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Research Article

A Mixture Theory for Micropolar Thermoelastic Solids

C. Galeş

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We derive a nonlinear theory of heat-conducting micropolar mixtures in Lagrangian description. The kinematics, balance laws, and constitutive equations are examined and utilized to develop a nonlinear theory for binary mixtures of micropolar thermoelastic solids. The initial boundary value problem is formulated. Then, the theory is linearized and a uniqueness result is established.

1. Introduction

The importance of the study of mixtures was recognized long ago when the basic concepts of the theory have been established and the possible applications of the mathematical models have been identified. The origin of modern formulations of continuum thermomechanical theories of mixtures goes back to articles of Truesdell and Toupin [1], Kelly [2], Eringen and Ingram [3, 4], Green and Naghdi [5, 6], Green and Laws [7], Müller [8], and Bowen and Wiese [9]. The theoretical progress in the field is discussed in detail in review articles by Bowen [10], Atkin and Craine [11, 12], Bedford and Drumheller [13], and in the books of Samohýl [14] and Rajagopal and Tao [15].

In general, the theories of mixtures describe the interaction between fluids and gases, and the Eulerian description is used. Thus, the motion of a mixture of two continua is described by two equations, $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ and $\mathbf{y} = \mathbf{y}(\mathbf{Y}, t)$, and the particles X and Y are assumed to occupy the same position at current time t, so that $\mathbf{x} = \mathbf{y}$. In contrast to mixtures of fluids, the theory on mixtures of solids is developed naturally in the Lagrangian description and it leads to different results. In this case, the motion of a binary mixture is described by the equations $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ and $\mathbf{y} = \mathbf{y}(\mathbf{Y}, t)$, where the particles under consideration occupy the same position in the reference configuration, so that $\mathbf{X} = \mathbf{Y}$.

The Lagrangian description has been used for the first time by Bedford and Stern [16, 17] in order to derive a mixture theory of binary elastic solids. In this theory, the independent constitutive variables are displacement gradients and the relative displacement. The Lagrangian description was used by Pop and Bowen [18] to establish a theory of mixtures with long-range spatial interaction, by Tiersten and Jahanmir [19] to derive a theory of composites, by Ieşan [20] to elaborate on a binary mixture theory of thermoelastic solids, and by Ieşan [21] to develop a theory of nonsimple elastic solids.

A special attention has been paid to include some terms in the basic formulation of the theory of mixtures in order to reflect the microstructure of the constituents. In this sense, we mention that the Eulerian approach has been used in the papers by Allen and Kline [22], Twiss and Eringen [23, 24], Dunwoody [25], Passman [26], and Eringen [27, 28]. On the other hand, Ieşan [29–31] studied the mixtures of granular materials, the porous viscoelastic mixtures, and the interacting micromorphic materials in Lagrangian description.

In this work, we consider a mixture consisting of two micropolar thermoelastic solids. By using the nonlinear theory of micropolar media [32–34] and the results established in [5, 16, 17, 20, 30, 31], we derive the basic equations of a nonlinear theory in Lagrangian description. According to [32–34], each material point of such materials can independently translate and rotate, so that it has 6 degrees of freedom. The rotation is described by a proper orthogonal tensor.

We recall that in [23], the authors have derived the micromorphic and micropolar equations and entropy production inequalities for a mixture of any number of constituents in Eulerian description. The theory of mixtures for micromorphic materials is more general than the micropolar mixture theory, in that it also considers deformation of material points, requiring 12 degrees of freedom. The results have been used in [24] to construct the general form of the nonlinear, anisotropic, elastic, constitutive equations for micromorphic, and micropolar mixtures; as an illustration of a special case, the specific linear equations for an isotropic two-constituent micropolar mixture have been presented.

Concerning the mixtures of solids, it is important to have an alternative form of the theory. It is well known that for a fluid, every configuration of the body can be taken as reference configuration. So, the present configuration is taken as reference configuration and the Eulerian description is used to formulate the basic concepts. In the case of a solid, a configuration of the body is supposed to be known and this configuration is taken as reference configuration. The Lagrangian description is usually used, although the Eulerian description leads to the same theory. The situation changes in the case of mixtures; the Lagrangian description and the Eulerian description lead to different theories. For mixtures of solids, it is important to have a theory in which the motion is to be referred to the known configuration of the body in order to measure the forces and the stresses and to prescribe the boundary conditions. In this sense, Ieşan [31] pointed out that "*when the continuum has a reference configuration B through which it passes at time* t_0 *it is convenient to have an alternative form of the theory in which the generalized forces and stresses are measured with respect to this configuration. In the nonlinear theory this fact is important for specification of the boundary conditions, since the boundary of current configuration is,* *in general, unknown.*" In [31], Ieşan presented the basic equations of the nonlinear theory for binary mixtures of micromorphic elastic solids in Lagrangian description. Then, the initial boundary value problems were formulated and some uniqueness and continuous-dependence results were established.

In this paper, we continue the research line initiated in [31] concerning the theory of micromorphic mixtures developed in Lagrangian description. We focus our attention to the micropolar elastic mixtures, but we consider in addition the thermal effects. The intended applications of these kind of theories (see [24]) are to granular composites, to polycrystalline mixtures, or to polyatomic or polymolecular crystal latices. The next section is devoted to a kinematical study of the motion in which we present the measures of deformation and their rates. Then, the balance laws for mass, microinertia, energy, and production of entropy are constructed. The balance equations are derived by using an idea of Green and Naghdi [5]. We assume that the two constituents have a common temperature and every thermodynamical process that takes place in mixture satisfies the Clausius-Duhem inequality. We consider the following constitutive variables: the displacement fields, displacement gradients, micromotion fields, micromotion gradients, temperature, and temperature gradient. By using the constitutive axioms, we express the dependent constitutive variables in an invariant form and then we use the Clausius-Duhem inequality to develop constitutive equations. Initial boundary value problems for the nonlinear theory are formulated. In the second part of the paper, the theory is linearized and a uniqueness result is established.

2. Kinematics

We consider a mixture of two interacting continua s_1 and s_2. The mixture is viewed as a superposition of two continua each following its own motion and at any time each place in the mixture is occupied simultaneously by different particles, one from each constituent.

