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Research Article

An Explicit Criterion for the Existence of Positive Solutions of the Linear Delayed Equation $\dot{x}(t) = -c(t)x(t - \tau(t))$

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The paper investigates an equation with single delay $\dot{x}(t) = -c(t)x(t-\tau(t))$, where $\tau : [t_0-r,\infty) \to (0,r]$, $r > 0$, $t_0 \in \mathbb{R}$, and $c : [t_0-r,\infty) \to (0,\infty)$ are continuous functions, and the difference $t-\tau(t)$ is an increasing function. Its purpose is to derive a new explicit integral criterion for the existence of a positive solution in terms of c and τ. An overview of known relevant criteria is provided, and relevant comparisons are also given.

1. Introduction

Let us consider an equation

$$\dot{x}(t) = -c(t)x(t - \tau(t)), \tag{1.1}$$

where $\tau : [t_0-r,\infty) \to \mathbb{R}^+$ is a continuous function, $t_0 \in \mathbb{R}$, $0 < r = \text{const}$ $\mathbb{R}^+ = (0,\infty)$, $\tau(t) \le r$ if $t \in [t_0-r,\infty)$, and $c : [t_0-r,\infty) \to \mathbb{R}^+$ is a continuous function. The symbol "$\cdot$" denotes the right-hand derivative.

A function x is called a *solution* of (1.1) corresponding to an initial point $t^* \in [t_0,\infty)$ if x is defined and is continuous on $[t^*-r,\infty)$ and differentiable on $[t^*,\infty)$ and satisfies (1.1) for $t \ge t^*$. We denote by $x(t^*,\varphi)(t)$ a solution of (1.1) corresponding to an initial point $t^* \in [t_0,\infty)$ generated by a continuous initial function $\varphi : [t^* - r, t^*] \to \mathbb{R}$. In the case of a linear equation

(1.1), the solution $x(t^*, \varphi)(t)$ is unique on its maximal existence interval $[t^*, \infty)$ (see, e.g., [1]). If in the paper an initial point is not indicated, t_0 is assumed.

As is customary, a solution of (1.1) corresponding to an initial point $t^* \in [t_0, \infty)$ is called *oscillatory* if it has arbitrarily large zeros. Otherwise it is called *nonoscillatory*. A nonoscillatory solution x of (1.1) corresponding to an initial point t^* is called *positive* (*negative*) if $x(t) > 0$ $(x(t) < 0)$ on $[t^* - r, \infty)$. A nonoscillatory solution x of (1.1) corresponding to an initial point t^* is called *eventually positive* (*eventually negative*) if there exists $t^{**} > t^*$ such that $x(t) > 0$ $(x(t) < 0)$ on $[t^{**}, \infty)$. Instead of the terms "eventually positive" ("eventually negative"), the phrase "positive for $t \to \infty$" ("negative for $t \to \infty$") is very often used.

In order to simplify the formulation of results in the paper, we do not mention all the technical conditions. Typically, we do not mention that inequalities are valid on an interval $[t_0, \infty)$, tacitly assuming that t_0 is so large that the necessary assumptions hold on the interval in question, nor do we mention an initial point t^* when we characterize a property of a solution.

In the paper, we derive a new explicit criterion for the existence of a positive solution of the scalar differential equation with delay (1.1). Although, by its form, (1.1) is a simple equation, it plays an important role in the theory of differential equations with delay. One of the reasons is that, because of its simplicity, it is often used for testing new results and comparing them with previous ones.

In the literature there are numerous criteria for positivity even for equations more general than (1.1).

Below, we give an overview of previous results (we assume that above assumptions are true in spite of the fact that some of criteria are valid even under weaker assumptions).

In [2] the following criterion for the existence of a nonoscillatory solution is given.

Theorem 1.1. *If*

$$\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} c(s)\mathrm{d}s < \frac{1}{\mathrm{e}}, \tag{1.2}$$

then (1.1) *has a nonoscillatory solution.*

In addition to this, according to a result in [2], we have the following.

Theorem 1.2. *If*

$$\int_{t-\tau(t)}^{t} c(s)\mathrm{d}s \le \frac{1}{\mathrm{e}}, \tag{1.3}$$

for all sufficiently large t, then (1.1) *has a nonoscillatory solution.*

In [3], the authors consider (1.1) in the form

$$\dot{x}(t) = -\left(\frac{1}{\mathrm{e}} + a(t)\right)x(t-1), \quad t \ge 1, \tag{1.4}$$

and prove the following.

