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Research Article

Oscillation and Asymptotic Behavior of Higher-Order Nonlinear Differential Equations

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The aim of this paper is to offer a generalization of the Philos and Staikos lemma. As a possible application of the lemma in the oscillation theory, we study the asymptotic properties and oscillation of the nth order delay differential equations (E) $(r(t)[x^{(n-1)}(t)]^{\gamma})' + q(t)x^{\gamma}(\tau(t)) = 0$. The results obtained utilize also the comparison theorems.

1. Introduction

In this paper, we will study the asymptotic and oscillation behavior of the solutions of the higher-order advanced differential equations:

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' + q(t)x^{\gamma}(\tau(t)) = 0. \tag{E}$$

Throughout the paper, we will assume $q, \tau, r \in C([t_0, \infty))$, and

(H_1) $n \geq 3$, γ is the ratio of two positive odd integers,

(H_2) $r(t) > 0$, $q(t) > 0$, $\tau(t) \leq t$, $\lim_{t\to\infty}\tau(t) = \infty$.

Whenever, it is assumed

$$R(t) = \int_{t_0}^{t} r^{-1/\gamma}(s)\mathrm{d}s \longrightarrow \infty \quad \text{as } t \longrightarrow \infty. \tag{1.1}$$

By a solution of (E) we mean a function $x(t) \in C^{n-1}([T_x,\infty))$, $T_x \geq t_0$, which has the property $r(t)(x^{(n-1)}(t))^{\gamma} \in C^1([T_x,\infty))$ and satisfies (E) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$, and otherwise it is called to be nonoscillatory.

The problem of the oscillation of higher-order differential equations has been widely studied by many authors, who have provided many techniques for obtaining oscillatory criteria for studied equations (see, e.g., [1–19]).

Philos and Staikos lemma (see [16, 17]) essentially simplifies the examination of nth-order differential equations of the form

$$y^{(n)}(t) + q(t)y^{\gamma}(\tau(t)) = 0, \tag{1.2}$$

since it provides needed relationship between $y(t)$ and $y^{(n-1)}(t)$, and this fact permit us to establish just one condition for asymptotic behavior of (1.2). If we try to apply the Philos and Staikos lemma to (E), the strong condition $r'(t) \geq 0$ appears (see, e.g., [2, 18, 20]). In this paper we offer such generalization of the Philos and Staikos lemma, where this restriction is relaxed. Moreover, the obtained lemma yields many applications in the oscillation theory. As an example of it, we offer its disposal in the comparison theory and we establish new oscillation criteria for (E).

2. Main Results

The following result is a well-known lemma of Kiguradze, see, for example, [6] or [13].

Lemma 2.1. *Let $z(t) \in C^{k-1}([t_0,\infty))$ and $r(t)(z^{(k-1)}(t))^{\gamma} \in C^1([t_0,\infty))$ with $z(t) > 0$, $(r(t)(z^{(k-1)}(t))^{\gamma})' \leq 0$, and not identically zero on a subray of $[t_0,\infty)$. Then there exist a $t_1 \geq t_0$ and an integer ℓ, $0 \leq \ell \leq k-1$, with $k+\ell$ odd so that*

$$\begin{gathered} (-1)^{\ell+j} z^{(j)}(t) > 0, \quad j = \ell, \dots, k-1, \\ z^{(i)}(t) > 0, \quad i = 1, \dots, \ell-1, \ \textit{when } \ell > 1, \end{gathered} \tag{2.1}$$

on $[t_1,\infty)$.

Now we are prepared to provide a generalization of the Philos and Staikos lemma.

Lemma 2.2. *Let $z(t)$ be as in Lemma 2.1 and numbers t_1 and ℓ assigned to $z(t)$ by Lemma 2.1. Then for $0 < \ell < k-1$,*

$$z(t) \geq \frac{r^{1/\gamma}(t) z^{(k-1)}(t)}{(k-2)!} \int_{t_1}^{t} r^{-1/\gamma}(s)(s-t_1)^{k-2} \mathrm{d}s, \tag{2.2}$$

for $\ell = k-1$*;*

$$z(t) \geq \frac{r^{1/\gamma}(t) z^{(k-1)}(t)}{(k-2)!} \int_{t_1}^{t} r^{-1/\gamma}(s)(t-s)^{k-2} \mathrm{d}s, \tag{2.3}$$

for $t \geq t_1$.

