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Research Article

Stabilizability and Motion Tracking Conditions for Mechanical Nonholonomic Control Systems

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This paper addresses formulation of stabilizability and motion tracking conditions for mechanical systems from the point of view of constraints put on them. We present a new classification of constraints, which includes nonholonomic constraints that arise in both mechanics and control. Based on our classification we develop kinematic and dynamic control models of systems subjected to these constraints. We demonstrate that a property of being a "hard-to-control" nonholonomic system may not be related to the nature of the constraints. It may result from the formulation of control objectives for a system. We examine two control objectives which are stabilization to the target equilibrium by a continuous static state feedback control and motion tracking. Theory is illustrated with examples of control objective formulations for systems with constraints of various types.

1. Introduction

A control design project does not begin when a control engineer is handed a model of a system. It begins at the onset of the model formulation. The paper is focused on the formulation of kinematic and dynamic control models of constrained systems and a subsequent specification of control objectives for them. We provide a new classification of constraints, which is a basis for the formulation of the models. We consider nonholonomic constraints, which may be of two types: material and non-material. Equations, which specify the non-material constraints, may be differential equations of high-order with respect to time derivatives of coordinates.

Dynamic models of mechanical systems with first-order nonholonomic constraints can be developed using classical methods of analytical mechanics, for example, Lagrange's equations with multipliers and their modifications. For systems with second-order constraints Appell's equations are available [1–4]. Recently, a method of the derivation of equations of motion of systems with nonholonomic constraints of high-order has been developed. This is a generalized programmed motion equations (GPME) method [5, 6]. The high-order constraints are referred to as programmed, since they are put by a designer to specify tasks that systems have to perform or they may arise from design and control objectives [5–7]. They are non-material constraints in contrast to materials that are given by nature. Also, an equation that specifies the angular momentum conservation is meant as a non-material nonholonomic constraint [1]. Constraints that arise from underactuation in a control system are non-material nonholonomic and second-order [8, 9].

Nonholonomic control systems are a class of nonlinear control systems, which are not amenable to methods of linear control theory even locally and they are not transformable into linear control problems in any meaningful way. They require different control approaches than other nonlinear control systems due to the presence of the nonholonomic constraints. Moreover, control systems in which the high-order constraints are present require different control approaches than systems with first-order constraints [1–3, 10–12].

Nonholonomic control systems can be presented in a general form [2]:

$$\dot{x} = F(x, u), \tag{*}$$

where $x \in M$, and M is a smooth n-dimensional manifold referred to as the state space, $u \in U$, $u(t)$ is a time-dependent map from the nonnegative reals $\mathbb{R}^+$ to a constraint set $\Sigma \subset \mathbb{R}^m$, F is assumed to be C^∞ (smooth) or C^ω (analytic) and is taken from $M \times \mathbb{R}^m$ into TM such that for each fixed u, F is a vector field on M. The map u is assumed to be piecewise smooth or piecewise analytic, that is, it is admissible. There are many generalizations and specializations of this definition, for example, for Hamiltonian and Lagrangian control systems; see [2] and references there. For the scope of this paper we may consider affine nonlinear control systems in the form [2, 11]

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, \tag{**}$$

where f is the drift vector field, g_i, $i = 1, \ldots, m$, are the control vector fields, and both are smooth on M. We assume that the constraint set Σ contains an open neighborhood of the origin in $\mathbb{R}^m$.

In this paper, we make connections between the control models (**) for systems with material and non-material nonholonomic constraints, and control objectives stated for them. We demonstrate that nonholonomically constrained systems are not "hard to control" when proper control objectives and strategies are employed.

We select two control objectives, that is, stabilization (local asymptotic stabilizability) by a continuous static-state feedback strategy and motion tracking.

The system (**) is said to be LAS if there exists a feedback $u(x)$ defined on a neighborhood of 0 such that $0 \in M$ is an asymptotically stable equilibrium of the closed-loop system. A feedback controller $u(x)$ is said to be a static-state feedback when it is a continuous map $u: M \to U: x \to u(x)$, $u(0) = 0$, such that the closed-loop system (**) has a unique solution $x(t, x(0))$, $t \geq 0$, for sufficiently small initial state $x(0)$. The asymptotic stabilizability of the target equilibrium holds only if the dimension of the equilibria set including the target is equal to the number of control inputs [13, 14]. This result is equivalent to Brockett's necessary condition for feedback stabilization [2, 15]. Based on Brockett's condition, control models of nonholonomic systems are not asymptotically stabilizable even locally. However, we can still formulate control objectives for some control problems that make them LAS.

Motion tracking consists in tracking a desired motion specified by algebraic or differential equations of constraints [16, 17]. This extended definition of tracking includes trajectory tracking as a peculiar case for which a trajectory is specified by an algebraic equation. Usually in nonlinear control, motion tracking means trajectory tracking. It is achieved using two kinds of models. One considers velocities of a system as control inputs and uses a kinematic model, and ignores a system dynamics (see [2, 18, 19] and references therein). The second uses the system dynamics, where control forces and torques as well as velocities can be control inputs [10–12]. For a nonholonomic system with first-order constraints, kinematic and dynamic control models are usually integrated. A control strategy developed in such a way has a two-level architecture. The lower control level operates within the kinematic model to stabilize the system motion to a desired trajectory. The upper control level uses the dynamic model and stabilizes feedback obtained on the lower control level [12]. Trajectory tracking for nonholonomic systems with first-order constraints can also be achieved using controllers based on dynamic models in a reduced-state form [3, 12, 20]. For a control objective other than trajectory tracking these nonlinear control strategies are not applicable and new strategies have to be pursued. A model reference tracking control strategy for programmed motion is a tool developed for tracking motions specified by equations of constraints of arbitrary order [6, 7, 16, 17]. The strategy uses only dynamic models of a system, both derived by the GPME method. This strategy can also be applied to underactuated systems additionally subjected to programmed constraints [21]. For this reason we do not have to distinguish the underactuated systems as a special class of systems with second-order nonholonomic constraints as it is usually done.

Contributions of the paper consist of the presentation of the new classification of nonholonomic constraints, formulation of control models and control objectives for systems with such constraints, and design control strategies to realize these objectives.