We assume that at time t_0 the body occupies the region B of Euclidean three-dimensional space and is bounded by piecewise smooth surface ∂B. The configuration of the body at time t_0 is taken as the reference configuration. We refer the motion of the body to the reference configuration and a fixed system of rectangular Cartesian axes. We use vector and Cartesian tensor notation with Latin indices having the values 1, 2, and 3. Greek indices are understood to range over integers $(1,2)$ and summation convention is not used for these indices.

The position of typical particles of s_1 and s_2 at time t are $\mathbf{x}$ and $\mathbf{y}$, where

$$\mathbf{x}=\mathbf{x}(\mathbf{X},t), \quad \mathbf{y}=\mathbf{y}(\mathbf{Y},t), \quad \mathbf{X},\, \mathbf{Y}\in B,\ t\in I. \tag{2.1}$$

Here $\mathbf{X}$ and $\mathbf{Y}$ are reference positions of the particles and $I=[t_0,t_1)$, where t_1 is some instant that may be infinity.

In this paper, we derive a theory for binary mixture of micropolar elastic solids where the particles under consideration occupy the same position in the reference configuration, so that $\mathbf{X}=\mathbf{Y}$. The translations of the body are described by

$$\mathbf{x}=\mathbf{x}(\mathbf{X},t), \quad \mathbf{y}=\mathbf{y}(\mathbf{X},t), \quad \mathbf{X}\in B,\ t\in I. \tag{2.2}$$

We stress that in the continuum theory of mixtures, the constituents are not physically separated and in consequence a great care must be exercised to preserve the generality that the constitutive relations for a given constituent depend upon the state variables of the other. This is not the case for multiphase materials or immiscible mixtures where the constituents remain physically separate on a scale which is large in comparison with molecular dimensions and in some local sense each constituent will obey the constitutive relations for that constituent alone.

We imagine that each particle of the mixture can also independently rotate. Just as the functions $x_i(\mathbf{X},t)$ and $y_i(\mathbf{X},t)$ specify the translation of each material particle of the mixture from the reference configuration, following [32, 33], we introduce proper orthogonal tensors that specify the rotation of each material particle from reference configuration. Let a particle of the constituent s_α occupy the position $\mathbf{X}$ in the reference configuration and let $\Xi_K^{(\alpha)}$ be an arbitrary vector at $\mathbf{X}$ associated with this material particle. In the context of micropolar continua, the above vector can rigidly rotate with the particle to the vector $\xi_k^{(\alpha)}$. Then we may write $\xi_k^{(\alpha)} = \chi_{kK}^{(\alpha)}\Xi_K^{(\alpha)}$, where $\boldsymbol{\chi}^{(\alpha)}$ is proper orthogonal, that is,

$$\chi_{iK}^{(\alpha)}\chi_{iL}^{(\alpha)} = \delta_{KL}, \qquad \chi_{iK}^{(\alpha)}\chi_{jK}^{(\alpha)} = \delta_{ij}, \qquad \epsilon_{LMN}\chi_{lL}^{(\alpha)}\chi_{mM}^{(\alpha)}\chi_{nN}^{(\alpha)} = \epsilon_{lmn} \tag{2.3}$$

and completely describes the rotation. In the above relations, δ_{ij} and δ_{KL} are Kronecker deltas and ϵ_{LMN} and ϵ_{lmn} are alternating symbols. The tensor $\chi_{iK}^{(\alpha)}(\mathbf{X},t)$ is usually called micromotion (see [33]). Thus, the motion of a micropolar mixture is described by

$$x_i = x_i(\mathbf{X},t), \quad y_i = y_i(\mathbf{X},t), \quad \chi_{iK}^{(\alpha)} = \chi_{iK}^{(\alpha)}(\mathbf{X},t), \quad \mathbf{X} \in B,\ t \in I,\ \alpha = 1,2. \tag{2.4}$$

We suppose that the above functions are sufficiently smooth for the ensuing analysis to be valid.

The velocity and acceleration fields associated with constituents s_1 and s_2 are

$$v_i^{(1)} = \dot{x}_i(\mathbf{X},t), \qquad a_i^{(1)} = \ddot{x}_i(\mathbf{X},t), \tag{2.5}$$

$$v_i^{(2)} = \dot{y}_i(\mathbf{X},t), \qquad a_i^{(2)} = \ddot{y}_i(\mathbf{X},t), \tag{2.6}$$

respectively, where $\dot{f}$ denotes differentiation of f with respect to t holding X_K fixed.

In view of the relation (2.3), we introduce the skew-symmetric tensor

$$\nu_{ij}^{(\alpha)}(\mathbf{X},t) = \dot{\chi}_{iK}^{(\alpha)}\chi_{jK}^{(\alpha)}, \quad \nu_{ij}^{(\alpha)} = -\nu_{ji}^{(\alpha)}. \tag{2.7}$$

The corresponding angular velocity is given by

$$\nu_i^{(\alpha)} = -\frac{1}{2}\epsilon_{ijk}\nu_{jk}^{(\alpha)}. \tag{2.8}$$

From (2.7) and (2.8), we deduce

$$\nu_i^{(\alpha)} = -\frac{1}{2}\epsilon_{ijk}\dot{\chi}_{jL}^{(\alpha)}\chi_{kL}^{(\alpha)}, \tag{2.9}$$

$$\dot{\chi}_{iK}^{(\alpha)} = \epsilon_{ijk}\nu_j^{(\alpha)}\chi_{kK}^{(\alpha)}. \tag{2.10}$$

We note that a rigid motion of the mixture is described by the relations

$$\begin{aligned} x_i(\mathbf{X},t) &= Q_{iK}(t)X_K + \widetilde{c}_i(t), \qquad & y_i(\mathbf{X},t) &= Q_{iK}(t)X_K + \widetilde{c}_i(t), \\ \chi^{(1)}_{iK}(\mathbf{X},t) &= Q_{iK}(t), \qquad & \chi^{(2)}_{iK}(\mathbf{X},t) &= Q_{iK}(t), \end{aligned} \tag{2.11}$$

where $\mathbf{Q}$ is proper orthogonal. For rigid motions, from (2.10) and (2.11), we obtain

$$v^{(1)}_i = \epsilon_{ijk}b_j x_k + c_i, \qquad v^{(2)}_i = \epsilon_{ijk}b_j y_k + c_i, \qquad \nu^{(1)}_i = \nu^{(2)}_i = b_i, \tag{2.12}$$

where b_i is the axial vector of the skew-symmetric tensor $\dot{Q}_{iK}Q_{jK}$ and $c_i = \widetilde{c}_i - \dot{Q}_{iK}Q_{jK}\widetilde{c}_j$.