Proof. Let ℓ be the integer assigned to function $z(t)$ as in Lemma 2.1. Assume that $\ell < k-1$, then, for any s, t with $t \geq s \geq t_1$,

$$\begin{aligned} -z^{(k-2)}(s) &\geq \int_s^t z^{(k-1)}(u) \mathrm{d}u = \int_s^t r^{1/\gamma}(u) z^{(k-1)}(u) r^{-1/\gamma}(u) \mathrm{d}u \\ &\geq r^{1/\gamma}(t) z^{(k-1)}(t) \int_s^t r^{-1/\gamma}(u) \mathrm{d}u. \end{aligned} \tag{2.4}$$

Repeated integration in s from s to t yields

$$z^{(\ell)}(s) \geq r^{1/\gamma}(t) z^{(k-1)}(t) \int_s^t r^{-1/\gamma}(u) \frac{(u-s)^{k-2-\ell}}{(k-2-\ell)!} \mathrm{d}u. \tag{2.5}$$

On the other hand, if $\ell \geq 1$, then, for every $t \geq t_1$,

$$z^{(\ell-1)}(t) \geq \int_{t_1}^{t} z^{(\ell)}(s) \mathrm{d}s. \tag{2.6}$$

Repeated integration from t_1 to t leads to

$$z(t) \geq \frac{1}{(\ell-1)!} \int_{t_1}^{t} z^{(\ell)}(s)(t-s)^{\ell-1} \mathrm{d}s. \tag{2.7}$$

Setting (2.5) into (2.7), one gets

$$\begin{aligned} z(t) &\geq \frac{r^{1/\gamma}(t) z^{(k-1)}(t)}{(\ell-1)!(k-\ell-2)!} \int_{t_1}^{t} (t-s)^{\ell-1} \int_s^t r^{-1/\gamma}(u)(u-s)^{k-2-\ell} \mathrm{d}u \, \mathrm{d}s \\ &\geq \frac{r^{1/\gamma}(t) z^{(k-1)}(t)}{(k-3)!} \int_{t_1}^{t} \int_s^t r^{-1/\gamma}(u)(u-s)^{k-3} \mathrm{d}u \, \mathrm{d}s \\ &= \frac{r^{1/\gamma}(t) z^{(k-1)}(t)}{(k-2)!} \int_{t_1}^{t} r^{-1/\gamma}(u)(u-t_1)^{k-2} \mathrm{d}s. \end{aligned} \tag{2.8}$$

We have verified the first part of the lemma. Now assume that $\ell = k-1$. It follows from (2.7) that

$$\begin{aligned} z(t) &\geq \frac{1}{(k-2)!}\int_{t_1}^{t} z^{(k-1)}(s)(t-s)^{k-2}\mathrm{d}u \\ &\geq \frac{1}{(k-2)!}\int_{t_1}^{t} r^{1/\gamma}(s)z^{(k-1)}(s)r^{-1/\gamma}(s)(t-s)^{k-2}\mathrm{d}s \\ &= \frac{r^{1/\gamma}(t)z^{(k-1)}(t)}{(k-2)!}\int_{t_1}^{t} r^{-1/\gamma}(s)(t-s)^{k-2}\mathrm{d}s. \end{aligned} \tag{2.9}$$

The proof is complete now. □

Employing additional conditions, we are able to joint (2.5) and (2.7) to just one estimate.

Lemma 2.3. *Let $z(t)$ be as in Lemma 2.1 and $\lim_{t\to\infty} z(t) \neq 0$. Let $r'(t) \geq 0$, Then for any $\lambda \in (0,1)$ there exists some $t_\lambda \geq t_1$ such that*

$$z(t) \geq \frac{\lambda}{(k-1)!}t^{k-1}z^{(k-1)}(t), \tag{2.10}$$

for $t \geq t_\lambda$.