The paper is organized as follows. In Section 2 we present the classification of constrained systems. In Sections 3 and 4 we address kinematic and dynamic control models for them. In Sections 5 and 6, based on examples, we review these models with respect to the stabilizability conditions and possibilities of motion tracking. Examples are illustrated with simulation results. The paper closes with conclusions and a list of references.

2. Classification of nonholonomic systems

Classifications of nonholonomic constraints known to the author capture material constraints and non-material that arise from the conservation law and underactuation, for example, [1, 8, 9, 12, 14, 22]. In Table 2.1 we present a new classification, which includes nonholonomic non-material high-order constraints. The high-order constraints enable specification of many tasks, control objectives, and motion requirements that are usually considered "side conditions" not the constraints. In the new classification they are treated in the same way as other constraints on systems, provided that they can be specified by algebraic or differential equations. An example is an equation of a desired trajectory, which we treat as a constraint.

We do not consider high-order constraints in biomechanical systems herein, see, for example, [23].

The most common constrained systems are these with first-order material constraints (group 1). They arise from the condition that vehicle wheels or fingers of multifinger hands grasping objects do not slip. There is a subgroup of the wheeled systems for which their wheels are not powered. These are a snakeboard [24, 25], a roller-racer [26, 27], a roller-blader [28], a roller-walker [29], or snake-like robots [30, 31]. All these systems can move their bodies due to the relative motion of their joints. This motion is referred to as snake-like motion. Control properties of systems with powered and idle wheels significantly differ.

The constraints from group 2 originate from the conservation law and have the form of nonholonomic constraint equations of first-order. They play the same role as the material constraints do, that is, they specify conditions, which system velocities have to satisfy. Usually, they are distinguished as the "conservation laws" not the constraints *per se* [1]. They arise for space vehicles and robots, for a falling cat [1], for a sportsman performing a summersault [32], and for an astronaut on a space walk [3]. Some of these systems may be underactuated; then we assign them to group 5.

Underactuated systems from group 3 are defined as systems for which the dimension of the configuration space exceeds that of the control input space. Dynamic models of underactuated systems are classified as second-order nonholonomic system models (see (2.3a) in Table 2.1). This is due to equations that represent unactuated degrees of freedom, which are second-order nonholonomic and nonintegrable in general [9].

The underactuated systems may be wheeled mobile robots, underactuated vehicles and manipulators with unactuated joints or space robots without jets or momentum wheels [8]. Sometimes specific properties of these systems are utilized to facilitate control design, for example, equipping unactuated joints with breaking mechanisms or including gravity terms make linearization of system models about equilibrium controllable (see (2.3b) in Table 2.1).

The constraints form group 4 are programmed and they are specified by (4). We assume that they are ideal constraints. Equations (4) may specify both material and non-material constraints on a system and for this reason they are referred to as a unified constraint formulation. We state the following proposition.

Proposition 2.1. *The unified constraint formulation* $B(t,q,\dot{q},\dots,q^{(p-1)})q^{(p)} + s(t,q,\dot{q},\dots,q^{(p-1)}) = 0$ *may specify both material and non-material constraints on mechanical systems.*

Proof. The proof is based upon the reasoning that the type of a constraint equation does not influence the derivation of equations of motion of a system subjected to this constraint. The only concern is the constraint order and whether it is ideal. Indeed, when

Table 2.1. Classification of nonholonomic constraints.

Kind of constraints	Systems/constraint equations	Number of degrees of freedom (m), number of control inputs (l)	LAS	Tracking
(1) First-order, material nonholonomic.	Car-like vehicles, mobile platforms with powered wheels, multifingered hands, nonholonomic manipulators, dexterous manipulation. $B_1(q,\dot{q}) = 0 \quad (2.1)$ B_1 is a $(k \times n)$ full rank matrix, $n > k$.	$m = n - k$; $m = l$	−	+
	Wheeled vehicles with idle wheels, nonholonomic toys, snake-like robots and manipulators. Constraints have the form (2.1), $n > k$.	$m = n - k$; $m \geq l$	−	+
(2) First-order, non-material nonholonomic (conservation law).	Space vehicles and robots, sportsman, falling cat. $B_2(q)\dot{q} + b_2(q) = 0 \quad (2.2)$ B_2 is a $(k \times n)$ full rank matrix, $n > k$	$m = n - k$; $m \geq l$	May be	+
(3) Second-order, non-material nonholonomic, (underactuated).	Manipulators, space systems, underwater vehicles. $M_{11}(q)\ddot{q}_1 + M_{12}(q)\ddot{q}_2 + C_1(q,\dot{q}) = T_1(q)\tau,$ $M_{21}(q)\ddot{q}_1 + M_{22}(q)\ddot{q}_2 + C_2(q,\dot{q}) = 0, \quad (2.3a)$	No gravity is present: $m = n$, $m > l$	−	+
	$M_{11}(q)\ddot{q}_1 + M_{12}(q)\ddot{q}_2 + C_1(q,\dot{q}) + D_1(q) = T_1(q)\tau,$ $M_{21}(q)\ddot{q}_1 + M_{22}(q)\ddot{q}_2 + C_2(q,\dot{q}) + D_2(q) = 0, \quad (2.3b)$	Gravity is present: $m = n$, $m > l$	+	+

Table 2.1. Continued.

(4) High-order, non-material nonholonomic (programmed).	Task specifications for any system: $$B(t,q,\dot{q},\ldots,q^{(p-1)})q^{(p)} + s(t,q,\dot{q},\ldots,q^{(p-1)}) = 0, \quad (2.4)$$ B is a $(k \times n)$ full rank matrix, $n \geq k$, s is a $(k \times 1)$ vector.	$m = n - k$, $m \geq l$	May be	+
(5) Different types of constraints put on a system.	Underactuated vehicles with idle wheels, manipulators and other systems with material and programmed constraints. The unified constraint (4), $n \geq k$.	$m = n - k$, $m \geq l$	May be	+

$p = 0$ we get a position constraint, which may be a material constraint that describes, for example, a constant distance between link ends or be a programmed constraint that specifies a desired trajectory. When $p = 1$, a constraint equation is in the form (2.1) or (2.2). It can be a material constraint, a specification of the conservation law, or a programmed constraint that specifies a desired velocity. For all examples of constraints of order $p = 1$, equations of motion are generated in the same way provided that constraints are ideal. Material constraints are of orders $p = 0$ or $p = 1$ and can be presented by (2.1). Equations for the conservation law are of order $p = 1$ and are specified by (2.2). Constraint equations for $p > 1$ are of the non-material type. Two or more constraint equations, each of a different type, may be listed in (4). The constraint (4) can be used then to specify constraint equations of any order and type. □

It should be emphasized that the constraint equations which have been investigated so far in nonlinear control were mostly in the so-called Chaplygin form, they were mostly driftless and differentially flat, and could be transformed into the power or chained forms or to their extensions [2, 3, 33]. A trajectory tracking control design for such systems can be considered a solved problem, at least theoretically [2, 11, 19]. Systems with both material and programmed constraints may be, in general, non-Chaplygin and may not be transformable into any special control form [34].