Later in this work, we shall need the concept of change of frame. We say that two motions of the mixture described by the functions $x_i(\mathbf{X},t)$, $y_i(\mathbf{X},t)$, $\chi^{(\alpha)}_{iK}(\mathbf{X},t)$, and $\widetilde{x}_i(\mathbf{X},t')$, $\widetilde{y}_i(\mathbf{X},t')$, $\widetilde{\chi}^{(\alpha)}_{iK}(\mathbf{X},t')$, respectively, are equivalent if

$$\begin{aligned} \widetilde{x}_i(\mathbf{X},t') &= Q_{ij}(t)x_j(\mathbf{X},t) + c_i(t), \quad & \widetilde{\chi}^{(1)}_{iK}(\mathbf{X},t') &= Q_{ij}(t)\chi^{(1)}_{jK}(\mathbf{X},t), \\ \widetilde{y}_i(\mathbf{X},t') &= Q_{ij}(t)y_j(\mathbf{X},t) + c_i(t), \quad & \widetilde{\chi}^{(2)}_{iK}(\mathbf{X},t') &= Q_{ij}(t)\chi^{(2)}_{jK}(\mathbf{X},t), \quad t' = t - a, \end{aligned} \tag{2.13}$$

where a is a constant, c_i represent a translation, and $\mathbf{Q}$ is an orthogonal tensor. In fact, the motions are equivalent if they differ by the reference frame and by the reference time.

Let us introduce the following measures of deformation:

$$\begin{aligned} E_{KL} &= x_{i,K}\chi^{(1)}_{iL} - \delta_{KL}, \qquad & G_{KL} &= y_{i,K}\chi^{(1)}_{iL} - \delta_{KL}, \qquad & D_K &= \chi^{(1)}_{iK}(x_i - y_j), \\ 2\Gamma^{(1)}_{KL} &= \epsilon_{LMN}\chi^{(1)}_{iN}\chi^{(1)}_{iM,K}, \qquad & 2\Gamma^{(2)}_{KL} &= \epsilon_{LMN}\chi^{(2)}_{iN}\chi^{(2)}_{iM,K}, \qquad & \Delta_{KL} &= \chi^{(1)}_{iK}\chi^{(2)}_{iL} - \delta_{KL}. \end{aligned} \tag{2.14}$$

In the case of a single micropolar medium, the strain measures are $\mathbf{E}$ and $\mathbf{\Gamma}^{(1)}$ and their rates as well as their geometrical significance are examined in detail in [32, 33]. $\mathbf{E}$ and $\mathbf{\Gamma}^{(1)}$ are called the Cosserat deformation tensor and wryness tensor, respectively. In the context of the theory of micropolar mixtures, we introduce further the measures $\mathbf{D}$ and $\mathbf{\Delta}$ which describe relative motions of constituents (translations and rotations). Moreover, for reasons that will be obvious in the next section we use the tensor G_{KL} for the second constituent instead of the Cosserat deformation tensor $E^{(2)}_{KL} = y_{i,K}\chi^{(2)}_{iL} - \delta_{KL}$. As regards the kinematics, this is unimportant since $E^{(2)}_{KL}$ can be written in terms of G_{KL} and Δ_{KL} in the form

$$E^{(2)}_{KL} = y_{i,K}\delta_{ij}\chi^{(2)}_{jL} - \delta_{KL} = y_{i,K}\chi^{(1)}_{iM}\chi^{(1)}_{jM}\chi^{(2)}_{jL} - \delta_{KL} = G_{KM}\Delta_{ML} + G_{KL} + \Delta_{KL}. \tag{2.15}$$

It is easy to see that two motions described by the functions $x_i(\mathbf{X},t)$, $y_i(\mathbf{X},t)$, $\chi^{(\alpha)}_{iK}(\mathbf{X},t)$, and $\widetilde{x}_i(\mathbf{X},t')$, $\widetilde{y}_i(\mathbf{X},t')$, $\widetilde{\chi}^{(\alpha)}_{iK}(\mathbf{X},t')$, respectively, and related to (2.13), produce equal measures of deformations.

The time rates of $\mathbf{E}$, $\mathbf{G}$, $\boldsymbol{\Gamma}^{(\alpha)}$, $\mathbf{D}$, and $\boldsymbol{\Delta}$ computed from (2.3), (2.5), (2.9), (2.10), and (2.14) are

$$\begin{gathered}
\dot{E}_{KL}=\chi_{iL}^{(1)}\left(v_{i,K}^{(1)}+\epsilon_{ijk}x_{j,K}v_k^{(1)}\right), \qquad \dot{G}_{KL}=\chi_{iL}^{(1)}\left(v_{i,K}^{(2)}+\epsilon_{ijk}y_{j,K}v_k^{(1)}\right),\\
\dot{\Gamma}_{KL}^{(1)}=\chi_{iL}^{(1)}v_{i,K}^{(1)}, \qquad \dot{\Gamma}_{KL}^{(2)}=\chi_{iL}^{(2)}v_{i,K}^{(2)},\\
\dot{D}_K=\chi_{iK}^{(1)}\left(v_i^{(1)}-v_i^{(2)}+\epsilon_{ijk}(x_j-y_j)v_k^{(1)}\right), \qquad \dot{\Delta}_{KL}=\epsilon_{ijm}\chi_{mK}^{(1)}\chi_{iL}^{(2)}\left(v_j^{(1)}-v_j^{(2)}\right).
\end{gathered} \tag{2.16}$$

If these rates are zero, then the differential equations can be integrated. Following [32], we deduce that the solution is a rigid motion of the form (2.11).