Theorem 1.3. *Assume that*

$$a(t) \le \frac{1}{8et^2}, \tag{1.5}$$

for all sufficiently large t. Then (1.4) *has a solution*

$$x(t) \ge \sqrt{t}\, e^{-t}. \tag{1.6}$$

In [4], Theorem 1.3 is improved to the following theorem.

Theorem 1.4. *Assume that*

$$a(t) \le \frac{1}{8et^2}\left(1 + \frac{1}{\ln^2 t}\right), \tag{1.7}$$

for all sufficiently large t. Then (1.4) *has a solution*

$$x(t) \ge \sqrt{t \ln t}\, e^{-t}. \tag{1.8}$$

The authors then demonstrate that, in terms of the values of the coefficient of the equation itself, inequalities (1.5), (1.7) are sharp in a sense because the following result holds.

Theorem 1.5 (see [4]). *Let* $\tau(t) = \tau = \text{const}$ *in* (1.1). *Assume that*

$$\begin{gathered}
\liminf_{t \to \infty} c(t) = \frac{1}{\tau e}, \\
\liminf_{t \to \infty} \left[\left(c(t) - \frac{1}{\tau e}\right) t^2\right] = \frac{\tau}{8e}, \\
\liminf_{t \to \infty} \left\{\left[\left(c(t) - \frac{1}{\tau e}\right) t^2 - \frac{\tau}{8e}\right] \ln^2 t\right\} = C > \frac{\tau}{8e}.
\end{gathered} \tag{1.9}$$

Then all solutions of (1.1) *oscillate.*

Motivated by Theorems 1.3–1.5, the second author gave in [5] a generalization of these results. To formulate them we need to introduce the concept of an iterated logarithm.

Definition 1.6. Let one calls the expression $\ln_k t$, $k \ge 1$, defined by the formula

$$\ln_k t = \underbrace{\ln \ln \cdots \ln t}_{k}, \quad k \ge 1, \tag{1.10}$$

a kth *iterated* logarithm if $t > \exp_{k-2} 1$, where

$$\exp_k t \equiv \underbrace{(\exp(\exp(\cdots \exp t)))}_{k}, \quad k \ge 1, \tag{1.11}$$

$\exp_0 t \equiv t$, and $\exp_{-1} t \equiv 0$. Moreover, let us define $\ln_0 t \equiv t$. Instead of expressions $\ln_0 t$, $\ln_1 t$, we will write only t and $\ln t$ in the sequel.

Theorem 1.7. *Let* $\tau(t) = \tau = \mathrm{const}$ *in* (1.1).

(A) *Let one assumes that*

$$c(t) \le \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \frac{\tau}{8e(t\ln t\ln_2 t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t\cdots\ln_k t)^2} \tag{1.12}$$

for $t \to \infty$ *and an integer* $k \ge 0$. *Then there is a positive solution* $x = x(t)$ *of* (1.1). *Moreover,*

$$x(t) < e^{-t/\tau}\sqrt{t\ln t\ln_2 t\cdots\ln_k t} \tag{1.13}$$

as $t \to \infty$.

(B) *Let one assumes that*

$$c(t) \ge \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t\cdots\ln_{k-1} t)^2} + \frac{\theta\tau}{8e(t\ln t\ln_2 t\cdots\ln_k t)^2} \tag{1.14}$$

for $t \to \infty$, *an integer* $k \ge 0$, *and a constant* $\theta > 1$. *Then all solutions of* (1.1) *oscillate.*

Recently, Theorem 1.7 has been generalized for variable delay in [6].

Theorem 1.8. (A) *Let* $0 \le \tau(t) \le r$ *for* $t \to \infty$, *and let condition* (A) *of Theorem 1.7 hold. Then* (1.1) *has a nonoscillatory solution.*

(B) *Let* $\tau(t) \ge r > 0$ *for* $t \to \infty$, *and let condition* (B) *of Theorem 1.7 hold. Then all solutions of* (1.1) *oscillate.*

For further criteria on the existence of positive solutions, we refer, for example, to papers [6–14], books [15–18], and relevant references therein.