For the unified constraint formulation (4) we introduce a definition.

Definition 2.2. The equations of constraints (4) are completely nonholonomic if they cannot be integrated with respect to time, that is, constraint equations of a lower-order cannot be obtained.

If we can integrate (4) $(p - 1)$ or less times, that is, we can obtain nonholonomic constraints of first-orders or orders lower than p, we say that (4) are partially integrable. If (4) can be integrated completely, we say that they are holonomic.

We assume that (4) are completely nonholonomic. Then they do not restrict positions $q(t)$ and their time derivatives up to $(p-1)$th-order. Our definition is an extension of a definition of completely nonholonomic first-order constraints [2] and completely nonholonomic second-order constraints [9]. Necessary and sufficient integrability conditions for differential constraints of arbitrary order such as (4) are formulated in [35].

The constraint equations (4) may be of different orders. From the point of view of a control strategy design they may be differentiated. For the numerical simulation the differentiated constraint equations have to be stabilized; for more details see [17].

Finally, the constraints belong to group 5 when they are of different types and also arise form underactuation in a system.

3. Kinematic control models of constrained systems

Kinematic control models of systems with the material constraints (2.1) have a form of driftless state equations [1, 2]

$$\dot{x} = \sum_{i=1}^{n-k} g_i(x)u_i, \tag{3.1}$$

where g_i, $i = 1,\dots n-k$, are control vector fields smooth on M. The vector $x \in M$, and M is a smooth n-dimensional manifold referred to as the state space, $u(t)$ is a time-dependent map from the nonnegative reals $\mathbb{R}^+$ to a constraint set $\Sigma \subset \mathbb{R}^{n-k}$, which contains an open neighborhood of the origin in $\mathbb{R}^{n-k}$. For systems from group 1 stabilizability conditions and trajectory tracking algorithms at kinematic and dynamic control levels are well established [1–3, 11, 18]. Nonholonomic systems with the constraints (2.1) are not LAS [13–15]. A trajectory tracking formulated as an asymptotic stabilization of a tracking error is LAS for them [14]. The same holds for motion tracking [6]. For some vehicles with idle wheels subjected to the constraints (2.1) no kinematic control models can be developed [26, 27].

Kinematic control models of systems with the constraints (2.2) and (4) may have the form

$$\dot{x} = f(x) + \sum_{i=1}^{n-k} g_i(x)u_i, \tag{3.2}$$

where f is the drift vector field smooth on M.

For these systems a trajectory tracking and motion tracking control formulated as an asymptotic stabilization of a tracking error is LAS. In Section 5 we show that we can select a control objective that may make (3.2) stabilizable at some equilibrium by a continuous static-state feedback.

For the unified constraints (4) we formulate the following theorem.

Theorem 3.1. *The unified constraint formulation (4) can be presented in the state space control form (3.2).*

Proof. Let us take a new p-vector $x = (x_1,\dots,x_p)$ such that $x_1 = q$, $\dot{x}_1 = x_2,\dots,\dot{x}_{p-1} = x_p$. If time t is present explicitly in (4), we reorder coordinates, assigning $x_0 = t$. With the new vector x (4) can be written as $(p-1+k)$ first-order equations

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \vdots \quad & \quad \vdots \\ \vdots \quad & \quad \vdots \\ \dot{x}_{p-1} &= x_p, \\ B(x_1,\dots,x_p)\dot{x}_p &= -s(x_1,\dots,x_p) \end{aligned} \tag{3.3a}$$

or in a matrix form

$$C(x)\dot{x} = b(x), \tag{3.3b}$$

where C is a $(p-1+k)\times p$ matrix and b is a $(p-1+k)$-dimensional vector. Let $f(x)$ be a particular solution of (3.3b) so that $C(x)f(x) = b(x)$. Let $g(x)$ be a full-rank matrix, whose column space is in the null space of $C(x)$, that is, $C(x)g(x) = 0$. Then, the solution of (3.3b) is given by $\dot{x} = f(x)+g(x)u(t)$ for any smooth vector $u(t)$. □

In the control models (3.1) or (3.2) the number of equations is less than the number of degrees of freedom of a system, to which they are related, that is, $n > k$. When constraints are programmed, we say that the program is partly specified. When the number of equations (4) and (2.1) or (2.2) is equal to the number of degrees of freedom, that is, $n = k$, a system motion is fully specified provided that the constraints are not mutually exclusive [7, 36]. In this paper, we consider partly specified programs.

4. Dynamic control models of constrained systems

Motions specified by equations of programmed constraints have to be controlled at a dynamic level. There are important reasons to formulate a motion tracking control problem at the dynamic level. The first reason, significant from the perspective of this paper, is that we consider constraints of high-order, which specify dynamic properties of systems. Secondly, this is the level at which control takes place in practice. Designing controllers at the dynamic level usually leads to significant improvements in performance and implementability, and can help in the early identification and resolution of difficulties. Finally, unmodeled dynamics, friction, and disturbances can be taken into account at that level. Also, for massive wheeled robots that operate at high speeds, dynamics-based control strategies are necessary to obtain realistic control results [19]. It is interesting to consider tracking for holonomically constrained systems in this regard; the kinematic control problem is trivial, but the dynamic control problem is still quite challenging [37]. For wheeled vehicles that perform the snake-like motion, control at the dynamic level is only possible. The reason is that we cannot determine their global motions by just the shape variations, since they do not posses a sufficient number of nonholonomic constraint equations for this [26, 27]. Dynamic control models of such systems consist of (2.3a) and (2.1). For underactuated systems dynamic control models are (2.3a) or (2.3b). For systems with the constraints (4) dynamic models can be derived by the GPME method only.