By using the relation (2.3) and the properties of the alternating symbol, from (2.16) we obtain

$$\begin{gathered}
v_{i,K}^{(1)}+\epsilon_{ijk}x_{j,K}v_k^{(1)}=\chi_{iL}^{(1)}\dot{E}_{KL}, \qquad v_{i,K}^{(2)}+\epsilon_{ijk}y_{j,K}v_k^{(1)}=\chi_{iL}^{(1)}\dot{G}_{KL},\\
v_{i,K}^{(1)}=\chi_{iL}^{(1)}\dot{\Gamma}_{KL}^{(1)}, \qquad v_{i,K}^{(2)}=\chi_{iL}^{(2)}\dot{\Gamma}_{KL}^{(2)},\\
v_i^{(1)}-v_i^{(2)}+\epsilon_{ijk}(x_j-y_j)v_k^{(1)}=\chi_{iK}^{(1)}\dot{D}_K, \quad v_i^{(1)}-v_i^{(2)}=\epsilon_{ijm}\chi_{jK}^{(1)}\chi_{mL}^{(2)}\dot{\Delta}_{KL}.
\end{gathered} \tag{2.17}$$

3. Basic laws

In this section, we postulate the balance laws of micropolar mixtures and then we derive the local balance equations. We consider the following balance laws: conservation of mass for each constituent, conservation of microinertia for each constituent, conservation of energy for the mixture as a whole, and law of entropy. We suppose that the mixture is chemical inert so that we do not consider the axiom of balance of mass (or microinertia) for the mixture. The balance of momentum and balance of moment of momentum for each constituent are derived following an idea by Green and Naghdi [5]. As regards the entropy production inequality, we suppose that the constituents have a common temperature, so that the axiom involves only the statement concerning the mixture as a whole.

We consider an arbitrary material region P_α of constituent s_α at time t bounded by the surface ∂P_α, and we suppose that P_0 is the corresponding region at time t_0 bounded by the surface ∂P_0. The equation of balance of mass for the constituent s_α is

$$\frac{d}{dt}\int_{P_\alpha}\rho_\alpha dv=\int_{P_\alpha}m_\alpha dv, \tag{3.1}$$

where ρ_α is the mass density of the constituent s_α; and m_α is the rate at which mass is supplied to s_α per unit volume from the other constituent. In this paper, we assume that mass elements of each constituent are conserved so that $m_1=0$ and $m_2=0$. The relation (3.1) can be written in the form

$$\frac{d}{dt}\int_{P_0}J_{(\alpha)}\rho_\alpha dV=0, \tag{3.2}$$

where

$$J_{(1)} = \det\left(\frac{\partial x_i}{\partial X_A}\right), \qquad J_{(2)} = \det\left(\frac{\partial y_i}{\partial X_A}\right). \tag{3.3}$$

With the usual assumptions, from (3.2) we deduce

$$J_{(1)}\rho_1 = \rho_1^0, \qquad J_{(2)}\rho_2 = \rho_2^0, \tag{3.4}$$

where ρ_α^0 is the mass density of the constituent s_α at time t_0.

Since each material particle of the micropolar mixture is envisioned as a rigid particle, we ascribe to the particle of constituent s_α that occupy the position $\mathbf{X}$ in the reference configuration a material inertia tensor $I_{KL}(\mathbf{X})$, which is symmetric and positive definite. At time t, the inertia tensor of the particle in consideration is denoted by $i_{kl}^{(1)}(\mathbf{x},t)$ (if the material particle belongs to constituent s_1) or $i_{kl}^{(2)}(\mathbf{y},t)$, otherwise. Conservation of microinertia is stated as [32, 33]

$$\frac{d}{dt}\int_{P_\alpha} \rho_\alpha i_{kl}^{(\alpha)} \chi_{kK}^{(\alpha)} \chi_{lL}^{(\alpha)} dv = 0, \tag{3.5}$$

If we proceed as above, from (3.4) and (3.5), we deduce

$$I_{KL}^{(\alpha)} = i_{kl}^{(\alpha)} \chi_{kK}^{(\alpha)} \chi_{lL}^{(\alpha)}. \tag{3.6}$$

In view of (2.10) and symmetry of $i_{ij}^{(\alpha)}$, the above relation leads to

$$\frac{di_{ij}^{(\alpha)}}{dt} = \left(\epsilon_{imn} i_{nj}^{(\alpha)} + \epsilon_{jmn} i_{ni}^{(\alpha)}\right) \nu_m^{(\alpha)}. \tag{3.7}$$

Following [7, 30], we postulate an energy balance at time t in the form

$$\begin{aligned} &\frac{d}{dt}\sum_{\alpha=1}^{2}\int_{P_\alpha} \rho_\alpha\left(e + \frac{1}{2}v_i^{(\alpha)}v_i^{(\alpha)} + \frac{1}{2}i_{ij}^{(\alpha)}\nu_i^{(\alpha)}\nu_j^{(\alpha)}\right)dv \\ &\quad = \sum_{\alpha=1}^{2}\left[\int_{P_\alpha} \rho_\alpha\left(F_i^{(\alpha)}v_i^{(\alpha)} + G_i^{(\alpha)}\nu_i^{(\alpha)} + r\right)dv + \int_{\partial P_\alpha}\left(t_i^{(\alpha)}v_i^{(\alpha)} + m_i^{(\alpha)}\nu_i^{(\alpha)} + q^{(\alpha)}\right)da\right], \end{aligned} \tag{3.8}$$

where e is the internal energy of the mixture per unit mass; $\mathbf{F}^{(\alpha)}$ is the body force per unit mass acting on the constituent s_α; $\mathbf{G}^{(\alpha)}$ is the body couple per unit mass; r is the external volume supply per unit mass per unit time; $\mathbf{t}^{(\alpha)}$ is the partial stress vector; $\mathbf{m}^{(\alpha)}$ is the partial couple stress vector; and $q^{(\alpha)}$ is the heat flux per unit area per unit time associated with the constituent s_α. By using (3.4), we may write (3.8) in the form

$$\frac{d}{dt}\sum_{\alpha=1}^{2}\int_{P_0}\rho_\alpha^0\left(e+\frac{1}{2}v_i^{(\alpha)}v_i^{(\alpha)}+\frac{1}{2}i_{ij}^{(\alpha)}\nu_i^{(\alpha)}\nu_j^{(\alpha)}\right)dV$$
$$=\sum_{\alpha=1}^{2}\left[\int_{P_0}\rho_\alpha^0\left(F_i^{(\alpha)}v_i^{(\alpha)}+G_i^{(\alpha)}\nu_i^{(\alpha)}+r\right)dV+\int_{\partial P_0}\left(T_i^{(\alpha)}v_i^{(\alpha)}+M_i^{(\alpha)}\nu_i^{(\alpha)}+Q^{(\alpha)}\right)dA\right], \tag{3.9}$$