Proof. Note that $r'(t) \geq 0$ implies that $r^{-1/\gamma}(t)$ is nonincreasing. Assume that ℓ is the integer associated with $z(t)$ in Lemma 2.1. If $0 < \ell < k-1$, then using (2.2), we have

$$z(t) \geq \frac{z^{(k-1)}(t)}{(k-2)!}\int_{t_1}^{t}(s-t_1)^{k-2}\mathrm{d}s = \frac{z^{(k-1)}(t)}{(k-1)!}(t-t_1)^{k-1}. \tag{2.11}$$

It is easy to see that for any $\lambda \in (0,1)$ there exists a $t_\lambda \geq t_1$ such that $t - t_1 \geq \lambda^{1/(k-1)}\, t$ for $t \geq t_\lambda$, which in view of (2.11) yields (2.10).

If $\ell = k-1$, then proceeding similarly as above it can be shown that (2.3) implies (2.10).

If $\ell = 0$, then it follows from (2.5) that

$$z(s) \geq r^{1/\gamma}(t)z^{(k-1)}(t)\int_{s}^{t} r^{-1/\gamma}(u)\frac{(u-s)^{k-2}}{(k-2)!}\mathrm{d}u = \frac{z^{(k-1)}(t)}{(k-1)!}(t-s)^{k-1}. \tag{2.12}$$

Setting $s = (1-\lambda^{1/2(k-1)})t$, we have

$$x\left(\left(1-\lambda^{1/2(k-1)}\right)t\right) \geq \frac{z^{(k-1)}(t)}{(k-1)!}\lambda^{1/2}t^{k-1}. \tag{2.13}$$

Moreover,

$$\lim_{t\to\infty} \frac{x(t)}{x\left(\left(1-\lambda^{1/2(k-1)}\right)t\right)} = 1 > \lambda^{1/2}. \tag{2.14}$$

Therefore, for all large t,

$$x(t) \geq \lambda^{1/2} x\left(\left(1-\lambda^{1/2(k-1)}\right)t\right) \geq \frac{\lambda}{(k-1)!} t^{k-1} z^{(k-1)}(t). \tag{2.15}$$

The proof is complete now. □

In the following result, we compare (2.2) and (2.3) to verify that both estimates are independent; that is, one does not result from the other, although for $r(t) \equiv 1$ they are equivalent.

Remark 2.4. Let $z(t)$ be as in Lemma 2.1, such that $z(t)$ is increasing. At first we consider $r(t) = t^3$, $\gamma = 3$, $t_1 = 1$, and $k = 3$ then (2.2), (2.3), and (2.10) yield

$$z(t) \geq z''(t)\left(t^2 - t\ln t - t\right), \qquad z(t) \geq z''(t)\left(t^2 \ln t - t^2 + t\right), \qquad z(t) \geq z''(t)\frac{\lambda t^2}{2}, \tag{2.16}$$

respectively.

Now we modify $r(t) = t^{-3}$. Then (2.2) and (2.3) reduce to

$$z(t) \geq z''(t)\left(\frac{t^2}{3} - \frac{t}{2} + \frac{1}{6t}\right), \qquad z(t) \geq z''(t)\left(\frac{t^2}{6} - \frac{1}{2} + \frac{1}{3t}\right), \tag{2.17}$$

respectively, and (2.10) is not applicable.

3. Applications

Lemma 2.2 can be applied in various techniques for investigations of the higher-order differential equations. We offer one such application in comparison theory.

Theorem 3.1. *Assume that both first-order delay differential equations*

$$y'(t) + \frac{q(t)}{((n-2)!)^{\gamma}} \left(\int_{t_1}^{\tau(t)} r^{-1/\gamma}(s)(s-t_1)^{n-2} \mathrm{d}s\right)^{\gamma} y(\tau(t)) = 0, \tag{E_1}$$

$$y'(t) + \frac{q(t)}{((n-2)!)^{\gamma}} \left(\int_{t_1}^{\tau(t)} r^{-1/\gamma}(s)(\tau(t)-s)^{n-2} \mathrm{d}s\right)^{\gamma} y(\tau(t)) = 0 \tag{E_2}$$

are oscillatory. Moreover, for n-odd assume that

$$\int_{t_0}^{\infty} u^{n-2} r^{-1/\gamma}(u)\left(\int_{u}^{\infty} q(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u = \infty. \tag{P_0}$$

Then

(i) *for n even, (E) is oscillatory;*

(ii) *for n odd, every nonoscillatory solution $x(t)$ of (E) satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that $x(t)$ is a nonoscillatory solution of (E); let us say positive. Then $(r(t)[x^{(n-1)}(t)]^{\gamma})' < 0$, and there exist a $t_1 \geq t_0$ and an integer ℓ with $n+\ell$ odd such that (2.1) holds.

