Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2008, Article ID 153280, 10 pages
doi:10.1155/2008/153280

Research Article

Second Hankel Determinant for a Class of Analytic Functions Defined by Fractional Derivative

A. K. Mishra and P. Gochhayat

Department of Mathematics, Berhampur University, Bhanja Bihar, Berhampur 760 007, Orissa, India

Correspondence should be addressed to A. K. Mishra, akshayam2001@yahoo.co.in

Received 4 April 2007; Accepted 21 November 2007

Recommended by Vladimir Mityushev

By making use of the fractional differential operator Ω_z^{λ} due to Owa and Srivastava, a class of analytic functions $\mathcal{R}_{\lambda}(\alpha,\rho)$ $(0 \le \rho \le 1,\ 0 \le \lambda < 1,\ |\alpha| < \pi/2)$ is introduced. The sharp bound for the nonlinear functional $|a_2a_4 - a_3^2|$ is found. Several basic properties such as inclusion, subordination, integral transform, Hadamard product are also studied.

1. Introduction

Let $\mathcal{A}$ denote the class of functions analytic in the *open* unit disc

$$\mathcal{U} := \{z : z \in \mathbb{C},\ |z| < 1\} \tag{1.1}$$

and let $\mathcal{A}_0$ be the class of functions f in $\mathcal{A}$ given by the *normalized* power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \tag{1.2}$$

Also let $\mathcal{S}$, $\mathcal{S}^*(\beta)$, $\mathcal{CV}(\beta)$, and $\mathcal{K}$ denote, respectively, the subclasses of $\mathcal{A}_0$ consisting of functions which are *univalent*, *starlike* of order β, *convex* of order β (cf. [1]), and *close-to-convex* (cf. [2]) in $\mathcal{U}$. In particular, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{CV}(0) = \mathcal{CV}$ are the familiar classes of starlike and convex functions in $\mathcal{U}$ (cf. [2]).

Given f and g in $\mathcal{A}$, the function f is said to be *subordinate* to g in $\mathcal{U}$ if there exits a function $\omega \in \mathcal{A}$ satisfying the conditions of the Schwarz Lemma such that $f(z) = g(\omega(z))$, $(z \in \mathcal{U})$. We denote the subordination by

$$f(z) \prec g(z) \quad (z \in \mathcal{U}) \text{ or } f \prec g \text{ in } \mathcal{U}. \tag{1.3}$$

It is well known [2] that if g is univalent in $\mathcal{U}$, then $f \prec g$ in $\mathcal{U}$ is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

For the functions f and g given by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in \mathcal{U}), \tag{1.4}$$

their Hadamard product (or *convolution*), denoted by $f*g$, is defined by

$$(f*g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g*f)(z) \quad (z \in \mathcal{U}). \tag{1.5}$$

Note that $f*g \in \mathcal{A}$.

By making use of the Hadamard product, Carlson-Shaffer [3] defined the linear operator $\mathcal{L}(a,c) : \mathcal{A} \to \mathcal{A}$ by

$$\big(\mathcal{L}(a,c)f\big)(z) := \Phi(a,c;z)*f(z) \quad (z \in \mathcal{U},\ f \in \mathcal{A}), \tag{1.6}$$

where

$$\Phi(a,c;z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \quad \left(z \in \mathcal{U},\ c \notin \mathbb{Z}_0^- = \{0\} \cup \{-1,-2,-3,\ldots\}\right) \tag{1.7}$$

and $(\lambda)_k$ is the Pochhammer symbol (or *shifted factorial*) defined in terms of the gamma function by

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0), \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1) & (k \in \mathbb{N} := \{1,2,\ldots\}). \end{cases} \tag{1.8}$$

It can be readily verified that $\mathcal{L}(a,a)\ (a \notin \mathbb{Z}_0^-)$ is the identity operator; the operators $\mathcal{L}(a,b)$, $\mathcal{L}(c,d)$ commute, where $b,\ d \notin \mathbb{Z}_0^-$, that is,

$$\mathcal{L}(a,b)\mathcal{L}(c,d)f = \mathcal{L}(c,d)\mathcal{L}(a,b)f \quad (f \in \mathcal{A}), \tag{1.9}$$

and the transitive property, that is,

$$\mathcal{L}(a,b)\mathcal{L}(b,c)f = \mathcal{L}(a,c)f \quad \left(b, c \notin \mathbb{Z}_0^-,\ f \in \mathcal{A}\right), \tag{1.10}$$

holds. Each of the following definitions will also be required in our present investigation.