where $\mathbf{T}^{(\alpha)}$, $\mathbf{M}^{(\alpha)}$, $Q^{(\alpha)}$ are the partial stress, the partial couple stress, and the heat flux, respectively, associated with the surface ∂P_α but measured per unit area of ∂P_0. From (3.9), we deduce

$$\sum_{\alpha=1}^{2}\int_{P_0}\rho_\alpha^0\left(\dot{e}+v_i^{(\alpha)}a_i^{(\alpha)}+i_{ij}^{(\alpha)}\nu_i^{(\alpha)}\dot{\nu}_j^{(\alpha)}\right)dV$$
$$=\sum_{\alpha=1}^{2}\left[\int_{P_0}\rho_\alpha^0\left(F_i^{(\alpha)}v_i^{(\alpha)}+G_i^{(\alpha)}\nu_i^{(\alpha)}+r\right)dV+\int_{\partial P_0}\left(T_i^{(\alpha)}v_i^{(\alpha)}+M_i^{(\alpha)}\nu_i^{(\alpha)}+Q^{(\alpha)}\right)dA\right]. \tag{3.10}$$

In (3.10), we used (3.7) to obtain $(1/2)(di_{ij}^{(\alpha)}/dt)\nu_i^{(\alpha)}\nu_j^{(\alpha)}=\epsilon_{imn}\nu_m^{(\alpha)}\nu_i^{(\alpha)}\nu_j^{(\alpha)}i_{nj}^{(\alpha)}=0$.

Let us now require that the expression given by (3.10) is invariant with respect to the rigid motion of the body (see (2.11) and (2.12)). Consider first motions of the mixture that differ from those given by (2.4) only by superposed uniform rigid body translational velocities, the continuum occupying the same position at time t. We assume that this motion does not lead to any changes of the quantities ρ_α^0, e, $i_{ij}^{(\alpha)}$, $F_i^{(\alpha)}$, $G_i^{(\alpha)}$, r, $T_i^{(\alpha)}$, $M_i^{(\alpha)}$, and $Q^{(\alpha)}$. Thus the relation (3.10) is also true when $\mathbf{v}^{(\alpha)}$ is replaced by $\mathbf{v}^{(\alpha)}+\mathbf{c}$, so that by substraction, we have

$$c_i\sum_{\alpha=1}^{2}\left[\int_{P_0}\rho_\alpha^0\left(a_i^{(\alpha)}-F_i^{(\alpha)}\right)dV-\int_{\partial P_0}T_i^{(\alpha)}dA\right]=0, \tag{3.11}$$

for all arbitrary constants c_i. We deduce

$$\sum_{\alpha=1}^{2}\left[\int_{P_0}\rho_\alpha^0\left(a_i^{(\alpha)}-F_i^{(\alpha)}\right)dV-\int_{\partial P_0}T_i^{(\alpha)}dA\right]=0, \tag{3.12}$$

for arbitrary region $P_0\subset B$. From (3.12), by usual procedures, we obtain

$$T_i^{(1)}+T_i^{(2)}=\left(T_{Ki}^{(1)}+T_{Ki}^{(2)}\right)N_K, \tag{3.13}$$

where $T_{Ki}^{(\alpha)}$ is the first Piola-Kirchhoff partial stress associated with constituent s_α; and N_K are the components of the unit outward normal vector to the surface ∂P_0. It follows from

(3.12) and (3.13) that

$$\sum_{\alpha=1}^{2}\left[T_{Ki,K}^{(\alpha)}+\rho_\alpha^0\left(F_i^{(\alpha)}-a_i^{(\alpha)}\right)\right]=0. \tag{3.14}$$

If we write the term $T_i^{(1)}v_i^{(1)}+T_i^{(2)}v_i^{(2)}$ in the form

$$T_i^{(1)}v_i^{(1)}+T_i^{(2)}v_i^{(2)}=\frac{1}{2}\left(T_i^{(1)}+T_i^{(2)}\right)\left(v_i^{(1)}+v_i^{(2)}\right)+\frac{1}{2}\left(T_i^{(1)}-T_i^{(2)}\right)\left(v_i^{(1)}-v_i^{(2)}\right), \tag{3.15}$$

and use (3.13) and (3.14), then (3.10) reduces to

$$\begin{aligned}\int_{P_0}&\left[\rho_0\dot{e}+\frac{1}{2}\left(\rho_1^0a_i^{(1)}-\rho_2^0a_i^{(2)}\right)\left(v_i^{(1)}-v_i^{(2)}\right)-\frac{1}{2}\left(\rho_1^0F_i^{(1)}-\rho_2^0F_i^{(2)}\right)\left(v_i^{(1)}-v_i^{(2)}\right)\right.\\&\qquad\left.-\frac{1}{2}\left(T_{Ki}^{(1)}+T_{Ki}^{(2)}\right)\left(v_{i,K}^{(1)}+v_{i,K}^{(2)}\right)\right]dV+\sum_{\alpha=1}^{2}\int_{P_0}\rho_\alpha^0\left(i_{ij}^{(\alpha)}v_i^{(\alpha)}\dot{v}_j^{(\alpha)}-G_i^{(\alpha)}v_i^{(\alpha)}-r\right)dV\\&=\int_{\partial P_0}\left[\frac{1}{2}\left(T_i^{(1)}-T_i^{(2)}\right)\left(v_i^{(1)}-v_i^{(2)}\right)+M_i^{(1)}v_i^{(1)}+M_i^{(2)}v_i^{(2)}+Q\right]dA,\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}\rho_0&=\rho_1^0+\rho_2^0,\\Q&=Q^{(1)}+Q^{(2)}.\end{aligned} \tag{3.17}$$