If $0 < \ell < n-1$, then by Lemma 2.2

$$x(t) \geq \frac{r^{1/\gamma}(t)x^{(n-1)}(t)}{(n-2)!}\int_{t_1}^{t} r^{-1/\gamma}(s)(s-t_1)^{n-2}\mathrm{d}s. \tag{3.1}$$

Setting to (E), we get

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' + \frac{q(t)r(\tau(t))\left[x^{(n-1)}(\tau(t))\right]^{\gamma}}{((n-2)!)^{\gamma}} \times \left(\int_{t_1}^{\tau(t)} r^{-1/\gamma}(s)(\tau(s)-t_1)^{n-2}\mathrm{d}s\right)^{\gamma} \leq 0. \tag{3.2}$$

Then $y(t) = r(t)[x^{(n-1)}(t)]^{\gamma}$ is positive and satisfies the differential inequality:

$$y'(t) + \frac{q(t)}{((n-2)!)^{\gamma}}\left(\int_{t_1}^{\tau(t)} r^{-1/\gamma}(s)(s-t_1)^{n-2}\mathrm{d}s\right)^{\gamma} y(\tau(t)) \leq 0. \tag{3.3}$$

By Theorem 1 in [15], the corresponding equation (E_1) has also a positive solution. A contradiction.

If $\ell = n-1$, then by Lemma 2.2

$$x(t) \geq \frac{r^{1/\gamma}(t)x^{(n-1)}(t)}{(n-2)!}\int_{t_1}^{t} r^{-1/\gamma}(s)(t-s)^{k-2}\mathrm{d}s, \tag{3.4}$$

and proceeding as above, we find out that (E_2) has a positive solution. A contradiction and the proof are finished for n even.

Assume that $\ell = 0$; note that it is possible only for n is odd. Since $x'(t) < 0$, then there exists a finite $\lim_{t\to\infty} x(t) = c \geq 0$. We claim that $c = 0$. If not, then $x(\tau(t)) \geq c > 0$, eventually, let us say, for $t \geq t_2$. An integration of (E) from t to ∞ yields

$$r(t)\left[x^{(n-1)}(t)\right]^{\gamma} \geq \int_{t}^{\infty} q(s)x^{\gamma}(\tau(s))\mathrm{d}s \tag{3.5}$$

or equivalently

$$x^{(n-1)}(t) \geq r^{-1/\gamma}(t)\left(\int_t^\infty q(s)x^\gamma(\tau(s))\mathrm{d}s\right)^{1/\gamma}. \tag{3.6}$$

Integrating $n-1$ times from t to ∞, we get

$$x(t) \geq \int_t^\infty \frac{(u-t)^{n-2}}{(n-2)!}\, r^{-1/\gamma}(u)\left(\int_u^\infty q(s)x^\gamma(\tau(s))\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u, \tag{3.7}$$

then

$$x(t_2) \geq \frac{c}{(n-2)!}\int_{t_2}^\infty (u-t_2)^{n-2}\, r^{-1/\gamma}(u)\left(\int_u^\infty q(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u, \tag{3.8}$$

which contradicts (P_0). The proof is complete. □

Employing any result (e.g., Theorem 2.1.1 in [13]) for the oscillation of (E_1) and (E_2), we immediately obtain criteria for studied properties of (E).

