prove Theorem 1.2. To do this, we need the following definitions and lemmas.

Lemma 2.1 (see [4, Lemma 1.4]). *Let $f \in C^0(I)$. If $I_0, I_1, \ldots, I_m$ are compact subintervals of I with $I_m = I_0$ such that $f(I_{k-1}) \supset I_k$ for $1 \leq k < m$, then there exists a point y such that $f^m(y) = y$ and $f^k(y) \in I_k$ for every $0 \leq k < m$.*

Lemma 2.2. *Let $f \in C^0(I)$. If there are points a, b, and c such that $f(c) \leq a = f(a) < b < c \leq f(b)$ (resp., $f(c) \geq a = f(a) > b > c \geq f(b)$), then for any integers m and n with $m/n \leq 1/2$ (resp. $1/2 < m/n < 1$), f has a cycle $Q = \{y_1 < y_2 < \cdots < y_n\}$ with rotation pair (m,n) satisfying $f(y_i) > y_i$ for all $1 \leq i \leq n-m$ and $f(y_i) < y_i$ for all $n-m+1 \leq i \leq n$.*

Proof. We only prove the case $f(c) \leq a = f(a) < b < c \leq f(b)$ (the proof for the case $f(c) \geq a = f(a) > b > c \geq f(b)$ is similar).

We may assume that $(a,b) \cap F(f) = \emptyset$, then $f(x) > x$ for all $x \in (a,b)$. Choose $p \in (b,c) \cap F(f)$. Then there exist points $a < e_1 < e_2 < \cdots < e_{n-2m+1} < b$ such that $f(e_k) = e_{k+1}$ for every $1 \leq k \leq n-2m$ and $f(e_{n-2m+1}) = p$. Let

$$\begin{gathered} I_k = [e_k, e_{k+1}] \quad \text{if } 1 \leq k \leq n-2m, \\ I_{n-2m+2r+1} = [e_{n-2m+1}, b] \quad \text{if } \ 0 \leq r \leq m-2, \\ I_{n-2m+2r+2} = [p, c] \quad \text{if } \ 0 \leq r \leq m-1, \\ I_{n-1} = [b, p]. \end{gathered} \tag{2.1}$$

Then $f(I_i) \supset I_{i+1}$ for $i \in \{1,2,\ldots,n-1\}$ and $f(I_n) \supset I_1$. By Lemma 2.1, there exists a cycle $Q = \{x_1, x_2, \ldots, x_n\}$ such that $x_i \in I_i (1 \leq i \leq n)$. Furthermore, Q can be renumbered so that $Q = \{y_1 < y_2 < \cdots < y_n\}$ with the desirable properties. □

Lemma 2.3. *Let $f \in \mathbf{P}(m,n)$; then f has a cycle $Q = \{y_1 < y_2 < \cdots < y_n\}$ with rotation pair (m,n) such that $f(y_i) > y_i$ for all $1 \leq i \leq n-m$ and $f(y_i) < y_i$ for all $n-m+1 \leq i \leq n$.*

Proof. Let $P = \{x_1 < x_2 < \cdots < x_n\}$ be a cycle of f with rotation pair (m,n). We may assume that $m/n \leq 1/2$ (the proof for the case $1/2 < m/n < 1$ is similar). Let $s = \min\{k : f(x_k) < x_k\}$; then $s \geq 2$, $(x_{s-1}, x_s) \cap F(f) \neq \emptyset$, and $f(x_i) > x_i$ for each $1 \leq i \leq s-1$. We may also assume that there exists some $s < j \leq n$ such that $f(x_j) > x_j$ otherwise; let $Q = P$ which completes the proof of Lemma 2.3.