2. An Integral Positivity Criterion, Comparisons, and Open Problems

Theorems 1.3–1.8 in part 1 are formulated in terms of the values of the coefficients c or a itself whereas Theorems 1.1 and 1.2 are formulated in terms of the average of the coefficient c. In accordance with the opinion presented, for example, in [4], we observe that, sometimes, conditions of the first of the types mentioned yield stronger results. However, conditions of the second type are, in general, preferable.

The following criterion is of an integral type (i.e., of the second type) and uses the term "average" of coefficient c of the form

$$\int_{t-\tau(t)}^{t} c(s)\omega(s)\,ds \tag{2.1}$$

with an appropriate weight function ω.

Theorem 2.1. *Let one assumes that*

$$\int_{t-\tau(t)}^{t} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s \le \frac{1}{\mathrm{e}} - \frac{\tau(t)}{2\mathrm{e}t}, \tag{2.2}$$

for $t \in [t_0, \infty)$*. Then there exists a positive solution* $x = x(t)$ *of* (1.1) *on* $[t_0, \infty)$*. Moreover,*

$$x(t) < \exp\left(-\mathrm{e}\int_{t_0}^{t} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s\right) \tag{2.3}$$

for $t \in [t_0, \infty)$.

The proof of this criterion, given in part 3, is not difficult and uses the retract technique. Now we consider a simple example of (1.1) to show that known criteria cannot be applied. For simplicity, a constant delay $\tau(t) = \tau = \text{const}$ is assumed. The symbol $\mathcal{O}$ (big "O") below stands for the Landau order symbol.

Example 2.2. Let

$$c(t) = \frac{1}{\tau\mathrm{e}} + \frac{1}{2\tau\mathrm{e}}\mathrm{e}^{\tau/(2t)} \sin\frac{2\pi t}{\tau} \tag{2.4}$$

be chosen in (1.1); that is, we consider an equation

$$\dot{x}(t) = -\left(\frac{1}{\tau\mathrm{e}} + \frac{1}{2\tau\mathrm{e}}\mathrm{e}^{\tau/(2t)} \sin\frac{2\pi t}{\tau}\right)x(t-\tau). \tag{2.5}$$

Let us apply Theorem 2.1. Then the integral on the left-hand side of inequality (2.2) equals

$$\int_{t-\tau}^{t} c(s)\mathrm{e}^{-\tau/(2s)}\mathrm{d}s = \int_{t-\tau}^{t}\left(\frac{1}{\tau\mathrm{e}} + \frac{1}{2\tau\mathrm{e}}\mathrm{e}^{\tau/(2s)} \sin\frac{2\pi s}{\tau}\right)\mathrm{e}^{-\tau/(2s)}\mathrm{d}s = I_1(t) + I_2(t), \tag{2.6}$$

where

$$\begin{aligned} I_1(t) &= \frac{1}{\tau\mathrm{e}}\int_{t-\tau}^{t} \mathrm{e}^{-\tau/(2s)}\mathrm{d}s, \\ I_2(t) &= \frac{1}{2\tau\mathrm{e}}\int_{t-\tau}^{t} \sin\frac{2\pi s}{\tau}\mathrm{d}s. \end{aligned} \tag{2.7}$$

Obviously $I_2(t) = 0$. It remains to estimate $I_1(t)$ in order to show that inequality (2.2) holds. We develop an asymptotic decomposition for $I_1(t)$ for $t \to \infty$ with the necessary degree of accuracy. We obtain

$$
\begin{aligned}
I_1(t) &= \frac{1}{\tau e}\int_{t-\tau}^{t} e^{-\tau/(2s)} ds = \frac{1}{\tau e}\int_{t-\tau}^{t}\left(1-\frac{\tau}{2s}+\frac{\tau^2}{8s^2}+\mathcal{O}\left(\frac{1}{s^3}\right)\right)ds \\
&= \frac{1}{e}-\frac{1}{2e}\ln\frac{1}{1-(\tau/t)}-\frac{\tau}{8e}\left(\frac{1}{t}-\frac{1}{t-\tau}\right)+\mathcal{O}\left(\frac{1}{t^3}\right) \\
&= \frac{1}{e}-\frac{1}{2e}\ln\left(1+\frac{1}{\tau}+\frac{1}{\tau^2}+\mathcal{O}\left(\frac{1}{t^3}\right)\right)-\frac{\tau}{8et}\left(1-\frac{1}{1-(\tau/t)}\right)+\mathcal{O}\left(\frac{1}{t^3}\right) \\
&= \frac{1}{e}-\frac{1}{2e}\left(\frac{\tau}{t}+\frac{\tau^2}{2t^2}\right)+\frac{\tau^2}{8et^2}+\mathcal{O}\left(\frac{1}{t^3}\right) \\
&= \frac{1}{e}-\frac{\tau}{2et}-\frac{\tau^2}{8et^2}+\mathcal{O}\left(\frac{1}{t^3}\right).
\end{aligned} \tag{2.8}
$$