Definition 1.1 (cf. [4, 5], see also [6]). Let the function f be analytic in a simply connected region of the z-plane containing the origin. The *fractional derivative of f of order λ* is defined by

$$\left(D_z^{\lambda} f\right)(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \le \lambda < 1), \tag{1.11}$$

where the multiplicity of $(z-\zeta)^{\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Using Definition 1.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [5] introduced the *fractional differintegral operator* $\Omega_z^{\lambda} : \mathcal{A}_0 \to \mathcal{A}_0$ defined by

$$\left(\Omega_z^{\lambda} f\right)(z) = \Gamma(2-\lambda) z^{\lambda} \left(D_z^{\lambda} f\right)(z) \quad (\lambda \neq 2, 3, \dots,\ z \in \mathcal{U}). \tag{1.12}$$

Note that $\Omega_z^0 f(z) = f(z)$, $\Omega_z^1 f(z) = z f'(z)$, and

$$\left(\Omega_z^{\lambda} f\right)(z) = \left(\mathcal{L}(2, 2-\lambda) f\right)(z) \quad (0 \leq \lambda < 1,\ z \in \mathcal{U}). \tag{1.13}$$

Definition 1.2 (cf. [7]). For the function f given by (1.2) and $q \in \mathbb{N} := \{1, 2, 3, \dots\}$, the qth Hankel determinant of f is defined by

$$\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.14}$$

We now introduce the following class of functions.

Definition 1.3. The function $f \in \mathcal{A}_0$ is said to be in the class $\mathcal{R}_{\lambda}(\alpha, \rho)$ $(0 \leq \lambda < 1,\ |\alpha| < \pi/2,\ 0 \leq \rho \leq 1)$ if it satisfies the inequality

$$\Re\left\{ e^{i\alpha} \frac{\Omega_z^{\lambda} f(z)}{z} \right\} > \rho \cos\alpha \quad (z \in \mathcal{U}). \tag{1.15}$$

Write

$$\mathcal{R}_{\lambda}(0, \rho) := \mathcal{R}_{\lambda}(\rho). \tag{1.16}$$

Let $\mathcal{P}$ be the family of functions $p \in \mathcal{A}$ satisfying $p(0) = 1$ and $\Re(p(z)) > 0$ $(z \in \mathcal{U})$.

It follows from (1.15) that

$$f \in \mathcal{R}_{\lambda}(\alpha, \rho) \Longleftrightarrow e^{i\alpha} \frac{\Omega_z^{\lambda} f(z)}{z} = \left[(1-\rho) p(z) + \rho\right] \cos\alpha + i \sin\alpha, \tag{1.17}$$

where α is real, $|\alpha| < \pi/2$, and $p(z) \in \mathcal{P}$.

We note that

$$\begin{aligned} \mathcal{R}_0(\alpha, \rho) &:= \left\{ f \in \mathcal{A}_0 \mid \Re\left\{ e^{i\alpha} \frac{f(z)}{z} \right\} > \rho \cos\alpha \right\}, \\ \mathcal{R}_1(\alpha, \rho) &:= \left\{ f \in \mathcal{A}_0 \mid \Re\left\{ e^{i\alpha} f'(z) \right\} > \rho \cos\alpha \right\}, \end{aligned} \tag{1.18}$$

and the class $\mathcal{R}_{\lambda}(\rho)$ has been studied in [8].

It is well known (cf. [2]) that for $f \in \mathcal{S}$ and given by (1.2), the sharp inequality $|a_3 - a_2^2| \leq 1$ holds. This corresponds to the Hankel determinant with $q = 2$ and $n = 1$. For a given family $\mathcal{F}$ of functions in $\mathcal{A}_0$, the more general problem of finding sharp estimates for $|\mu a_2^2 - a_3|$ $(\mu \in \mathbb{R}$ or $\mu \in \mathbb{C})$ is popularly known as the Fekete-Szegö problem for $\mathcal{F}$. The Fekete-Szegö problem for the families $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{CV}$, $\mathcal{K}$ has been completely solved by many authors including [9–12].