The GPME method can also be used to derive the dynamic models (2.3a) and (2.3b), and dynamic models of systems with the constraints (2.1) or (2.2). To demonstrate this, recall that dynamic control models used in control theory are mostly based on Lagrange's equations with multipliers [2, 7], that is,

$$\begin{gathered} M(q)\ddot{q}+C(q,\dot{q})+D(q)=J(q)^T\lambda+E(q)\tau, \\ J(q)\dot{q}=0, \end{gathered} \tag{4.1}$$

where q is a n-vector of generalized coordinates, $M(q)$ is a $(n\times n)$ positive definite symmetric inertia matrix, $C(q,\dot{q})$-vector of centripetal and Coriolis forces, $D(q)$-vector of gravitational forces, $E(q)$-vector of an input transformation, $J(q)$ is a full-rank $(k\times n)$ matrix of the constraint equations, $2\le n-k<n$, λ is a k-vector of Lagrange multipliers, $E(q)\tau$ is a vector of generalized forces applied to a system, and τ is an r-vector of control inputs. For control applications, the dynamic control model (4.1) has to be transformed to the reduced-state form [2, 20, 32]. The reduced-state equations characterize the control dependent motion on the constraint manifold. The reduction procedure consists in the elimination of the constraint reaction forces. To this end let $q=(q_1,q_2)$ be a partition of the configuration variables corresponding to the partitioning of the matrix function $J(q)$ as $J(q)=[J_1(q),J_2(q)]$, $\det J_1(q)\neq 0$, and $q_1\in\mathbb{R}^k$, $q_2\in\mathbb{R}^{n-k}$. The second time derivative of a vector of dependent coordinates q_1 extracted from the constraint equations and inserted into the first of (4.1) yields equations of motion decoupled into two sets, from which one is used to design a control strategy

$$\begin{gathered} M_{22}(q)\ddot{q}_2+C_{22}(q,\dot{q}_2)\dot{q}_2+D_2(q)=E_2\tau, \\ \dot{q}_1=-J_1^{-1}(q)J_2(q)\dot{q}_2, \end{gathered} \tag{4.2a}$$

and the second when one wishes to retrieve the constraint reaction forces

$$M_{12}(q)\ddot{q}_2+C_{12}(q,\dot{q}_2)\dot{q}_2+D_1(q)=E_1\tau+J_1^T\lambda. \tag{4.2b}$$

The dynamic control model (4.2a) can be written in the extended kinematic control form [2]

$$\dot{q}=g_1(q)v_1+\cdots+g_{n-k}(q)v_{n-k},\qquad i=1,\dots,n-k,\ 2\le n-k<n, \tag{4.3a}$$

$$v_i^{r_i}=u_i, \tag{4.3b}$$

where $r_i,\dots,r_m$ denote an order of time differentiation and v is the output of a linear system consisting of chains of integrators. Equations (4.3a), (4.3b) form a dynamic model, since in applications from mechanics $r_i=1$, $i=1,\dots,n-k$, controls are typically generalized forces and the model consists of the constraint (4.3a) and the equations of motion (4.3b), which reduce to $\dot{v}=u$.

The dynamic control model (4.2a) is applicable to systems with the constraints (2.1) and for trajectory tracking. A desired trajectory is specified by $q_{2p}=q_{2p}(t)$, where "p" stands for "program." It is enough then to control $q_2(t)$ and $q_1(t)$ is also controlled, since it satisfies the constraint equations. The resulting tracking is state tracking. Using the same reduced-state dynamics (4.2a), the input-output decoupling procedure can be applied for output tracking [10]. In what follows we address state tracking strategies. For systems with the constraints (4) a new tracking control strategy is designed, that is, the model reference tracking control strategy for programmed motion. All details about this strategy can be found in [5, 6, 17] and here we report it briefly. Its architecture consists of three blocks. One is a control law block with feedback and the two are dynamic models. The first one is a reference dynamic model for programmed motion. It is a constrained dynamics that incorporates effects of all constraints on a system and has the form

$$\begin{gathered} M(q)\ddot{q} + V(q,\dot{q}) + D(q) = Q(t,q,\dot{q}), \\ B(t,q,\dot{q},\dots,q^{(p-1)})q^{(p)} + s(t,q,\dot{q},\dots,q^{(p-1)}) = 0. \end{gathered} \tag{4.4}$$

The matrix $M(q)$ is a $(n-k\times n)$ matrix, $B(t,q,\dot{q},\dots,q^{(p-1)})$ is a full-rank $(k\times n)$ matrix. $V(q,\dot{q})$, $D(q)$, and $Q(t,q,\dot{q})$ are all $(n-k\times 1)$ vectors and they stand, respectively, for centripetal, Coriolis and friction forces, for gravitational forces, and for other external forces applied to a system. Equations (4.4) form a reference block that plans a programmed motion.

The second dynamic model in the strategy is a dynamic control model, which incorporates effects of material constraints and conservation laws only, that is,

$$\begin{gathered} M_c(q)\ddot{q} + V_c(q,\dot{q})\dot{q} + D_c(q) = E_c(q)\tau, \\ B_1(q)\dot{q} = 0. \end{gathered} \tag{4.5}$$

Equations (4.5) consist of $(n-k)$ equations of motion and k equations of the constraints. They form a "plant" block in the strategy. Both models are derived by the GPME so they are in the reduced-state form. Outputs $q_{ip}(t)$, $i=1,\dots,n$, of (4.4) are inputs to the control law τ in (4.5). We can demonstrate that (4.5) are equivalent to (4.2a).

Theorem 4.1. *The dynamic control model (4.5) is equivalent to the reduced-state dynamic control model (4.2a).*

Proof. The reduction procedure that results in (4.2a) can be accomplished in several ways [3, 20]. We start from Lagrange's equations with multipliers (4.1), which we write as

$$\begin{gathered} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = J^T(q)\lambda + Q(q,\dot{q}), \\ J(q)\dot{q} = 0, \end{gathered} \tag{4.6}$$

where we assume that $Q(q,\dot{q})$ stands for all external forces applied to a system.

To eliminate constraint forces from (4.6) we project these equations onto the linear subspace generated by the null space of $J(q)$. Since $(J^T(q)\lambda)\cdot\delta q = 0$, Lagrange's equations become

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} - Q\right]\cdot\delta q = 0, \tag{4.7}$$

where $\delta q \in \mathbb{R}^n$ and satisfies $J(q)\delta q = 0$. We partition the coordinate vector q and the $J(q)$ matrix such that $q = (q_1,q_2) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, and $J = [J_1(q)J_2(q)]$, $J_1(q) \in \mathbb{R}^{k\times k}$ is invertible.