Let $t = \min\{k : k > s \text{ and } f(x_k) > x_k\}$ and $p = \max\{(x_s, x_t) \cap F(f)\}$. Then $f(x) > x$ for all $x \in (p, x_t)$. Let $j = \min\{k : f^{k+1}(x_t) \leq p\}$ and $i = \min\{k : k \leq j \text{ and } f^{k+1}(x_t) \geq f^j(x_t)\}$. Then $f^{j+1}(x_t) < p < f^i(x_t) < f^j(x_t) \leq f^{i+1}(x_t)$. It follows from Lemma 2.2 that f has a cycle $Q = \{y_1 < y_2 < \cdots < y_n\}$ such that Q with the desirable properties. □

Definition 2.4 (see [4]). Let $f \in C^0(I)$. A cycle P of f with odd period $n > 1$ is called a cycle of Stefan type if

$$P = \left\{f^{n-1}(c) < \cdots < f^2(c) < c < f(c) < \cdots < f^{n-2}(c)\right\} \tag{2.2}$$

or

$$P = \left\{f^{n-2}(c) < \cdots < f(c) < c < f^2(c) < \cdots < f^{n-1}(c)\right\}. \tag{2.3}$$

Definition 2.5 (see [4, 5]). Let $f \in C^0(I)$ and $P = \{x_1 < x_2 < \cdots < x_{n2^m}\}$ be a cycle with period $n2^m$, where $n \geq 1$ is odd and $m \geq 0$ is an integer. For each $0 \leq i \leq m$ and each $1 \leq j \leq 2^i$, write $A_{2^i}^j = \{x_{(j-1)2^{m-i}n+1} < x_{(j-1)2^{m-i}n+2} < \cdots < x_{j2^{m-i}n}\}$. We call P a strongly simple cycle if one of the following three conditions hold.

(1) If $m = 0$, then either $n = 1$ or $n > 1$ and P is a cycle of f of Stefan type, that is,

$$P = \left\{f^{n-1}(c) < \cdots < f^2(c) < c < f(c) < \cdots < f^{n-2}(c)\right\} \tag{2.4}$$

or

$$P = \left\{ f^{n-2}(c) < \cdots < f(c) < c < f^2(c) < \cdots < f^{n-1}(c) \right\}. \tag{2.5}$$

(2) If $n = 1$ and $m > 0$, then for each $1 \le i \le m$ and each $1 \le 2k \le 2^i$, $f^{2^{i-1}}(A_{2^i}^{2k-1}) = A_{2^i}^{2k}$ and $f^{2^{i-1}}(A_{2^i}^{2k}) = A_{2^i}^{2k-1}$.

(3) If $n > 1$ and $m > 0$, then the following three conditions hold.

(i) For each $1 \le i \le m$ and each $1 \le 2k \le 2^i$, $f^{2^{i-1}}(A_{2^i}^{2k-1}) = A_{2^i}^{2k}$, and $f^{2^{i-1}}(A_{2^i}^{2k}) = A_{2^i}^{2k-1}$.

(ii) For each $1 \le j \le 2^m$, $A_{2^m}^j$ is a cycle of f^{2^m} of Stefan type.

(iii) f maps each $A_{2^m}^i$ monotonically onto another $A_{2^m}^j$, with one exception.

Lemma 2.6 (see [4, 5]). *If $f \in C^0(I)$ has a cycle with period n, then f has a strongly simple cycle with period n.*

Let $P = \{x_1 < x_2 < \cdots < x_n\}$ be a cycle of f with period $n > 1$. Then there is a unique map $g : [x_1, x_n] \to [x_1, x_n]$, which is called the linearization of P, satisfying

(1) *$g(x_i) = f(x_i)$ for all $1 \le i \le n$,*

(2) *$g|_{[x_i, x_{i+1}]}$ is linear for all $1 \le i \le n-1$.*

By Theorem 7.5 of [4], we know that if g has a strongly simple cycle with rotation pair (p, q), then f has also a strongly simple cycle with rotation pair (p, q).