In the present paper, we consider the Hankel determinant for $q = 2$ and $n = 2$ and we find the sharp bound for the functional $|a_2 a_4 - a_3^2|$ $(f \in \mathcal{R}_{\lambda}(\alpha, \rho))$. We also obtain some basic properties of the class $\mathcal{R}_{\lambda}(\alpha, \rho)$. Our investigation includes a recent result of Janteng et al. [13]. We also generalize some results of Ling and Ding [8].

2. Preliminaries

To establish our results, we recall the following.

Lemma 2.1 (see [2]). *Let the function* $p \in \mathcal{P}$ *and be given by the series*

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathcal{U}). \tag{2.1}$$

Then, the sharp estimate

$$\left|c_k\right| \leq 2 \quad (k \in \mathbb{N}) \tag{2.2}$$

holds.

Lemma 2.2 (cf. [14, page 254], see also [15]). *Let the function* $p \in \mathcal{P}$ *be given by the power series* (2.1). *Then,*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right) \tag{2.3}$$

for some x, $|x| \leq 1$, *and*

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z \tag{2.4}$$

for some z, $|z| \leq 1$.

Lemma 2.3 (see [16]). *Let* F *and* G *be univalent convex functions in* $\mathcal{U}$. *Then, the Hadamard product* $F*G$ *is also a univalent convex function in* $\mathcal{U}$.

Lemma 2.4 (see [17]). *Let* F *and* G *be univalent convex functions in* $\mathcal{U}$. *Also let* $f \prec F$ *and* $g \prec G$ *in* $\mathcal{U}$. *Then,* $f*g \prec F*G$ *in* $\mathcal{U}$.

Lemma 2.5 (see [16], also see [8]). *Let* f *and* g *be starlike of order* $1/2$. *Then, for each function* $F(z)$, *satisfying* $\Re(F(z)) > \alpha$ $(0 \leq \alpha < 1,\ z \in \mathcal{U})$, *one has*

$$\Re\left(\frac{f(z)*F(z)g(z)}{f(z)*g(z)}\right) > \alpha \quad (z \in \mathcal{U}). \tag{2.5}$$

Lemma 2.6 (see [8]). *Let the function* $h(z) = 1 + h_1 z + h_2 z^2 + \cdots$ *be univalent convex in* $\mathcal{U}$. *For* $0 \leq \lambda < 1$ *if* $\Omega_z^{\lambda} f(z)/z \prec h(z)$ $(z \in \mathcal{U})$, *then*

$$\frac{f(z)}{z} \prec \left\{\mathcal{L}(2-\lambda, 2)\left[zh(z)\right]\right\} \quad (z \in \mathcal{U}). \tag{2.6}$$

3. Main results

We prove the following.

Theorem 3.1. *Let the function* f *given by* (1.2) *be in the class* $\mathcal{R}_\lambda(\alpha, \rho)$ $(0 \leq \lambda < 1,\ -\pi/2 < \alpha < \pi/2,$ *and* $0 \leq \rho \leq 1)$. *Then,*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2\cos^2\alpha}{9}. \tag{3.1}$$

The estimate (3.1) *is sharp.*

Proof. Let $f \in \mathcal{R}_\lambda(\alpha,\rho)$ $(0 \le \lambda < 1,\ -\pi/2 < \alpha < \pi/2,$ and $0 \le \rho \le 1)$. Then, by (1.17),

$$e^{i\alpha}\frac{\Omega_z^\lambda f(z)}{z} = \left[(1-\rho)p(z)+\rho\right]\cos\alpha + i\sin\alpha \quad (z \in \mathcal{U}), \tag{3.2}$$

where $p \in \mathcal{P}$ and is given by (2.1). Using (1.6), (1.7), and (1.13), we write

$$\Omega_z^\lambda f(z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}a_n z^n, \quad (z \in \mathcal{U}). \tag{3.3}$$

Comparing the coefficients, we get

$$\begin{aligned} e^{i\alpha}\frac{2}{(2-\lambda)}a_2 &= (1-\rho)c_1\cos\alpha, \\ e^{i\alpha}\frac{6}{(2-\lambda)(3-\lambda)}a_3 &= (1-\rho)c_2\cos\alpha, \\ e^{i\alpha}\frac{24}{(2-\lambda)(3-\lambda)(4-\lambda)}a_4 &= (1-\rho)c_3\cos\alpha. \end{aligned} \tag{3.4}$$

Therefore, (3.4) yields

$$\left|a_2a_4 - a_3^2\right| = \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)(\cos^2\alpha)}{12}\left|\left(\frac{(4-\lambda)c_1c_3}{4} - \frac{(3-\lambda)c_2^2}{3}\right)\right|. \tag{3.5}$$

Since the functions $p(z)$ and $p(e^{i\theta}z)$, $(\theta \in \mathbb{R})$ are members of the class $\mathcal{P}$ simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c$ $(c \in [0,2])$.