Then the relation $\delta q_1 = -J_1^{-1}(q)J_2(q)\delta q_2$ holds. Inserting it to (4.7) we obtain

$$J_1^{-1}J_2\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} - Q_1\right] - \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} - Q_2\right] = 0. \tag{4.8}$$

Equations (4.8) are second-order differential equations in terms of q. They can be simplified by reusing the constraint equation $\dot{q}_1 = -J_1^{-1}(q)J_2(q)\dot{q}_2$ to eliminate $\dot{q}_1$ and $\ddot{q}_1$. The evolution of q_1 can be retrieved by reapplication of the constraint equations. Equations (4.8) are equivalent to Nielsen's equations in Maggi's form [7], that is,

$$J_1^{-1}J_2\left[\frac{\partial \dot{T}}{\partial \dot{q}_1} - 2\frac{\partial T}{\partial q_1} - Q_1\right] - \frac{\partial \dot{T}}{\partial \dot{q}_2} - 2\frac{\partial T}{\partial q_2} - Q_2 = 0 \tag{4.9a}$$

which are the GPME for $p = 1$, that is, they are (4.5). It is enough to show that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\sigma}\right) = \frac{\partial^2 T}{\partial \dot{q}_\sigma \partial t} + \sum_{\rho=1}^{n} \frac{\partial^2 T}{\partial \dot{q}_\sigma \partial q_\rho}\dot{q}_\rho + \sum_{\rho=1}^{n} \frac{\partial^2 T}{\partial \dot{q}_\sigma \partial \dot{q}_\rho}\ddot{q}_\rho, \tag{4.10a}$$

$$\dot{T} = \frac{\partial T}{\partial t} + \sum_{\rho=1}^{n} \frac{\partial T}{\partial q_\rho}\dot{q}_\rho + \sum_{\rho=1}^{n} \frac{\partial T}{\partial \dot{q}_\rho}\ddot{q}_\rho. \tag{4.10b}$$

Based on (4.10b) we have

$$\frac{\partial \dot{T}}{\partial \dot{q}_\sigma} = \frac{\partial^2 T}{\partial t \partial \dot{q}_\sigma} + \sum_{\rho=1}^{n} \frac{\partial^2 T}{\partial q_\rho \partial \dot{q}_\sigma}\dot{q}_\rho + \sum_{\rho=1}^{n} \frac{\partial^2 T}{\partial \dot{q}_\rho \partial \dot{q}_\sigma}\ddot{q}_\rho + \frac{\partial T}{\partial q_\sigma} \tag{4.11}$$

and comparing (4.10a) and (4.11) we obtain that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\sigma}\right) = \frac{\partial \dot{T}}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma}. \tag{4.12}$$

Relations (4.12) inserted into (4.8) for q_1 and q_2 yield that terms in brackets in (4.8) are equal to $(\partial \dot{T}/\partial \dot{q}_\sigma - 2(\partial T/\partial q_\sigma))$, $\sigma = 1,2$, and (4.8) are equivalent to (4.9a), that is, equivalent to the GPME for $p = 1$. □

THEOREM 4.2. *There exists a static-state feedback $U(\dot{q}_1, q, u) : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ such that the dynamics (4.5) can be transformed to the state space control formulation (4.3a), (4.3b).*

Proof. First, transform (4.5) to the state space control formulation. To this end, present the constraint equation as

$$\ddot{q} = G(q)\ddot{q}_1 + \dot{G}(q)\dot{q}_1, \tag{4.13}$$

where partition of the vector q is $q = (q_1, q_2)$ and $q_1 \in \mathbb{R}^{n-k}$, $q_2 \in \mathbb{R}^k$, $m = n - k$, and q_1, q_2 are the vectors of independent and dependent coordinates, respectively. Columns of the matrix $G(q)$ span the right null space of $B_1(q)$. It is the $(n\times m)$ matrix of the form

$$G=\begin{bmatrix} I_{(m\times m)} \\ -B_{12}^{-1}(q)B_{11}(q)\end{bmatrix}, \tag{4.14}$$

where I is a $(m\times m)$ identity matrix, $B_{12}^{-1}(q)B_{11}(q)$ is a locally smooth $(k\times m)$ matrix function, and the matrix $B_1(q)$ is expressed as $B_1=[B_{11}(q),B_{12}(q)]$, and $B_{11}(q)$ is a $k\times(n-k)$ matrix function, and $B_{12}(q)$ is a $(k\times k)$ locally nonsingular matrix function. Elimination of second-order derivatives of dependent coordinates from the first of (4.5) yields

$$\begin{gathered} M_c(q)G(q)\ddot{q}_1+[M_c(q)\dot{G}(q)+V_c(q,\dot{q})G(q)]\dot{q}_1+D_c(q)=E_c(q)\tau, \\ \dot{q}=G(q)\dot{q}_1. \end{gathered} \tag{4.15}$$

Equations (4.15) are exactly the reduced-state dynamic model of a nonholonomic system [4, 20].

Now, introduce in (4.15) a new state variable vector $x=(q,\dot{q}_1)=(x_1,x_2)$ such that $\dot{x}_1=\dot{q}=(\dot{q}_1,\dot{q}_2)$, $\dot{x}_2=\ddot{q}_1$ and $x_1\in\mathbb{R}^n$, $x_2\in\mathbb{R}^m$. Then, (4.15) takes the form

$$\begin{gathered} M_c(x_1)G(x_1)\dot{x}_2+[M_c(x_1)\dot{G}(x_1)+V_c(x_1,\dot{x}_1)G(x_1)]x_2+D_c(x_1)=E_c(x_1)\tau, \\ \dot{x}_1=G(x_1)x_2. \end{gathered} \tag{4.16}$$

Now, select for the dynamics (4.16) a static-state feedback $U(x_2,x_1,u):\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m$ defined by the relation $M_c(x_1)G(x_1)u+[M_c(x_1)\dot{G}(x_1)+V_c(x_1,\dot{x}_1)G(x_1)]x_2+D_c(x_1)=E_c(x_1)\tau$. Application of this static-state feedback to (4.16) transforms it to the form

$$\begin{gathered} \dot{x}_1=G(x_1)x_2, \\ \dot{x}_2=u, \end{gathered} \tag{4.17}$$

which is a desired state space control formulation with $f(x)=(G(x_1),0)$ and $g(x)=(0,e_i)$, and e_i is the standard basis vector in $\mathbb{R}^{n-k}$. □