This means that

$$
\int_{t-\tau}^{t} c(s)e^{-\tau/(2s)} ds = \frac{1}{e}-\frac{\tau}{2et}-\frac{\tau^2}{8et^2}+\mathcal{O}\left(\frac{1}{t^3}\right) < \frac{1}{e}-\frac{\tau}{2et} \tag{2.9}
$$

and inequality (2.2) holds. By Theorem 2.1, (2.5) has a positive solution.

We use this example to compare Theorem 2.1 with known criteria mentioned in Section 1.

Comparison with Theorems 1.1 and 1.2

We show that the results mentioned cannot be applied to Example 2.2. For this, we will analyse the integral

$$
\int_{t-\tau}^{t} c(s) ds = \int_{t-\tau}^{t}\left(\frac{1}{\tau e}+\frac{1}{2\tau e}e^{\tau/(2s)}\sin\frac{2\pi s}{\tau}\right)ds. \tag{2.10}
$$

For an arbitrarily large integer k, we set $t_k = 10k\tau$. Then $t_k - \tau = (10k-1)\tau$ and

$$
\int_{t_k-\tau}^{t_k} c(s) ds = \int_{(10k-1)\tau}^{10k\tau}\left(\frac{1}{\tau e}+\frac{1}{2\tau e}e^{\tau/(2s)}\sin\frac{2\pi s}{\tau}\right)ds = J_1(k)+J_2(k), \tag{2.11}
$$

where

$$J_1(k) = \int_{(10k-1)\tau}^{10k\tau} \frac{1}{\tau e} ds = \frac{1}{e},$$
$$J_2(k) = \frac{1}{2\tau e} \int_{(10k-1)\tau}^{10k\tau} e^{\tau/(2s)} \sin \frac{2\pi s}{\tau} ds. \tag{2.12}$$

Then

$$\begin{aligned} J_2(k) &= \frac{1}{2\tau e} \int_{(10k-1)\tau}^{(10k-(1/2))\tau} e^{\tau/(2s)} \sin \frac{2\pi s}{\tau} ds + \frac{1}{2\tau e} \int_{(10k-(1/2))\tau}^{10k\tau} e^{\tau/(2s)} \sin \frac{2\pi s}{\tau} ds \\ &> \frac{1}{2\tau e} e^{1/(20k-1)} \int_{(10k-1)\tau}^{(10k-(1/2))\tau} \sin \frac{2\pi s}{\tau} ds + \frac{1}{2\tau e} e^{1/(20k-1)} \int_{(10k-(1/2))\tau}^{10k\tau} \sin \frac{2\pi s}{\tau} ds \\ &= \frac{1}{2\tau e} e^{1/(20k-1)} \int_{(10k-1)\tau}^{10k\tau} \sin \frac{2\pi s}{\tau} ds = 0. \end{aligned} \tag{2.13}$$

We conclude that, for every k, the inequality

$$\int_{t_k-\tau}^{t_k} c(s) ds = J_1(k) + J_2(k) > \frac{1}{e} \tag{2.14}$$

holds. Since k can be sufficiently large with $\lim_{k\to\infty} t_k = \infty$, inequalities (1.2) and (1.3) in Theorems 1.1 and 1.2 are not valid for all sufficiently large t because at least the values $t = t_k$ must be excluded. We conclude that Theorems 1.1 and 1.2 are not applicable to (2.5).

Comparison with Theorems 1.3–1.8

Using Example 2.2, we show again that the results mentioned cannot be applied. We will demonstrate the nonapplicability of Theorem 1.7. The same arguments can be used for the remaining theorems.