Lemma 2.7. *Let $f \in \mathbf{P}(ks, kt)$, where s, t are coprime, $k = n2^m$, $n \ge 1$ is odd, and $m \ge 0$ is an integer. Then f has a cycle $P = \{z_1 < z_2 < \cdots < z_{kt}\}$ with rotation pair (ks, kt) satisfying*

(1) *$f(y) < y$ if $y \in B_i = \{z_{(i-1)2^m n+1}, \ldots, z_{i2^m n}\}$ for $t - s + 1 \le i \le t$ and $f(y) > y$ if $y \in B_i = \{z_{(i-1)2^m n+1}, \ldots, z_{i2^m n}\}$ for $1 \le i \le t - s$;*

(2) *B_1 is a strongly simple cycle of f^t;*

(3) *f cyclically permutes the sets B_i ($i = 1, 2, \ldots, t$).*

Proof. By Lemma 2.3, we may assume that $R = \{x_1 < x_2 < \cdots < x_{kt}\}$ is a cycle of f with rotation pair (ks, kt) satisfying

$$\begin{aligned} f(y) > y \quad \forall y \in \{x_1, x_2, \ldots, x_{kt-ks}\}, \\ f(y) < y \quad \forall y \in \{x_{kt-ks+1}, \ldots, x_{kt}\}. \end{aligned} \tag{2.6}$$

Furthermore, we may assume that f is the linearization of R, $I = [x_1, x_{kt}]$, and p be the unique fixed point of f. Obviously, we have that $f(x) > x$ for all $x \in [x_1, p)$ and $f(x) < x$ for all $x \in (p, x_{kt}]$.

We may assume that $s/t \le 1/2$ (the proof for the case $1/2 < s/t < 1$ is similar). If $s/t = 1/2$, then it follows from Theorem 7.18 of [4] that Lemma 2.7 holds. Now we assume $s/t < 1/2$.

By Theorem C, f has a cycle $Q = \{y_1 < y_2 < \cdots < y_t\}$ $(t > 2)$ with rotation pair (s, t) satisfying $f(y) > y$ for all $y \in \{y_1, \ldots, y_{t-s}\}$ and $f(y) < y$ for all $y \in \{y_{t-s+1} \ldots, y_t\}$.

We can assume $k \geq 2$ since otherwise there is nothing to prove. Furthermore, we may assume $\#\{x \in Q : x > p\} \geq 2$ (the proof for the case $\#\{x \in Q : x < p\} \geq 2$ is similar). Write $x = \max Q$; then $f^t(x) = x$.

Claim 1. We may assume that there exists a positive number $\varepsilon > 0$ such that $f^t(y) > y$ for all $y \in (x, x + \varepsilon)$.

Proof of Claim 1. Since $x \notin R$, there exists a positive number $\varepsilon > 0$ such that $(f^t(y) - y)(y - x) > 0$ for all $y \in (x - \varepsilon, x + \varepsilon) - \{x\}$ or $(f^t(y) - y)(y - x) < 0$ for all $y \in (x - \varepsilon, x + \varepsilon) - \{x\}$. If $(f^t(y) - y)(y - x) > 0$ for all $y \in (x - \varepsilon, x + \varepsilon) - \{x\}$, then Claim 1 holds. Now we assume $(f^t(y) - y)(y - x) < 0$ for all $y \in (x - \varepsilon, x + \varepsilon) - \{x\}$. Write $u = \max\{y \in (p, x) : f^t(y) = y\}$ since $\#\{x \in Q : x > p\} \geq 2$.

We claim that for all $1 \leq i \leq t$, $(f^i(u) - p)(f^i(x) - p) > 0$. Indeed, if $(f^i(u) - p)(f^i(x) - p) < 0$ for some $1 \leq i \leq t$, then there exists a point $v \in (u, x)$ such that $f^i(v) = p$; thus $f^t(v) = p$, which implies $(u, v) \cap F(f^t) \neq \emptyset$. This is a contradiction.