Using (2.3) along with (2.4), we get

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| &= \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)(\cos^2\alpha)}{12} \\ &\quad\times\left|\frac{(4-\lambda)c}{16}\left\{c^3 + 2(4-c^2)cx - c(4-c^2)x^2 + 2(4-c^2)(1-|x|^2)z\right\}\right| \\ &= \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)(\cos^2\alpha)}{48} \\ &\quad\times\left|\left(\frac{(4-\lambda)}{4} - \frac{3-\lambda}{3}\right)c^4 + \left(\frac{(4-\lambda)(4-c^2)c^2}{2} - \frac{2c^2(3-\lambda)(4-c^2)}{3}\right)x\right. \\ &\qquad\left. - \left(\frac{(4-\lambda)(4-c^2)c^2}{4} + \frac{(3-\lambda)(4-c^2)^2}{3}\right)x^2 + \frac{(4-\lambda)(4-c^2)c(1-|x|^2)z}{2}\right| \\ &= \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)(\cos^2\alpha)}{48} \\ &\quad\times\left|\frac{\lambda c^4}{12} + \frac{\lambda(4-c^2)c^2x}{6} - \left(\frac{48-\lambda(16-c^2)}{12}\right)(4-c^2)x^2 + \frac{(4-\lambda)(4-c^2)c(1-|x|^2)z}{2}\right|. \end{aligned} \tag{3.6}$$

An application of triangle inequality and replacement of $|x|$ by μ give

$$\begin{aligned}
\left|a_2a_4-a_3^2\right| &\leq \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)\left(\cos^2\alpha\right)}{48} \\
&\quad\times\left[\frac{\lambda c^4}{12}+\frac{\lambda\left(4-c^2\right)c^2\mu}{6}+\frac{\left(4-c^2\right)\left[48-\lambda\left(16-c^2\right)\right]\mu^2}{12}+\frac{(4-\lambda)\left(4-c^2\right)c}{2}\right. \\
&\quad\quad\left.-\frac{(4-\lambda)\left(4-c^2\right)c\mu^2}{2}\right] \\
&= \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)\left(\cos^2\alpha\right)}{48} \\
&\quad\times\left[\frac{\lambda c^4}{12}+\frac{(4-\lambda)\left(4-c^2\right)c}{2}+\frac{\lambda\left(4-c^2\right)c^2\mu}{6}\right. \\
&\quad\quad\left.+\frac{\lambda\left[c^2-6(4-\lambda)c/\lambda+16(3-\lambda)/\lambda\right]\left(4-c^2\right)\mu^2}{12}\right] \\
&= \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)\left(\cos^2\alpha\right)}{48} \\
&\quad\times\left[\frac{\lambda c^4}{12}+\frac{(4-\lambda)\left(4-c^2\right)c}{2}+\frac{\lambda\left(4-c^2\right)c^2\mu}{6}+\frac{\lambda\left(c-\beta_1\right)\left(c-\beta_2\right)\left(4-c^2\right)\mu^2}{12}\right] \\
&:= F(c,\mu)\ \text{(say)},
\end{aligned} \tag{3.7}$$

where

$$\beta_1=2,\quad \beta_2=\frac{8(3-\lambda)}{\lambda},\quad 0\leq c\leq 2,\quad 0\leq\mu\leq 1. \tag{3.8}$$

We next maximize the function $F(c,\mu)$ on the closed square $[0,2]\times[0,1]$. Since

$$\frac{\partial F}{\partial\mu}=\frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)\cos^2\alpha}{48}\left[\frac{\lambda\left(4-c^2\right)c^2}{6}+\frac{\lambda\left(4-c^2\right)(c-2)(c-8(3-\lambda)/\lambda)\mu}{6}\right], \tag{3.9}$$