For an arbitrarily large integer p, we set $t_p = (10p - (3/4))\tau$. Then,

$$\begin{aligned} c(t_p) &= \frac{1}{\tau e} + \frac{1}{2\tau e} e^{\tau/(2t_p)} \sin \frac{2\pi t_p}{\tau} = \frac{1}{\tau e} + \frac{1}{2\tau e} e^{1/(20p-(3/4))} \sin\left(20p - \left(\frac{3}{2}\right)\right)\pi \\ &= \frac{1}{\tau e} + \frac{1}{2\tau e} e^{1/(20p-(3/4))}. \end{aligned} \tag{2.15}$$

Let $t = t_p$. Then inequality (1.14) becomes

$$\begin{aligned} c(t_p) &= \frac{1}{\tau e} + \frac{1}{2\tau e} e^{1/(20p-(3/4))} \\ &\le \frac{1}{e\tau} + \frac{\tau}{8et_p^2} + \frac{\tau}{8e(t_p \ln t_p)^2} + \frac{\tau}{8e(t_p \ln t_p \ln_2 t_p)^2} + \cdots + \frac{\tau}{8e(t_p \ln t_p \ln_2 t_p \cdots \ln_k t_p)^2}. \end{aligned} \tag{2.16}$$

As $p \to \infty$, we have $t_p \to \infty$. Inequality (2.16) does not hold because, for $p \to \infty$, the left-hand side and the right-hand side tend to $3/(2e\tau)$ and $1/(e\tau)$, respectively, but the inequality

$$\frac{3}{2e\tau} \leq \frac{1}{e\tau} \tag{2.17}$$

does not hold.

The sharpness of Theorem 2.1 will be illustrated by the following example.

Example 2.3. Let $\tau(t) = \tau = \text{const}$ and

$$c(t) = \frac{1}{\tau \mathrm{e}} + \frac{\tau\varepsilon}{8\mathrm{e}t^2}, \tag{2.18}$$

where a parameter $\varepsilon < 1$ is chosen in (1.1); that is, we consider an equation

$$\dot{x}(t) = -\left(\frac{1}{\tau \mathrm{e}} + \frac{\tau\varepsilon}{8\mathrm{e}t^2}\right)x(t-\tau). \tag{2.19}$$

Let us apply Theorem 2.1. Then the integral on the left-hand side of inequality (2.2) equals

$$I(t) = \int_{t-\tau}^{t} c(s)\mathrm{e}^{-\tau/(2s)}\mathrm{d}s = \int_{t-\tau}^{t}\left(\frac{1}{\tau \mathrm{e}} + \frac{\tau\varepsilon}{8\mathrm{e}t^2}\right)\mathrm{e}^{-\tau/(2s)}\mathrm{d}s. \tag{2.20}$$

We develop an asymptotic decomposition for $I(t)$ for $t \to \infty$ with the necessary degree of accuracy. Applying some similar asymptotic decompositions obtained by calculating the integral $I_1(t)$ in Example 2.2, we get

$$\begin{aligned}
I(t) &= \int_{t-\tau}^{t}\left(\frac{1}{\tau \mathrm{e}} + \frac{\tau\varepsilon}{8\mathrm{e}s^2}\right)\mathrm{e}^{-\tau/(2s)}\mathrm{d}s \\
&= \int_{t-\tau}^{t}\left(\frac{1}{\tau \mathrm{e}} + \frac{\tau\varepsilon}{8\mathrm{e}s^2}\right)\left(1 - \frac{\tau}{2s} + \frac{\tau^2}{8s^2} + \mathcal{O}\left(\frac{1}{s^3}\right)\right)\mathrm{d}s \\
&= \int_{t-\tau}^{t}\left(\frac{1}{\tau \mathrm{e}} + \frac{\tau\varepsilon}{8\mathrm{e}s^2}\right)\left(1 - \frac{\tau}{2s} + \frac{\tau^2}{8s^2} + \mathcal{O}\left(\frac{1}{s^3}\right)\right)\mathrm{d}s \\
&= \int_{t-\tau}^{t}\left(\frac{1}{\tau \mathrm{e}} - \frac{1}{2\mathrm{e}s} + \frac{\tau^2(1+\varepsilon)}{8\mathrm{e}s^2}\right)\mathrm{d}s \\
&= \frac{1}{\mathrm{e}} - \frac{1}{2\mathrm{e}}\left(\frac{\tau}{t} + \frac{\tau^2}{2t^2}\right) + \frac{\tau^2(1+\varepsilon)}{8\mathrm{e}t^2} + \mathcal{O}\left(\frac{1}{t^3}\right) \\
&= \frac{1}{\mathrm{e}} - \frac{\tau}{2\mathrm{e}t} + \frac{\tau^2(\varepsilon-1)}{8\mathrm{e}t^2} + \mathcal{O}\left(\frac{1}{t^3}\right).
\end{aligned} \tag{2.21}$$