The first of (4.17) is the constraint equation. The second is the motion equation, which transforms immediately to the linear controllable dynamics [11]

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ \dot{q}_1\end{bmatrix}=\begin{bmatrix} 0 & I_m \\ 0 & 0\end{bmatrix}\begin{bmatrix} q_1 \\ \dot{q}_1\end{bmatrix}+\begin{bmatrix} 0 \\ I_m\end{bmatrix}u. \tag{4.18}$$

Equations (4.17) can be transformed to the normal form equivalent to the one obtained for instance in [2]. Taking new state variables $z_1=q_1$, $z_2=q_2$, $z_3=\dot{q}_1$, which are related to x_1 and x_2 such that $\dot{x}_1=(\dot{z}_1,\dot{z}_2)$, $\dot{z}_3=\dot{x}_2$, (4.17) can be written as

$$\begin{gathered} \dot{z}_1=z_3, \\ \dot{z}_2=G^*(z_1,z_2)z_3, \\ \dot{z}_3=u. \end{gathered} \tag{4.19}$$

This form is equivalent to the control form (3.2) where $f(z) = (z_3, G^*(z_1,z_2)z_3, 0), g_i = (0,0,e_i)$ and e_i is the standard basis vector in $\mathbb{R}^{n-k}$. The matrix G^* in (4.19) is a $(k \times n-k)$ submatrix of the matrix G defined in (4.14).

We demonstrated that the dynamic control model derived with the aid of the GPME can be presented in a standard state space representation (4.17) or (4.19). This allows us to reformulate for our dynamics (4.5) all theoretical control results obtained for the classical control models [1–3, 11, 12, 19, 20, 33].

A main motivation to design the model reference tracking control strategy for programmed motion is that a variety of equations of the non-material constraints (4) disables designing a general algorithm for a tracking controller. Instead, we separate programmed constraints from material and conservation laws. All constraint equations on a system, that is, (2.1), (2.2), and (4) are merged into the reference dynamic model (4.4). Material constraints and conservation laws are merged into the dynamic control model (4.5). This separation yields that (4.5) can be derived once for a given system and different reference dynamic models (4.4) that specify different programmed motions can be plugged into (4.5) each time. Also, this separation makes motion tracking analog to trajectory tracking and enables application of controllers originally dedicated to holonomic systems. This latter property of the tracking strategy significantly increases its scope of applications.

5. Stabilizability conditions for constrained systems

The control model (3.1) is not LAS due to Brockett's condition [2, 13, 14]. For the control model (3.2) we may formulate a control objective, for which we may design a continuous static-state feedback that makes (3.2) LAS. To show this, consider a model of a free-floating space robot presented in Figure 5.1. The angular momentum conservation yields the constraint equation

$$[J+(m_1+m_2)l_1^2+m_2l_2^2]\dot{\phi}+[(m_1+m_2)l_1^2+m_2l_2^2]\dot{\theta}_1+m_2l_2^2\dot{\theta}_2+m_2l_1l_2\cos\theta_2(2\dot{\phi}+2\dot{\theta}_1+\dot{\theta}_2)=K_0 \tag{5.1}$$

which can be written as

$$B_{1\varphi}\dot{\varphi} + B_{1\theta 1}\dot{\theta}_1 + B_{1\theta 2}\dot{\theta}_2 = K_0, \tag{5.2}$$

where K_o is the initial angular momentum that may or may not be zero and

$$\begin{aligned} B_{1\varphi} &= J + (m_1+m_2)l_1^2 + m_2l_2^2 + 2m_2l_1l_2\cos\theta_2, \\ B_{1\theta 1} &= (m_1+m_2)l_1^2 + m_2l_2^2 + 2m_2l_1l_2\cos\theta_2, \\ B_{1\theta 2} &= m_2l_2^2 + m_2l_1l_2\cos\theta_2. \end{aligned} \tag{5.3}$$

In (5.1) J is the inertia of the base body, and m_1, m_2-masses of links concentrated at their ends. We assume that no external forces act on the space robot model. Let us select $\dot{\varphi} = u_1$, $\dot{\theta}_2 = u_2$ as controls and introduce a state vector $x \in \mathbb{R}^3$ such that $x_1 = \varphi - \varphi_p$, $x_2 = \theta_1 - \theta_{1p}$, $x_3 = \theta_2 - \theta_{2p}$. It quantifies the error between current values $(\varphi, \theta_1, \theta_2)$ and

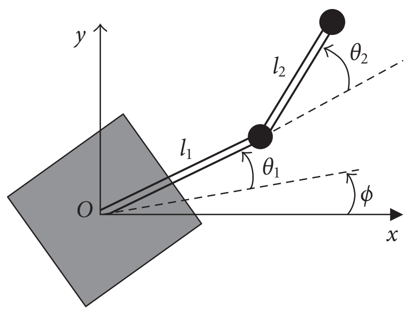

Figure 5.1. Free-floating space robot.

desired values $(\varphi_p, \theta_{1p}, \theta_{2p})$ of the coordinates. Then the control model (3.2) for the space robot becomes

$$\frac{dx}{dt} = \begin{bmatrix} 0 \\ \overline{K}_o(x) \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -\overline{K}_1(x) & -\overline{K}_2(x) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{5.4}$$

with $\overline{K}_o(x) = (K_o(x))/(B_{1\theta1}(x))$, $\overline{K}_1(x) = (B_{1\varphi}(x))/(B_{1\theta1}(x))$, $\overline{K}_2(x) = (B_{1\theta2}(x))/(B_{1\theta1}(x))$. When K_o is zero it seems natural to formulate a control objective as to asymptotically stabilize the equilibrium $x = 0$. Then the system (5.4) is driftless and the number of states $n = 3$ and $n - k = 2$. The equilibrium is not LAS. When K_o is not zero, the drift term never vanishes and $x = 0$ is not an equilibrium. It implies that asymptotic stabilization of $x = 0$ is not an appropriate control objective.