Corollary 3.2. *Let*

$$\begin{gathered}
\liminf_{t\to\infty}\int_{\tau(t)}^t q(u)\left(\int_{t_1}^{\tau(u)} r^{-1/\gamma}(s)(s-t_1)^{n-2}\mathrm{d}s\right)^\gamma \mathrm{d}u > \frac{((n-2)!)^\gamma}{\mathrm{e}}, \\
\liminf_{t\to\infty}\int_{\tau(t)}^t q(u)\left(\int_{t_1}^{\tau(u)} r^{-1/\gamma}(s)(\tau(s)-t)^{n-2}\mathrm{d}s\right)^\gamma \mathrm{d}u > \frac{((n-2)!)^\gamma}{\mathrm{e}}.
\end{gathered} \tag{3.9}$$

Moreover, for n-odd assume that (P_0) holds. Then

(i) *for n even, (E) is oscillatory;*

(ii) *for n odd, every nonoscillatory solution $x(t)$ of (E) satisfies* $\lim_{t\to\infty}x(t) = 0$.

If we replace (P_0) by stronger condition, we can establish oscillation of (E) even if n is odd.

Theorem 3.3. *Let $\tau'(t) \geq 0$. Assume that both first-order delay differential equations (E_1) and (E_1) are oscillatory. Moreover, for n-odd assume that*

$$\limsup_{t\to\infty}\int_{\tau(t)}^t (u-\tau(t))^{n-2}\, r^{-1/\gamma}(u)\left(\int_u^\infty q(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u > (n-2)!. \tag{P_1}$$

Then (E) is oscillatory.

Proof. Assume that $x(t)$ is a positive solution of (E). Then there exists an integer ℓ assigned with $x(t)$ by Lemma 2.1. If $1 \le \ell \le n-1$, then proceeding as in the proof of Theorem 3.1 we eliminate $x(t)$. If $\ell = 0$, then it follows from (3.7) that

$$\begin{aligned} x(\tau(t)) &\ge \int_{\tau(t)}^{\infty} \frac{(u-\tau(t))^{n-2}}{(n-2)!} r^{-1/\gamma}(u) \left(\int_u^{\infty} q(s) x^{\gamma}(\tau(s)) \mathrm{d}s \right)^{1/\gamma} \mathrm{d}u \\ &\ge x(\tau(t)) \int_{\tau(t)}^{t} \frac{(u-\tau(t))^{n-2}}{(n-2)!} r^{-1/\gamma}(u) \left(\int_u^{t} q(s) \mathrm{d}s \right)^{1/\gamma} \mathrm{d}u, \end{aligned} \tag{3.10}$$

which contradicts (P_1). The proof is complete. □

Corollary 3.4. *Let* (3.9) *hold. Moreover, for n-odd assume that* (P_1) *holds. Then* (E) *is oscillatory.*

Example 3.5. We consider the forth-order delay differential equation:

$$\left(t \left(x''(t) \right)^3 \right)' + \frac{a}{t^6} x^3(\lambda t) = 0, \quad a > 0, \ 0 < \lambda < 1, \ t \ge 1. \tag{E_3}$$

It is easy to verify that (P_0) holds for (E_3); moreover, conditions (3.9) reduce to

$$a\lambda^5 \ln\left(\frac{1}{\lambda}\right) > \frac{1}{\mathrm{e}} \left(\frac{5}{3}\right)^3, \tag{3.11}$$

$$a27\lambda^2 \left[\frac{\lambda^6}{125} \ln\left(\frac{1}{\lambda}\right) - \frac{3\lambda^3(1-\lambda)}{50} + \frac{3\left(1-\lambda^2\right)}{40} - \frac{(1-\lambda)^3}{24\lambda^3} \right] > \frac{1}{\mathrm{e}}, \tag{3.12}$$

respectively. Thus, by Corollary 3.4, if both (3.11) and (3.12) hold, then all nonoscillatory solutions of (E_3) tend to zero. For $a = \alpha^3(\alpha+1)^3(3\alpha+5)\lambda^{3\alpha}$, with $\alpha > 0$, one such solution is $x(t) = t^{-\alpha}$. On the other hand, condition (P_1) takes the form

$$a\left(\ln\left(\frac{1}{\lambda}\right) + 1 - \lambda \right)^3 > 5. \tag{3.13}$$

Therefore, by Corollary 3.4, (E_3) is oscillatory, provided that all (3.11), (3.12), and (3.13) hold.

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