$c-2<0$, and $c-8(3-\lambda)/\lambda<0$, we have $\partial F/\partial\mu>0$ for $0<c<2$, $0<\mu<1$. Thus $F(c,\mu)$ cannot have a maximum in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$,

$$\max_{0\leq\mu\leq 1}F(c,\mu)=F(c,1)=G(c)\ \text{(say)}. \tag{3.10}$$

Next,

$$G'(c)=\frac{-(1-\rho)^2(2-\lambda)^2(3-\lambda)\left(c^2-(7\lambda-12)\right)c\cos^2\alpha}{72}, \tag{3.11}$$

so that $G'(c)<0$ for $0<c<2$ and has real critical point at $c=0$. Also $G(c)>G(2)$. Therefore, $\max_{0\leq c\leq 2}$ occurs at $c=0$. Therefore, the upper bound of (3.7) corresponds to $\mu=1$ and $c=0$. Hence,

$$\left|a_2a_4-a_3^2\right|\leq\frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2\cos^2\alpha}{9} \tag{3.12}$$

which is the assertion (3.1). Equality holds for the function

$$f(z) = \Phi(2-\lambda, 2; z) * e^{-i\alpha}\left[z\left(\frac{1+(1-2\rho)z^2}{1-z^2}\cos\alpha + i\sin\alpha\right)\right]. \tag{3.13}$$

The proof of Theorem 3.1 is complete. □

The choice of $\alpha = 0$ yields what follows.

Corollary 3.2. *Let the function f given by* (1.2) *be a member of the class $\mathcal{R}_\lambda(\rho)$. Then,*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2}{9}. \tag{3.14}$$

Equality holds for the function

$$f(z) = \mathcal{L}(2-\lambda, 2) * \frac{z\left(1+(1-2\rho)z^2\right)}{1-z^2}. \tag{3.15}$$

Remark 3.3. Taking $\lambda \to 1$, $\alpha = 0$, and $\rho = 0$, we get a recent result due to Janteng et al. [13].

Theorem 3.4. *Suppose $-\pi/2 < \alpha < \pi/2$, $0 \leq \rho < 1$, and $0 \leq \mu < \lambda < 1$. Then,*

$$\mathcal{R}_\lambda(\alpha, \rho) \subset \mathcal{R}_\mu(\alpha, \rho). \tag{3.16}$$

Proof. Let

$$f \in \mathcal{R}_\lambda(\alpha, \rho) \quad \left(0 \leq \mu < \lambda < 1,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2},\ 0 \leq \rho \leq 1\right). \tag{3.17}$$

Using the associative and commutative properties of the operator $\mathcal{L}$, we write

$$\begin{aligned}
\Omega_z^\mu f(z) &= \mathcal{L}(2, 2-\mu) f(z) \\
&= \mathcal{L}(2-\lambda, 2)\mathcal{L}(2, 2-\lambda)\mathcal{L}(2, 2-\mu) f(z) \\
&= \mathcal{L}(2-\lambda, 2-\mu)\Omega_z^\lambda f(z) \\
&= \Phi(2-\lambda, 2-\mu; z) * \Omega_z^\lambda f(z),
\end{aligned} \tag{3.18}$$

where the function Φ is defined by (1.7). Therefore,

$$\begin{aligned}
\frac{e^{i\alpha}\Omega_z^\mu f(z)}{z} &= \frac{\Phi(2-\lambda, 2-\mu; z) * \left(e^{i\alpha}\Omega_z^\lambda f(z)/z\right)\cdot z}{\Phi(2-\lambda, 2-\mu; z) * z} \\
&= \frac{f(z) * F(z) g(z)}{f(z) g(z)},
\end{aligned} \tag{3.19}$$

where $f(z) = \Phi(2-\lambda, 2-\mu; z)$, $g(z) = z$, $F(z) = e^{i\alpha}\Omega_z^\lambda f(z)/z$. We note that $g \in \mathcal{S}^*(1/2)$, and $\Re(F(z)) > \rho\cos\alpha\ (0 \leq \rho \leq 1,\ -\pi/2 < \alpha < \pi/2)$. Moreover, it is well known (cf. [18]) that

$\Phi(2-\lambda, 2-\mu; z) \in \mathcal{S}^*(1/2)$. Therefore, by Lemma 2.5,

$$\Re\left(\frac{e^{i\alpha}\Omega_z^{\mu} f(z)}{z}\right) > \rho\cos\alpha \quad \left(-\frac{\pi}{2} < \alpha < \frac{\pi}{2},\ z \in \mathcal{U},\ 0 \le \rho \le 1\right). \tag{3.20}$$