This means that Theorem 2.1 will be applicable to (2.19) if inequality (2.2), that is, the inequality

$$\int_{t-\tau}^{t} c(s)\mathrm{e}^{-\tau/(2s)}\,\mathrm{d}s = \frac{1}{\mathrm{e}} - \frac{\tau}{2\mathrm{e}t} + \frac{\tau^2(\varepsilon-1)}{8\mathrm{e}t^2} + \mathcal{O}\left(\frac{1}{t^3}\right) < \frac{1}{\mathrm{e}} - \frac{\tau}{2\mathrm{e}t}, \tag{2.22}$$

holds. This is true since $\varepsilon < 1$. We finish this example by concluding that, in the case of a constant delay, Theorem 2.1 gives a result not equivalent to that given by Theorems 1.3–1.8. Nevertheless, as can be seen from the form of the function c in (2.18), which almost coincides with the first two terms of the auxiliary comparison function on the right-hand side of inequality (1.12) in Theorem 1.7, the result of Theorem 2.1 is not so far from a "sharp" criterion.

At the end of this part, we will formulate open problem not answered in this paper as a mathematical challenge for further research in this field.

Open Problem 1. Is it possible to improve the result of Theorem 2.1 in such a way that the new result completely covers the parts (A) of Theorems 1.7 and 1.8?

3. Proof of Theorem 2.1

Let $C_r([-r,0],\mathbb{R})$, be the Banach space of continuous functions from the interval $[-r,0]$ into $\mathbb{R}$ equipped with the supremum norm. If $\sigma \in \mathbb{R}$, $A \geq 0$, and $x : [\sigma - r, \sigma + A] \to \mathbb{R}$ is a continuous function, then, for each $t \in [\sigma, \sigma + A]$, we define the function $x_t \in C_r$ by $x_t(\theta) := x(t+\theta)$, $\theta \in [-r, 0]$. Let us consider a retarded functional differential equation

$$\dot{x}(t) = f(t, x_t), \tag{3.1}$$

where $f : \Omega \mapsto \mathbb{R}$, $\Omega \subseteq \mathbb{R} \times C_r$ is a *continuous quasi-bounded map* which satisfies a *local Lipschitz condition* with respect to the second argument. We assume that the derivatives in the system (3.1) are at least right-sided.

In accordance with [1], a function x is said to be a *solution* of (3.1) *on* $[\sigma - r, \sigma + A)$ with $A > 0$ if $x : [\sigma - r, \sigma + A) \to \mathbb{R}$ is a continuous function, $(t, x_t) \in \Omega$ for $t \in [\sigma, \sigma + A)$, and x satisfies (3.1) on $[\sigma, \sigma + A)$. In view of the above conditions, each element $(\sigma, \varphi) \in \Omega$ determines a unique noncontinuable solution $x(\sigma, \varphi)$ of system (3.1) through $(\sigma, \varphi) \in \Omega$ on its maximal existence interval. This solution depends continuously on the initial data [19].

For continuously differentiable functions $\rho, \delta : [t_0 - r, \infty) \to \mathbb{R}$ with $\rho(t) < \delta(t)$ for $t \in [t_0 - r, \infty)$, we introduce the sets

$$\begin{aligned} \Omega &:= [t_0 - r, \infty) \times C_r, \\ \omega &:= \left\{ (t, x) : t \geq t_0 - r,\ \rho(t) < x < \delta(t) \right\}. \end{aligned} \tag{3.2}$$

In the sequel, we employ the following particular case of Theorem 1.1 from [10]. Its proof uses the topological (Ważewski's) *retract principle* known in the theory of ordinary differential equations. The relevant references can be found, for example, in [10].