We also claim $u = \max\{f^i(u) : 0 \leq i \leq t\}$. Indeed, if $f^i(u) > u$ for some $1 \leq i \leq t - 1$, then there exists a point $v \in (u, x)$ such that $f^i(v) = v$ since $x = \max Q$. Let $w = \max\{v \in (u, x) : f^i(v) = v\}$; then $f^{t-i}(w) = f^t(w) > w$. Since $f^{t-i}(x) < x$, there exists a point $e \in (w, x)$ such that $f^{t-i}(e) = e$, which implies $f^i(e) = f^t(e) > e$ and $(e, x) \cap F(f^i) \neq \emptyset$. This is a contradiction.

By using u to replace x, we know that Claim 1 holds. Claim 1 is proven. □

Write $S = \{y : f^t(y) = x\} \cap (x, x_{kt}]$. Let $T = \min S$ if $S \neq \emptyset$ and $T = x_{kt}$; otherwise. Put $J = (x, T)$.

Claim 2. $J, f(J), \ldots, f^{t-1}(J)$ are pairwise disjoint and $p \notin \bigcup_{i=0}^{t-1} f^i(J)$.

Proof of Claim 2. We first prove that $J, f(J), \ldots, f^{t-1}(J)$ are pairwise disjoint. Suppose that there exist $0 \leq i < j \leq t - 1$ and $u, v \in J$ such that $f^i(u) = f^j(v)$, then $f^{t-j+i}(u) = f^t(v) > x$. Since $f^{t-j+i}(x) < x$, there exists a point $y \in (x, u)$ such that $f^{t-j+i}(y) = x$, which implies $x > f^{j-i}(x) = f^t(y) > x$. This is a contradiction.

Now we prove $p \notin \bigcup_{i=0}^{t-1} f^i(J)$. Suppose that there exist some $0 \leq i \leq t - 1$ and $u \in J$ such that $f^i(u) = p$, then $f^t(u) = p$, hence $x \in f^t((x, u))$, which contradicts definition of T. Claim 2 is proven. □

By definition of T, it follows that $R \cap (\bigcup_{i=0}^{t-1} f^i(J)) \neq \emptyset$ since otherwise we have $f^t(T) > T$, which is impossible.

If $f^t(J) \subset J$, then $f^t|_J$ has a cycle with period k. It follows from Claim 2 and Lemma 2.6 that f has a cycle $P = \{z_1 < z_2 < \cdots < z_{kt}\}$ with rotation pair (ks, kt) satisfying conditions (1), (2), and (3) of Lemma 2.7.

If $f^t(J) \not\subset J$, then $f^t(T) = x$ and there exists a point $y \in J$ such that $f^t(y) \geq T$. Thus $f^t([x, y]) \cap f^t([y, T]) \supset [x, T]$. By Lemma 2.3 of [4], f^t has a cycle of period 3 on J. It follows from Claim 2, Theorem A, and Lemma 2.6 that f has a cycle $P = \{z_1 < z_2 < \cdots < z_{kt}\}$ with rotation pair (ks, kt) satisfying conditions (1), (2), and (3) of Lemma 2.7. Lemma 2.7 is proven. □

3. Proof of Theorem 1.2

In this section, we will give the proof of Theorem 1.2.

Proof of Theorem 1.2. We may assume $s/t \leq 1/2$ (the proof for the case $1/2 < s/t < 1$ is similar). Let $f \in \mathbf{P}(2^m ns, 2^m nt)$. We wish to show that there exists a neighbourhood U of f in $C^0(I)$ such that every $g \in U$ has a cycle with rotation pair (γ, λ). The proof will be carried out in a number of stages. □

Claim 3. If $m \geq 0$ and $n \geq 3$, then there exists a neighbourhood U of f in $C^0(I)$ such that every $g \in U$ has a cycle with rotation pair $(2^m(n+2)s, 2^m(n+2)t)$.

Proof of Claim 3. By Lemma 2.7, we know that f has a cycle $\{z_1 < z_2 < \cdots < z_{2^m nt}\}$ with rotation pair $(2^m ns, 2^m nt)$ satisfying

(1) $f(y) < y$ if $y \in B_i = \{z_{(i-1)2^m n+1}, \dots, z_{i2^m n}\}$ for $t - s + 1 \leq i \leq t$ and $f(y) > y$ if $y \in B_i = \{z_{(i-1)2^m n+1}, \dots, z_{i2^m n}\}$ for $1 \leq i \leq t - s$;

(2) B_1 is a strongly simple cycle of f^t;

(3) f cyclically permutes the sets B_i $(i = 1, 2, \dots, t)$.