Hence, $f(z) \in \mathcal{R}_{\mu}(\alpha, \rho)$, and the proof of Theorem 3.4 is complete. □

Theorem 3.5. *Let* $f \in \mathcal{S}^*(1/2)$ *and* $g \in \mathcal{R}_{\lambda}(\alpha, \rho)$ $(0 \le \rho \le 1,\ -\pi/2 < \alpha < \pi/2,\ 0 \le \lambda < 1)$. *Then the Hadamard product*

$$f * g \in \mathcal{R}_{\lambda}(\alpha, \rho). \tag{3.21}$$

Proof. Since the Hadamard product is associative and commutative, we have

$$\Omega_z^{\lambda}(f * g)(z) = f(z) * \Omega_z^{\lambda} g(z). \tag{3.22}$$

Therefore,

$$\frac{e^{i\alpha}\Omega_z^{\lambda}(f * g)(z)}{z} = \frac{f(z) * \left(e^{i\alpha}\Omega_z^{\lambda} g(z)/z\right) \cdot z}{f(z) * z}. \tag{3.23}$$

Now applying Lemma 2.5, we get

$$\Re\left(\frac{e^{i\alpha}\Omega_z^{\lambda}(f * g)(z)}{z}\right) > \rho\cos\alpha. \tag{3.24}$$

Hence, $f * g \in \mathcal{R}_{\lambda}(\alpha, \rho)$, and the proof of Theorem 3.5 is complete. □

Theorem 3.6. *Let* $f \in \mathcal{R}_{\lambda}(\alpha, \rho)$ $(0 \le \lambda < 1,\ -\pi/2 < \alpha < \pi/2,\ 0 \le \rho \le 1)$. *Then, the function* $\mathcal{I}(f)$ *defined by the integral transform*

$$\mathcal{I}(f)(z) = \frac{\gamma + 1}{z^{\gamma}} \int_0^z t^{\gamma - 1} f(t)\,dt \quad (z \in \mathcal{U},\ \gamma > -1) \tag{3.25}$$

is also in $\mathcal{R}_{\lambda}(\alpha, \rho)$.

Proof. The Integral transform $\mathcal{I}(f)$ can be written in terms of Carlson-Shaffer operator as

$$\left(\mathcal{I}(f)\right)(z) = \left(\mathcal{L}(\gamma + 1, \gamma + 2) f\right)(z). \tag{3.26}$$

Hence,

$$\left(\Omega_z^{\lambda}\mathcal{I}(f)\right)(z) = \mathcal{L}(\gamma + 1, \gamma + 2)\Omega_z^{\lambda} f(z) = \Phi(\gamma + 1, \gamma + 2; z) * \Omega_z^{\lambda} f(z). \tag{3.27}$$

Therefore,

$$\frac{e^{i\alpha}\left(\Omega_z^{\lambda}\mathcal{I}(f)\right)(z)}{z} = \frac{\Phi(\gamma + 1, \gamma + 2; z) * \left(e^{i\alpha}\Omega_z^{\lambda} f(z)/z\right) z}{\Phi(\gamma + 1, \gamma + 2; z) * z}. \tag{3.28}$$

Using a result of Bernardi [19], it can be verified that $\Phi(\gamma + 1, \gamma + 2; z) \in \mathcal{S}^*(1/2)$. Thus by applying Lemma 2.5, the proof of Theorem 3.6 is complete. □

Theorem 3.7. *Let* $f \in \mathcal{R}_\lambda(\alpha, \rho)$, $(0 \le \lambda < 1,\ -\pi/2 < \alpha < \pi/2,\ 0 \le \rho \le 1)$. *Then,*

$$\frac{f(z)}{z} \prec \mathcal{G}(z) \quad (z \in \mathcal{U}), \tag{3.29}$$

where

$$\begin{aligned} \mathcal{G}(z) &= \frac{e^{-i\alpha}}{z}\left\{\Phi(2-\lambda, 2; z) * [zh(z)]\right\}, \\ h(z) &= \left(\frac{1+(1-2\rho)z}{1-z} cos\,\alpha + i\sin\alpha\right), \end{aligned} \tag{3.30}$$

and Φ *is defined by* (1.7). *Moreover,* $\mathcal{G}$ *is a univalent convex function in* $\mathcal{U}$.