Theorem 3.1. *Let for $t \geq t_0$, $\phi \in C_r$, and $(t+\theta, \phi(\theta)) \in \omega$, $\theta \in [-r, 0)$, the inequalities*

$$\delta'(t) < f(t, \phi) \quad \textit{when } \phi(0) = \delta(t), \tag{3.3}$$

$$\rho'(t) > f(t, \phi) \quad \textit{when } \phi(0) = \rho(t) \tag{3.4}$$

hold. Then there exists a solution of (3.1) *on* $[t_0 - r, \infty)$ *such that*

$$\rho(t) < x(t) < \delta(t), \quad t \in [t_0 - r, \infty). \tag{3.5}$$

Proof of Theorem 2.1. In our case $f(t, x_t) := -c(t)x(t-\tau(t))$ and $f(t, \phi) := -c(t)\phi(-\tau(t))$. We will apply Theorem 3.1 with

$$\begin{gathered} f(t, \phi) = -c(t)\phi(-\tau(t)), \\ \rho(t) = 0, \\ \delta(t) = \exp\left(-\mathrm{e}\int_{t_0}^{t} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s\right), \\ \omega = \left\{(t, x) : t \geq t_0 - r,\ 0 < x < \exp\left(-\mathrm{e}\int_{t_0}^{t} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s\right)\right\}. \end{gathered} \tag{3.6}$$

First we verify inequality (3.3). We get

$$\begin{aligned} \delta'(t) - f(t, \phi) &= \left[\exp\left(-\mathrm{e}\int_{t_0}^{t} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s\right)\right]' + c(t)\phi(-\tau(t)) \\ &= -\mathrm{e}\cdot c(t)\mathrm{e}^{-\tau(t)/(2t)}\exp\left(-\mathrm{e}\int_{t_0}^{t} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s\right) + c(t)\phi(-\tau(t)) = (*). \end{aligned} \tag{3.7}$$

Since, in accordance with the assumptions of the theorem, $(t+\theta, \phi(\theta)) \in \omega$ if $\theta \in [-r, 0)$, we can estimate the last term $\phi(-\tau(t))$:

$$\phi(-\tau(t)) < \delta(t - \tau(t)) = \exp\left(-\mathrm{e}\int_{t_0}^{t-\tau(t)} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s\right). \tag{3.8}$$

Continuing, we have

$$\begin{aligned}(*) &< -\mathrm{e}\cdot c(t)\mathrm{e}^{-\tau(t)/(2t)}\exp\left(-\mathrm{e}\int_{t_0}^{t} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s\right)\\ &\quad + c(t)\exp\left(-\mathrm{e}\int_{t_0}^{t-\tau(t)} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s\right)\\ &= c(t)\exp\left(-\mathrm{e}\int_{t_0}^{t-\tau(t)} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s\right)\\ &\quad\times\left[-\mathrm{e}\cdot\mathrm{e}^{-\tau(t)/(2t)}\exp\left(-\mathrm{e}\int_{t-\tau(t)}^{t} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s\right)+1\right] = (**).\end{aligned} \tag{3.9}$$

Using inequality (2.2), we can obtain the estimate

$$\exp\left(-\mathrm{e}\int_{t-\tau(t)}^{t} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s\right) \ge \mathrm{e}^{-1}\cdot\mathrm{e}^{\tau(t)/(2t)}. \tag{3.10}$$

Therefore,

$$(**) \le c(t)\exp\left(-\mathrm{e}\int_{t_0}^{t-\tau(t)} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s\right)\left[-\mathrm{e}\cdot\mathrm{e}^{-\tau(t)/(2t)}\cdot\mathrm{e}^{-1}\cdot\mathrm{e}^{\tau(t)/(2t)}+1\right] = 0, \tag{3.11}$$

and inequality (3.3) holds because $\delta'(t) - f(t,\phi) < 0$. It is much easier is to verify inequality (3.4) since

$$\rho'(t) - f(t,\phi) = 0 - \left(-c(t)\phi(-\tau(t))\right) = c(t)\phi(-\tau(t)) > 0. \tag{3.12}$$

All assumptions of Theorem 3.1 are fulfilled, and, therefore, by (3.5), there exists a solution $x : [t_0 - r, \infty) \to \mathbb{R}^*$ such that

$$0 < x(t) < \exp\left(-\mathrm{e}\int_{t_0}^{t} c(s)\mathrm{e}^{-\tau(s)/(2s)}\mathrm{d}s\right). \tag{3.13}$$

□

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