For each $1 \leq l \leq 2^m$, let $z_1(l)$ denote the midpoint of the n points in $C_l = \{x_{(l-1)n+1} < \cdots < x_{ln}\}$ and $z_j(l) = f^{2^m t(j-1)}(z_1(l))$ $(1 < j \leq n)$. Then for each $1 \leq l \leq 2^m$, we have either

$$z_n(l) < z_{n-2}(l) < \cdots < z_3(l) < z_1(l) < z_2(l) < \cdots < z_{n-3}(l) < z_{n-1}(l) \tag{3.1}$$

or

$$z_n(l) > z_{n-2}(l) > \cdots > z_3(l) > z_1(l) > z_2(l) > \cdots > z_{n-3}(l) > z_{n-1}(l). \tag{3.2}$$

Furthermore, the blocks C_l can be renumbered so that $f^t(z_1(l)) = z_1(l+1)$ for $1 \leq l < 2^m$. Then

$$\begin{gathered} f^t(z_j(l)) = z_j(l+1) \quad \text{if } 1 \leq l < 2^m,\ 1 \leq j \leq n, \\ f^t(z_j(2^m)) = z_{j+1}(1) \quad \text{if } 1 \leq j < n, \\ f^t(z_n(2^m)) = z_1(1). \end{gathered} \tag{3.3}$$

Since $f^t([z_1(l); z_2(l)]) \supset [z_1(l+1); z_2(l+1)]$ for $1 \leq l < 2^m$ and $f^t([z_1(2^m); z_2(2^m)]) \supset [z_3(1); z_2(1)]$, there exist points $z_{-3}, z_{-2}, z_{-1}, z_0$ such that $f^{2^m t}(z_{-i}) = z_{-i+1}$ $(i = 0, 1, 2, 3)$ satisfying

(1) for $1 \leq l \leq 2^m$ either

$$z_1(l) < f^{(l-1)t}(z_{-1}) < f^{(l-1)t}(z_{-3}) < f^{(l-1)t}(z_{-2}) < f^{(l-1)t}(z_0) < z_2(l) \tag{3.4}$$

or

$$z_1(l) > f^{(l-1)t}(z_{-1}) > f^{(l-1)t}(z_{-3}) > f^{(l-1)t}(z_{-2}) > f^{(l-1)t}(z_0) > z_2(l); \tag{3.5}$$

(2) $f^{(l-1)t+i}(z_{-j}) \in [f^i(B_1)]$ for $1 \le l \le 2^m$ and $0 \le i < t$ and $j = 0, 1, 2, 3$.

Let

$$\varepsilon = \frac{\min\{\left|f^i(z_{-3}(1)) - f^j(z_{-3}(1))\right| : 0 \le i < j < 2^m(n+2)t\}}{10} \tag{3.6}$$

and $U = \{g \in C^0(I) : d(f^i, g^i) < \varepsilon$ for all $1 \le i \le 2^m(n+2)t\}$. Then for every $g \in U$ and $0 \le i < j \le 2^m(n+2)t-1$, we have $f^i(z_{-3}(1)) < f^j(z_{-3}(1))$ if and only if $g^i(z_{-3}(1)) < g^j(z_{-3}(1))$, and $g^{2^m(n+4)t}(z_{-3}(1)) \in [g^{2^m 2t}(z_{-3}(1)); g^{2^m 6t}(z_{-3}(1))]$. Put

$$I_l = \left[g^{l-1}(z_{-3}(1)); g^{2^m 2t+l-1}(z_{-3}(1))\right] \quad \text{for } 1 \le l \le 2^m(n+2)t. \tag{3.7}$$