Instead, we can formulate a control problem as follows: make a system achieve $x(t_p) = 0$ for a given initial time $t = 0$ and some final time $t = t_p$. For this formulation of the control goal, we can apply the following time-varying transformation. Select $\xi \in \mathbb{R}^3$, $\xi = (\xi_1, \xi_2, \xi_3)$ such that

$$\begin{aligned} \xi_1 &= x_1, \\ \xi_2 &= x_2 + \overline{K}_1(0)x_1 + \overline{K}_2(0)x_3 - \overline{K}_o(0)(t - t_p), \\ \xi_3 &= x_3. \end{aligned} \tag{5.5}$$

In the new coordinates the control model (5.4) has the form

$$\frac{d\xi}{dt} = \begin{bmatrix} 0 \\ \overline{K}_o(\xi) - \overline{K}_o(0) \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -[\overline{K}_1(\xi) - \overline{K}_1(0)] & -[\overline{K}_2(\xi) - \overline{K}_2(0)] \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{5.6}$$

and the system has equilibrium at $\xi = 0$. It can be verified that the dimension of the equilibrium set is 2 and $n - k = 2$. Then the system is stabilizable by continuous static-state feedbacks.

We conclude that the control problem formulation is significant as well as the structure of the system and constraints on it.

6. Motion tracking conditions for constrained systems

Tracking a desired motion becomes a control objective in a case of the programmed constraints put on a system. Trajectory tracking can be formulated as asymptotic stabilization of a tracking error and the tracking error dynamics are LAS [13, 14]. Consider an example of a system subjected to the high-order constraint (4). Take a two-link planar manipulator model presented in Figure 6.1 [16, 36]. We put a constraint on the manipulator end effector, which specifies the rate of change of the curvature $\Phi(t)$ of its trajectory. In the joint coordinates the constraint has the form

$$F_2\dddot{\Theta}_1 + \dddot{\Theta}_2 - F_1 = 0, \tag{6.1}$$

where

$$F_1 = \frac{A_\phi - A_1 - A_2 a_o}{a_2 + a_4 a_o}, \qquad F_2 = \frac{a_1 + a_2 + a_o(a_3 + a_4)}{a_2 + a_4 a_o}, \qquad a_o = \frac{a_5}{a_6},$$

$$A_\phi = \frac{-\Phi(a_5^2 + a_6^2)^2\left[\dot{\Phi}(a_5^2 + a_6^2) + 3\Phi(a_5 a_7 + a_6 a_8)\right]}{a_6(a_5 a_8 - a_7 a_6)},$$

$$A_1 = 3a_3\dot{\Theta}_1\ddot{\Theta}_1 + 3a_4(\ddot{\Theta}_1 + \ddot{\Theta}_2)(\dot{\Theta}_1 + \dot{\Theta}_2) - a_1\dot{\Theta}_1^3 - a_2(\dot{\Theta}_1 + \dot{\Theta}_2)^3,$$

$$A_2 = 3a_3\dot{\Theta}_1\ddot{\Theta}_1 + 3a_2(\ddot{\Theta}_1 + \ddot{\Theta}_2)(\dot{\Theta}_1 + \dot{\Theta}_2) + a_3\dot{\Theta}_1^3 + a_4(\dot{\Theta}_1 + \dot{\Theta}_2)^3,$$

$$a_1 = -l_1 \sin\Theta_1, \qquad a_3 = -l_1\cos\Theta_1,$$

$$a_2 = -l_2\sin(\Theta_1 + \Theta_2), \qquad a_4 = -l_2\cos(\Theta_1 + \Theta_2),$$

$$a_5 = a_1\dot{\Theta}_1 + a_2(\dot{\Theta}_1 + \dot{\Theta}_2), \qquad a_7 = a_1\dot{\Theta}_1 + a_3\dot{\Theta}_1^2 + a_2(\ddot{\Theta}_1 + \ddot{\Theta}_2) + a_4(\dot{\Theta}_1 + \dot{\Theta}_2)^2,$$

$$a_6 = -a_3\dot{\Theta}_1 - a_4(\dot{\Theta}_1 + \dot{\Theta}_2), \qquad a_8 = -a_3\ddot{\Theta}_1 + a_1\dot{\Theta}_1^2 - a_4(\ddot{\Theta}_1 + \ddot{\Theta}_2) + a_2(\dot{\Theta}_1 + \dot{\Theta}_2)^2. \tag{6.2}$$

For this constraint $n = 2$, $n - k = 1$. The kinematic control model (3.2) generated for (6.1) has a drift that does not vanish. One option is to look for one control input that can steer a system to the desired motion consistent with (6.1). The other is to apply the model reference tracking control for programmed motion based on (4.4) and (4.5). The reference dynamic model of the manipulator subjected to the third-order constraint (6.1) and developed by the GPME is

$$\begin{gathered}(b_1 - b_2F_2)\ddot{\Theta}_1 + (b_2 - \delta F_2)\ddot{\Theta}_2 + c = 0,\\ \dddot{\Theta}_2 = F_1 - F_2\dddot{\Theta}_1,\end{gathered} \tag{6.3}$$

where $\alpha = I_{z1} + I_{z2} + m_1 r_1^2 + m_2(l_1^2 + r_2^2)$, $\beta = m_2 l_1 r_2$, $\delta = I_{z2} + m_2 r_2^2$, $b_1 = \alpha + 2\beta\cos\Theta_2$, $b_2 = \delta + \beta\cos\Theta_2$, and $c = -\beta\dot{\Theta}_2(\dot{\Theta}_2 + 2\dot{\Theta}_1)\sin\Theta_2 - 4/3\beta\dot{\Theta}_1^2 F_2\sin\Theta_2$.

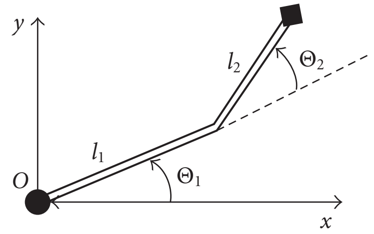

Figure 6.1. Two-link planar manipulator.