By using an argument similar to that used in obtaining the relation (3.13) from (3.16), we deduce

$$\begin{aligned}&\frac{1}{2}\left[T_i^{(1)}-T_i^{(2)}-\left(T_{Ki}^{(1)}-T_{Ki}^{(2)}\right)N_K\right]\left(v_i^{(1)}-v_i^{(2)}\right)+\left(M_i^{(1)}-M_{Ki}^{(1)}N_K\right)v_i^{(1)}\\&\qquad+\left(M_i^{(2)}-M_{Ki}^{(2)}N_K\right)v_i^{(2)}+Q-Q_KN_K=0,\end{aligned} \tag{3.18}$$

where $M_{Ki}^{(\alpha)}$ is the partial couple stress tensor associated with the constituent s_α; and Q_K is the heat flux vector. Using (3.18) in (3.16) and applying the resulting equation to an arbitrary region P_0, we obtain

$$\rho_0\dot{e}=\sum_{\alpha=1}^{2}\left(T_{Ki}^{(\alpha)}v_{i,K}^{(\alpha)}+M_{Ki}^{(\alpha)}v_{i,K}^{(\alpha)}+R_i^{(\alpha)}v_i^{(\alpha)}\right)+P_i\left(v_i^{(1)}-v_i^{(2)}\right)+Q_{K,K}+\rho_0r, \tag{3.19}$$

where

$$\begin{aligned}P_i&=\frac{1}{2}\left[T_{Ki,K}^{(1)}+\rho_1^0F_i^{(1)}-\rho_1^0a_i^{(1)}-T_{Ki,K}^{(2)}-\rho_2^0F_i^{(2)}+\rho_2^0a_i^{(2)}\right],\\R_i^{(\alpha)}&=M_{Ki,K}^{(\alpha)}+\rho_\alpha^0G_i^{(\alpha)}-\rho_\alpha^0i_{ij}^{(\alpha)}\dot{v}_j^{(\alpha)}.\end{aligned} \tag{3.20}$$

Assume now that the expression (3.19) is invariant with respect to a rotation of the body with a constant angular velocity **b**. Thus, by using (2.12), we suppose that

$$\frac{\partial v_i^{(1)}}{\partial x_j} \longrightarrow \frac{\partial v_i^{(1)}}{\partial x_j} + \epsilon_{jis} b_s, \qquad \frac{\partial v_i^{(2)}}{\partial y_j} \longrightarrow \frac{\partial v_i^{(2)}}{\partial y_j} + \epsilon_{jis} b_s,$$
$$v_i^{(1)} \longrightarrow v_i^{(1)} + b_i, \qquad v_i^{(2)} \longrightarrow v_i^{(2)} + b_i, \qquad v_i^{(1)} - v_i^{(2)} \longrightarrow v_i^{(1)} - v_i^{(2)} + \epsilon_{jis} b_s (x_j - y_j). \tag{3.21}$$

Introducing the above into (3.19) and assuming that the quantities ρ_0, e, $T_{Ki}^{(\alpha)}$, $M_{Ki}^{(\alpha)}$, P_i, $R_i^{(\alpha)}$, r, and $Q_{K,K}$ remain invariant, by substraction, we obtain

$$R_s^{(1)} + R_s^{(2)} + \epsilon_{sji} [T_{Ki}^{(1)} x_{j,K} + T_{Ki}^{(2)} y_{j,K} + P_i (x_j - y_j)] = 0. \tag{3.22}$$

By using the relation (3.22), we have

$$\begin{aligned} R_s^{(1)} v_s^{(1)} + R_s^{(2)} v_s^{(2)} &= (R_s^{(1)} + R_s^{(2)}) v_s^{(1)} - R_s^{(2)} (v_s^{(1)} - v_s^{(2)}) \\ &= \epsilon_{sij} [T_{Ki}^{(1)} x_{j,K} + T_{Ki}^{(2)} y_{j,K} + P_i (x_j - y_j)] v_s^{(1)} - R_s^{(2)} (v_s^{(1)} - v_s^{(2)}). \end{aligned} \tag{3.23}$$

From (3.23), we can write the energy equation (3.19) in the form

$$\begin{aligned} \rho_0 \dot{e} = {} & T_{Ki}^{(1)} (v_{i,K}^{(1)} + \epsilon_{ijk} x_{j,K} v_k^{(1)}) + T_{Ki}^{(2)} (v_{i,K}^{(2)} + \epsilon_{ijk} y_{j,K} v_k^{(1)}) + M_{Ki}^{(1)} v_{i,K}^{(1)} + M_{Ki}^{(2)} v_{i,K}^{(2)} \\ & + P_i [v_i^{(1)} - v_i^{(2)} + \epsilon_{ijk} (x_j - y_j) v_k^{(1)}] - R_i^{(2)} (v_i^{(1)} - v_i^{(2)}) + Q_{K,K} + \rho_0 r. \end{aligned} \tag{3.24}$$

Moreover it follows from (3.14), (3.20), and (3.22) that the equations of motion are

$$\begin{aligned} T_{Ki,K}^{(1)} - P_i + \rho_1^0 F_i^{(1)} &= \rho_1^0 a_i^{(1)}, \\ T_{Ki,K}^{(2)} + P_i + \rho_2^0 F_i^{(2)} &= \rho_2^0 a_i^{(2)}, \\ M_{Ki,K}^{(1)} + \epsilon_{ijs} [T_{Ks}^{(1)} x_{j,K} + T_{Ks}^{(2)} y_{j,K} + P_s (x_j - y_j)] + R_i^{(2)} + \rho_1^0 G_i^{(1)} &= \rho_1^0 i_{ij}^{(1)} \dot{v}_j^{(1)}, \\ M_{Ki,K}^{(2)} - R_i^{(2)} + \rho_2^0 G_i^{(2)} &= \rho_2^0 i_{ij}^{(2)} \dot{v}_j^{(2)}. \end{aligned} \tag{3.25}$$

We assume that the constituents have a common temperature and we adopt the following entropy production inequality(see [30, 35]):

$$\sum_{\alpha=1}^{2} \left[\frac{d}{dt} \int_{P_\alpha} \rho_\alpha \eta \, dv - \int_{P_\alpha} \frac{\rho_\alpha r}{\theta} dv - \int_{\partial P_\alpha} \frac{q^{(\alpha)}}{\theta} da \right] \geq 0, \tag{3.26}$$

where η is the entropy per unit mass of the mixture and θ is the absolute temperature. The relations (3.4) and (3.26) yield

$$\int_{P_0} \rho_0 \dot{\eta} dV - \int_{P_0} \frac{\rho_0 r}{\theta} dV - \int_{\partial P_0} \frac{Q}{\theta} dA \geq 0. \tag{3.27}$$

From (3.18), we deduce

$$Q = Q_K N_K, \tag{3.28}$$

and the inequality (3.27) reduces to

$$\rho_0 \theta \dot{\eta} - \rho_0 r - Q_{K,K} + \frac{1}{\theta} Q_K \theta_{,K} \geq 0. \tag{3.29}$$