Then we have

$$\begin{gathered} g(I_l) \supset I_{l+1}, \quad \text{if } 1 \le l \le 2^m(n+2)t-1, \\ g\left(I_{2^m(n+2)t}\right) \supset I_1. \end{gathered} \tag{3.8}$$

This yields a cycle $O = \{y, g(y), \dots, g^{2^m(n+2)t}(y)\}$ such that $g^{i-1}(y) \in I_i$ for $i = 1, 2, \dots, 2^m(n+2)t$. it is easy to verify that the rotation pair of O is $(2^m(n+2)s, 2^m(n+2)t)$. Claim 3 is proven. □

Claim 4. If $m \ge 1$ and $n = 1$, then there exists a neighourhood U of f in $C^0(I)$ such that every $g \in U$ has a cycle with rotation pair $(2^{m-1}s, 2^{m-1}t)$.

Proof of Claim 4. By Lemma 2.7, we know that f has a cycle $\{x_1 < x_2 < \cdots < x_{2^m t}\}$ with rotation pair $(2^m s, 2^m t)$ satisfying

(1) if $y \in B_i = \{x_{(i-1)2^m+1}, \dots, x_{i2^m}\}$ and $t - s + 1 \le i \le t$, then $f(y) < y$; if $y \in B_i = \{x_{(i-1)2^m+1}, \dots, x_{i2^m}\}$ and $1 \le i \le t - s$, then $f(y) > y$;

(2) B_1 is a strongly simple cycle of f^t;

(3) f cyclically permutes the sets B_i $(i = 1, 2, \dots, t)$.

Since $f^{2^{m-1}t}(x_1) = x_2$ and $f^{2^{m-1}t}(x_2) = x_1$, there exist points $x_1 \le a < b \le x_2$ such that $f^{2^{m-1}t}(b) = x_1 < x_2 = f^{2^{m-1}t}(a)$ and $f^i(a), f^i(b) \in [f^i(x_1); f^i(x_2)]$ for $0 \le i \le 2^{m-1}t - 1$. Let

$$\varepsilon = \frac{\min\{b - a, \min\{\left|f^i(x_1) - f^j(x_1)\right| : 0 \le i < j < 2^m t\}\}}{10} \tag{3.9}$$

and $U = \{g \in C^0(I) : d(f^i, g^i) < \varepsilon$ for all $1 \le i \le 2^{m-1}t\}$. Then for every $g \in U$, we have

$$g^{2^{m-1}t}(a) > a, \qquad g^{2^{m-1}t}(b) < b. \tag{3.10}$$

This yields a cycle $O = \{y, f(y), \dots, f^{2^{m-1}t-1}(y)\}$ such that $g^i(y) \in [g^i(a); g^i(b)]$ for $i = 0, 1, \dots, 2^{m-1}t - 1$. it is easy to verify that the rotation pair of O is $(2^{m-1}s, 2^{m-1}t)$. Claim 4 is proven. □

Claim 5. If $m = 0$ and $n = 1$, then there exists a neighourhood U of f in $C^0(I)$ such that every $g \in U$ has a cycle with rotation pair (γ, λ).

Proof of Claim 5. By Lemma 2.7, f, has a cycle $P = \{x_1 < x_2 < \cdots < x_t\}$ with rotation pair (s, t) such that $f(x_i) > x_i$ for all $1 \le i \le t - s$ and $f(x_i) < x_i$ for all $t - s + 1 \le i \le t$.