The parameters above consist of inertia and geometric data for the manipulator model. The dynamic control model of the manipulator is as follows:

$$\begin{bmatrix} \alpha+2\beta\cos\Theta_2 & \delta+\beta\cos\Theta_2 \\ \delta+\beta\cos\Theta_2 & \delta \end{bmatrix} \begin{bmatrix} \ddot{\Theta}_1 \\ \ddot{\Theta}_2 \end{bmatrix} + \begin{bmatrix} -\dot{\Theta}_2\beta\sin\Theta_2 & -\beta\sin\Theta_2(\dot{\Theta}_1+\dot{\Theta}_2) \\ \dot{\Theta}_1\beta\sin\Theta_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}, \tag{6.4}$$

since the manipulator with no programmed constraints is holonomic.

The reference dynamics (6.3) produces programmed outputs Θ_{1p}, Θ_{2p}, and their derivatives, which are inputs to the control dynamics (6.4). Two control inputs $\tau = (\tau_1, \tau_2)$ are torques, which have to be applied at manipulator joints to track the desired motion specified by (6.1). Furthermore, they can be static-state feedbacks designed in the same way as for any holonomic system, specifically for any manipulator [37, Chapter 3.4]. Indeed, when to select computed torque controllers τ_1, τ_2, and the PD controller for the outer loop, the tracking error is asymptotically stable as long as the PD controller gains are all positive. Specifically, we have

$$\tau = M_c(\Theta)u + V_c(\Theta,\dot{\Theta})\dot{\Theta}, \tag{6.5}$$

where $\Theta = (\Theta_1,\Theta_2)$, $M_c(\Theta)$, $V_c(\Theta,\dot{\Theta})$ are matrices that furnish (6.4), and u is a new input. The PD controller can be defined as $u = \ddot{\Theta}_p - 2\sigma\dot{e} - \sigma^2 e$ and a vector of a tracking error as $e(t) = \Theta(t) - \Theta_p(t)$. The tracking error satisfies the equation $\ddot{e} + 2\sigma\dot{e} + \sigma^2 e = 0$ in which σ is a convergence rate diagonal matrix. It converges to zero exponentially, that is, the end-effector motion converges to the programmed motion.

In a general case of a dynamic control model of a nonholonomic system, according to Theorem 4.2, the computed torque applied to (4.5) results in (4.17) that can be written as

$$\begin{aligned} \ddot{q}_1 &= u, \\ \ddot{q}_2 &= -B_{12}^{-1}(q)B_{11}(q)\ddot{q}_1 - \frac{d}{dt}\left[B_{12}^{-1}(q)B_{11}(q)\right]\dot{q}_1. \end{aligned} \tag{6.6}$$

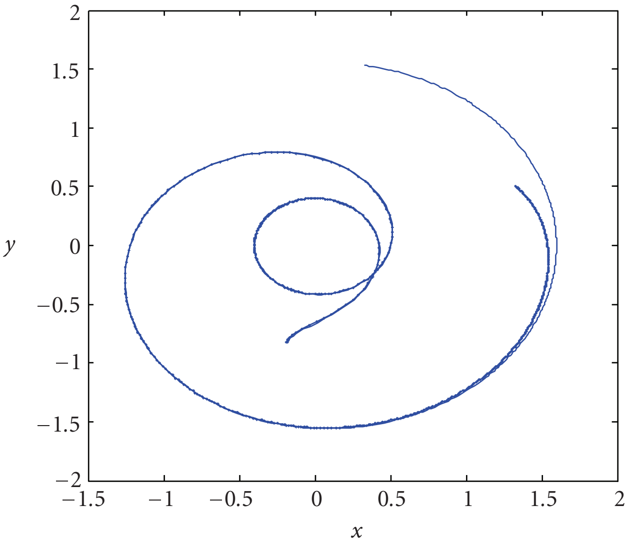

Figure 6.2. Programmed motion tracking by the PD controller.

A vector of a new input is u and it can be selected as

$$u = \ddot{q}_{1p} - 2\sigma\dot{\tilde{q}} - \sigma^2\tilde{q}, \tag{6.7}$$

where $\tilde{q} = q_1 - q_{1p}$ is a position tracking error. The tracking error satisfies the equation $\ddot{\tilde{q}} + 2\sigma\dot{\tilde{q}} + \sigma^2\tilde{q} = 0$ and converges to zero exponentially. This simple sample of a controller design illustrates the philosophy of the application of the reference dynamic model in the model reference tracking control strategy for programmed motion.

Simulation results for tracking the programmed motion specified by (6.1) by the PD controller and tracking errors are presented in Figures 6.2 and 6.3. Position and velocity errors are denoted by $e_1 = \Theta_1 - \Theta_{1p}$, $e_2 = \Theta_2 - \Theta_{2p}$, and e_3 and e_4 for the angle time derivatives, respectively.

This tracking strategy can be employed in the same way with the application of other static-state feedback controllers [17].

7. Conclusions

In this paper, we have presented the new constraint classification with respect to kinds of constraints put on mechanical systems. This classification reflects the extended constraint concept that includes non-material nonholonomic constraints of high-order. The general form of equations of constraints referred to as the unified constraint formulation follows this classification. For systems subjected to the unified high-order constraints kinematic and dynamic control models have been developed and examined from the point of view of stabilizability and motion tracking conditions. We have demonstrated that constrained systems are not “hard to control” when appropriate control objectives are formulated

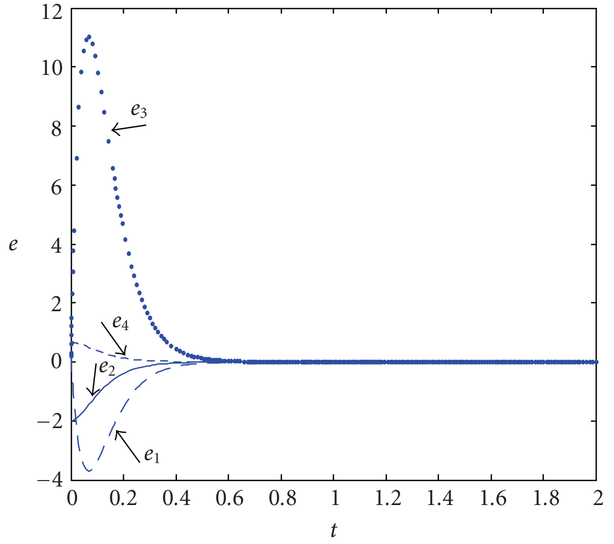

Figure 6.3. Position and velocity tracking errors versus time.

and control strategies are applied. In this paper, we applied the model reference tracking control strategy for programmed motion to track motions specified by the constraint equations.

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