Proof. Since $\Omega_z^\lambda f(z)/z \prec e^{-i\alpha} h(z)$, by an application of Lemma 2.6, we get

$$\frac{f(z)}{z} \prec \frac{e^{-i\alpha}}{z}\left\{\mathcal{L}(2-\lambda, 2) * [zh(z)]\right\} = \mathcal{G}(z). \tag{3.31}$$

The assertion (3.29) is proved.

It is well known (cf. [18]) that $\Phi(2-\lambda, 2; z)/z$ is a univalent convex function. Therefore, by Lemma 2.4, $\mathcal{G}(z)$ is univalent convex function. □

Remark 3.8. For $\alpha = 0$, Theorem 3.7(i) gives a result of Ling and Ding [8, Theorem 2].

Acknowledgments

The present investigation is partially supported by National Board for Higher Mathematics, Department of Atomic Energy, Government of India under Grant no. 48/2/2003-R&D-II.

References

[1] M. I. S. Robertson, “On the theory of univalent functions,” *Annals of Mathematics*, vol. 37, no. 2, pp. 374–408, 1936.

[2] P. L. Duren, *Univalent Functions*, vol. 259 of *Grundlehren der Mathematischen Wissenschaften*, Springer, New York, NY, USA, 1983.

[3] B. C. Carlson and D. B. Shaffer, “Starlike and prestarlike hypergeometric functions,” *SIAM Journal on Mathematical Analysis*, vol. 15, no. 4, pp. 737–745, 1984.

[4] S. Owa, “On the distortion theorems. I,” *Kyungpook Mathematical Journal*, vol. 18, no. 1, pp. 53–59, 1978.

[5] S. Owa and H. M. Srivastava, “Univalent and starlike generalized hypergeometric functions,” *Canadian Journal of Mathematics*, vol. 39, no. 5, pp. 1057–1077, 1987.

[6] H. M. Srivastava and S. Owa, “An application of the fractional derivative,” *Mathematica Japonica*, vol. 29, no. 3, pp. 383–389, 1984.

[7] J. W. Noonan and D. K. Thomas, “On the second Hankel determinant of areally mean p-valent functions,” *Transactions of the American Mathematical Society*, vol. 223, no. 2, pp. 337–346, 1976.

[8] Y. Ling and S. S. Ding, “A class of analytic functions defined by fractional derivation,” *Journal of Mathematical Analysis and Applications*, vol. 186, no. 2, pp. 504–513, 1994.

[9] M. Fekete and G. Szegö, “Eine Bemerkung über ungerade schlichte Funktionen,” *Journal of the London Mathematical Society*, vol. 8, pp. 85–89, 1933.

[10] F. R. Keogh and E. P. Merkes, “A coefficient inequality for certain classes of analytic functions,” *Proceedings of the American Mathematical Society*, vol. 20, pp. 8–12, 1969.

[11] W. Koepf, "On the Fekete-Szegö problem for close-to-convex functions," *Proceedings of the American Mathematical Society*, vol. 101, no. 1, pp. 89–95, 1987.

[12] W. Koepf, "On the Fekete-Szegö problem for close-to-convex functions—II," *Archiv der Mathematik*, vol. 49, no. 5, pp. 420–433, 1987.

[13] A. Janteng, S. A. Halim, and M. Darus, "Coefficient inequality for a function whose derivative has positive real part," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 2, p. 50, 2006.

[14] R. J. Libera and E. J. Złotkiewicz, "Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$," *Proceedings of the American Mathematical Society*, vol. 87, no. 2, pp. 251–257, 1983.

[15] R. J. Libera and E. J. Złotkiewicz, "Early coefficients of the inverse of a regular convex function," *Proceedings of the American Mathematical Society*, vol. 85, no. 2, pp. 225–230, 1982.

[16] St. Ruscheweyh and T. Sheil-Small, "Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture," *Commentarii Mathematici Helvetici*, vol. 48, pp. 119–135, 1973.

[17] St. Ruscheweyh and J. Stankiewicz, "Subordination under convex univalent functions," *Bulletin of the Polish Academy of Sciences. Mathematics*, vol. 33, no. 9-10, pp. 499–502, 1985.

[18] D. G. Yang, "Subclasses of starlike functions of order α," *Chinese Annals of Mathematics. Series A*, vol. 8, no. 6, pp. 687–692, 1987.

[19] S. D. Bernardi, "Convex and starlike univalent functions," *Transactions of the American Mathematical Society*, vol. 135, pp. 429–446, 1969.