Choose two integers u, v with u, v coprime such that $s/t < u/v < \gamma/\lambda$. Without loss of generality, we can assume $f(x_{t-s+1}) < x_{t-s}$. Take $w \in F(f) \cap (x_{t-s}, x_{t-s+1})$. Put $1 \le l = tu - sv$, then there exist points $y_i \in (x_{t-s}, x_{t-s+1})$ $(i = 0, 1, \ldots, 2l-1)$ such that $x_{t-s} < y_{2l-2} < y_{2l-4} < \cdots < y_2 < y_0 < w < y_1 < y_3 < \cdots < y_{2l-3} < y_{2l-1} < x_{t-s+1}$ with $f(y_i) = y_{i+1}$ $(i = 0, 1, \ldots, 2l-2)$ and $f(y_{2l-1}) = x_{t-s}$. Let

$$\varepsilon = \frac{\min\{\left|f^i(y_0) - f^j(y_0)\right| : 0 \le i < j < 2l + t\}}{10} \tag{3.11}$$

and $U = \{g \in C^0(I) : d(f^i, g^i) < \varepsilon \text{ for all } 1 \le i \le (t - 2s)v\}$. Then for every $g \in U$, we have $g^{(t-2s)v}(y_0) < y_0 < g(y_0)$ and $g^2(y_0) < g(y_0)$.

Let $z = \max\{x \in I : x < y_0 \text{ and } g^{(t-2s)v}(x) = x\}$, $\alpha = \min\{z, g(z), \ldots, g^{(t-2s)v-1}(z)\}$, and $w_1 \in (y_0, g(y_0)) \cap F(g)$. □

Claim 6. $(g^i(z) - w_1)(g^i(y_0) - w_1) > 0$ for any $i \in \{0, 1, \ldots, (t - 2s)v\}$.

Proof of Claim 6. Assume on the contrary that $(g^i(z) - w_1)(g^i(y_0) - w_1) \le 0$ for some $i \in \{0, 1, \ldots, (t - 2s)v\}$; then there exists a point $c \in [z, y_0)$ such that $g^i(c) = w_1$; thus $g^{(t-2s)v}(c) = w_1$, which implies $(z, y_0) \cap F(g^{(t-2s)v}) \ne \emptyset$, a contradiction. □

Claim 7. If $[\alpha, y_0] \cap F(g) \ne \emptyset$, then g has a cycle with rotation pair (γ, λ).

Proof of Claim 7. Indeed, if $[\alpha, y_0] \cap F(g) \ne \emptyset$, let $z_0 = \max\{[\alpha, y_0] \cap F(g)\}$, then $z_0 \in [\alpha, z)$. Since $g(y_0) > y_0$ and $[z, y_0] \cap F(g) = \emptyset$, we have $g(z) > z$. Let $j = \min\{k : f^{k+1}(z) \le z_0\}$ and $i = \min\{k : k \le j \text{ and } f^{k+1}(z) \ge f^j(z)\}$. Then $f^{j+1}(z) \le z_0 < f^i(z) < f^j(z) \le f^{i+1}(z)$. It follows from Lemma 2.2 that g has a cycle with rotation pair (γ, λ). □

In the following, we assume that $[\alpha, y_0] \cap F(g) = \emptyset$.

Claim 8. $g^{i+1}(z) \in [g^i(z); w_1]$ or $w_1 \in [g^{i+1}(z); g^i(z)]$ for any $i \in \{0, 1, \ldots, (t - 2s)v\}$.

Proof of Claim 8. Assume on the contrary that $g^i(z) \in [g^{i+1}(z); w_1]$ for some $i \in \{0, 1, \ldots, (t - 2s)v\}$; then $g^i(z) > y_0$. Since $g^{i+1}(y_0) \in [g^i(y_0); w_1]$, we have $[g^i(z); g^i(y_0)] \cap F(g) \ne \emptyset$. Let $w_2 \in [g^i(z); g^i(y_0)] \cap F(g)$; then $w_2 > y_0$ and there exists a point $d \in [z, y_0]$ such that $g^{(t-2s)v}(d) = w_2$; thus, $(d, y_0] \cap F(g^{(t-2s)v}) \ne \emptyset$, a contradiction.

By Claims 7 and 8, we know that g has a cycle with rotation number u/v. It follows from Theorem C that g has a cycle with rotation pair (γ, λ), which completes the proof of Claim 5. □

Theorem 1.2 now follows immediately from Claim 3, Claim 4, Claim 5, and Theorem C.

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