The relation (2.16) and the second Piola-Kirchhoff quantities, defined by

$$\begin{gathered} T_{Ki}^{(1)} = T_{KL}^{(1)} \chi_{iL}^{(1)}, \qquad T_{Ki}^{(2)} = T_{KL}^{(2)} \chi_{iL}^{(1)}, \qquad M_{Ki}^{(1)} = M_{KL}^{(1)} \chi_{iL}^{(1)}, \qquad M_{Ki}^{(2)} = M_{KL}^{(2)} \chi_{iL}^{(2)}, \\ P_i = \mathcal{P}_K \chi_{iK}^{(1)}, \qquad R_j^{(2)} = -\epsilon_{ijm} \chi_{mK}^{(1)} \chi_{iL}^{(2)} \mathcal{R}_{KL}, \end{gathered} \tag{3.30}$$

may be used to write the equation of energy (3.24) in the following material form:

$$\begin{aligned} \rho_0 \dot{e} = {} & T_{KL}^{(1)} \dot{E}_{KL} + T_{KL}^{(2)} \dot{G}_{KL} + M_{KL}^{(1)} \dot{\Gamma}_{KL}^{(1)} + M_{KL}^{(2)} \dot{\Gamma}_{KL}^{(2)} \\ & + \mathcal{P}_K \dot{D}_K + \mathcal{R}_{KL} \dot{\Delta}_{KL} + Q_{K,K} + \rho_0 r. \end{aligned} \tag{3.31}$$

Let us introduce the Helmholtz free energy $\Upsilon = e - \eta\theta$. Then the energy equation (3.31) may be written in the form

$$\begin{aligned} \rho_0 (\dot{\Upsilon} + \dot{\theta}\eta + \theta\dot{\eta}) = {} & T_{KL}^{(1)} \dot{E}_{KL} + T_{KL}^{(2)} \dot{G}_{KL} + M_{KL}^{(1)} \dot{\Gamma}_{KL}^{(1)} + M_{KL}^{(2)} \dot{\Gamma}_{KL}^{(2)} \\ & + \mathcal{P}_K \dot{D}_K + \mathcal{R}_{KL} \dot{\Delta}_{KL} + Q_{K,K} + \rho_0 r. \end{aligned} \tag{3.32}$$

With the help of (3.32), the inequality (3.29) becomes

$$T_{KL}^{(1)} \dot{E}_{KL} + T_{KL}^{(2)} \dot{G}_{KL} + M_{KL}^{(1)} \dot{\Gamma}_{KL}^{(1)} + M_{KL}^{(2)} \dot{\Gamma}_{KL}^{(2)} + \mathcal{P}_K \dot{D}_K + \mathcal{R}_{KL} \dot{\Delta}_{KL} - \rho_0 \dot{\Upsilon} - \rho_0 \dot{\theta} \eta + \frac{1}{\theta} Q_K \theta_{,K} \geq 0. \tag{3.33}$$

The quantities $T_{KL}^{(\alpha)}$, $M_{KL}^{(\alpha)}$, $\mathcal{P}_K$, $\mathcal{R}_{KL}$, Υ, η, and Q_K must be prescribed by constitutive equations.

4. Constitutive equations

In this section, we will state the constitutive equations that serve to classify the particular types of mixtures to be studied throughout the remainder of the paper. To be concrete, we assume that the mixture is consisting of two simple thermoelastic solids. According to basic postulates, the independent constitutive variables are

$$\mathcal{A} = \left(x_i, x_{i,K}, y_i, y_{i,K}, \chi_{iK}^{(1)}, \chi_{iK,L}^{(1)}, \chi_{iK}^{(2)}, \chi_{iK,L}^{(2)}, \theta, \theta_{,K}; X_K\right). \tag{4.1}$$

So that, the constitutive equations are

$$\begin{gathered} T_{KL}^{(\alpha)} = T_{KL}^{(\alpha)}(\mathcal{A}), \qquad M_{KL}^{(\alpha)} = M_{KL}^{(\alpha)}(\mathcal{A}), \qquad \mathcal{P}_K = \mathcal{P}_K(\mathcal{A}), \qquad \mathcal{R}_{KL} = \mathcal{R}_{KL}(\mathcal{A}), \\ \Upsilon = \Upsilon(\mathcal{A}), \qquad \eta = \eta(\mathcal{A}), \qquad Q_K = Q_K(\mathcal{A}), \end{gathered} \tag{4.2}$$

where the constitutive functionals are assumed to be sufficiently smooth. For homogeneous continua, the response functionals do not depend on X_K explicitly.

Let us now consider the restrictions imposed by the axiom of material frame-indifference. Following this principle, the constitutive functionals are form-invariant under the rigid body motions of the spatial frame of reference (see (2.13)). So that, by considering a translational motion of the frame of reference described by the vector $c_i(t) = y_i(X_K, t)$, then we conclude that the constitutive functionals depend on x_i and y_i only through the relative displacement $x_i - y_i$. Moreover, if we consider a rotation of the form $\mathbf{Q} = [\boldsymbol{\chi}^{(1)}]^T$, where $[\boldsymbol{\chi}^{(1)}]^T$ denotes the transpose of $\boldsymbol{\chi}^{(1)}$, and take into account the fact that

$$\chi_{iM}^{(1)} \chi_{iK,L}^{(2)} = \chi_{iM}^{(1)} \delta_{ij} \chi_{jK,L}^{(2)} = \chi_{iM}^{(1)} \chi_{iN}^{(2)} \chi_{jN}^{(2)} \chi_{jK,L}^{(2)} = f\left(\Delta_{KL}, \Gamma_{KL}^{(2)}\right), \tag{4.3}$$

then, in view of (2.14), it follows that $T_{KL}^{(\alpha)}$, $M_{KL}^{(\alpha)}$, $\mathcal{P}_K$, $\mathcal{R}_{KL}$, Υ, η, and Q_K must be expressible in the following invariant form:

$$\begin{gathered} T_{KL}^{(\alpha)} = T_{KL}^{(\alpha)}\left(E_{MN}, G_{MN}, \Gamma_{MN}^{(1)}, \Gamma_{MN}^{(2)}, D_M, \Delta_{MN}, \theta, \theta_{,M}, ; X_M\right), \\ M_{KL}^{(\alpha)} = M_{KL}^{(\alpha)}\left(E_{MN}, G_{MN}, \Gamma_{MN}^{(1)}, \Gamma_{MN}^{(2)}, D_M, \Delta_{MN}, \theta, \theta_{,M}, ; X_M\right), \\ \mathcal{P}_K^{(\alpha)} = \mathcal{P}_K^{(\alpha)}\left(E_{MN}, G_{MN}, \Gamma_{MN}^{(1)}, \Gamma_{MN}^{(2)}, D_M, \Delta_{MN}, \theta, \theta_{,M}, ; X_M\right), \\ \mathcal{R}_{KL}^{(\alpha)} = \mathcal{R}_{KL}^{(\alpha)}\left(E_{MN}, G_{MN}, \Gamma_{MN}^{(1)}, \Gamma_{MN}^{(2)}, D_M, \Delta_{MN}, \theta, \theta_{,M}, ; X_M\right), \\ \Upsilon = \Upsilon\left(E_{MN}, G_{MN}, \Gamma_{MN}^{(1)}, \Gamma_{MN}^{(2)}, D_M, \Delta_{MN}, \theta, \theta_{,M}, ; X_M\right), \\ \eta = \eta\left(E_{MN}, G_{MN}, \Gamma_{MN}^{(1)}, \Gamma_{MN}^{(2)}, D_M, \Delta_{MN}, \theta, \theta_{,M}, ; X_M\right), \\ Q_K = Q_K\left(E_{MN}, G_{MN}, \Gamma_{MN}^{(1)}, \Gamma_{MN}^{(2)}, D_M, \Delta_{MN}, \theta, \theta_{,M}, ; X_M\right). \end{gathered} \tag{4.4}$$

With the help of (4.4), inequality (3.33) becomes

$$\left(T_{KL}^{(1)} - \frac{\partial\sigma}{\partial E_{KL}}\right)\dot{E}_{KL} + \left(T_{KL}^{(2)} - \frac{\partial\sigma}{\partial G_{KL}}\right)\dot{G}_{KL} + \left(M_{KL}^{(1)} - \frac{\partial\sigma}{\partial\Gamma_{KL}^{(1)}}\right)\dot{\Gamma}_{KL}^{(1)}$$
$$+ \left(M_{KL}^{(2)} - \frac{\partial\sigma}{\partial\Gamma_{KL}^{(2)}}\right)\dot{\Gamma}_{KL}^{(2)} + \left(\mathcal{P}_K - \frac{\partial\sigma}{\partial D_K}\right)\dot{D}_K + \left(\mathcal{R}_{KL} - \frac{\partial\sigma}{\partial\Delta_{KL}}\right)\dot{\Delta}_{KL} \tag{4.5}$$
$$- \left(\rho_0\eta + \frac{\partial\sigma}{\partial\theta}\right)\dot{\theta} - \frac{\partial\sigma}{\partial\theta_{,K}}\dot{\theta}_{,K} + \frac{1}{\theta}Q_K\theta_{,K} \geq 0,$$

where $\sigma = \rho_0\Upsilon$.

From (4.5), we deduce

$$\sigma = \sigma\left(E_{MN}, G_{MN}, \Gamma_{MN}^{(1)}, \Gamma_{MN}^{(2)}, D_M, \Delta_{MN}, \theta; X_M\right), \tag{4.6}$$

$$T_{KL}^{(1)} = \frac{\partial\sigma}{\partial E_{KL}}, \qquad T_{KL}^{(2)} = \frac{\partial\sigma}{\partial G_{KL}}, \qquad M_{KL}^{(\alpha)} = \frac{\partial\sigma}{\partial\Gamma_{KL}^{(\alpha)}},$$
$$\mathcal{P}_K = \frac{\partial\sigma}{\partial D_K}, \qquad \mathcal{R}_{KL} = \frac{\partial\sigma}{\partial\Delta_{KL}}, \qquad \rho_0\eta = -\frac{\partial\sigma}{\partial\theta}, \tag{4.7}$$

$$Q_K\theta_{,K} \geq 0. \tag{4.8}$$

The inequality (4.8) implies

$$Q_K\left(E_{MN}, G_{MN}, \Gamma_{MN}^{(1)}, \Gamma_{MN}^{(2)}, D_M, \Delta_{MN}, \theta, \mathbf{0}, ; X_M\right) = 0. \tag{4.9}$$

From (3.30) and (4.7), we obtain

$$T_{Ki}^{(1)} = \chi_{iL}^{(1)}\frac{\partial\sigma}{\partial E_{KL}}, \qquad T_{Ki}^{(2)} = \chi_{iL}^{(1)}\frac{\partial\sigma}{\partial G_{KL}}, \qquad M_{Ki}^{(1)} = \chi_{iL}^{(1)}\frac{\partial\sigma}{\partial\Gamma_{KL}^{(1)}}, \qquad M_{Ki}^{(2)} = \chi_{iL}^{(2)}\frac{\partial\sigma}{\partial\Gamma_{KL}^{(2)}},$$
$$P_i = \chi_{iK}^{(1)}\frac{\partial\sigma}{\partial D_K}, \qquad R_j^{(2)} = -\epsilon_{ijm}\chi_{mK}^{(1)}\chi_{iL}^{(2)}\frac{\partial\sigma}{\partial\Delta_{KL}}, \qquad \rho_0\eta = -\frac{\partial\sigma}{\partial\theta}. \tag{4.10}$$

In conclusion, the constitutive equations are (4.6), (4.10), and the equation

$$Q_K = Q_K\left(E_{MN}, G_{MN}, \Gamma_{MN}^{(1)}, \Gamma_{MN}^{(2)}, D_M, \Delta_{MN}, \theta, \theta_{,M}, ; X_M\right). \tag{4.11}$$

We note that by using the relations (4.6) and (4.7), the energy balance (3.32) reduces to

$$\rho_0\theta\dot{\eta} = Q_{K,K} + \rho_0 r. \tag{4.12}$$

The complete system of field equations of nonlinear theory consist of the equations of conservation of microinertia (3.6), equations of motion (3.25), energy equation (4.12), constitutive equations (4.6), (4.10), (4.11), and the geometric equations (2.14). To these equations, we adjoin boundary conditions and initial conditions. In the case of first boundary value problem, the boundary conditions are

$$x_i = \tilde{x}_i, \quad y_i = \tilde{y}_i, \quad \chi_{iK}^{(1)} = \tilde{\chi}_{iK}^{(1)}, \quad \chi_{iK}^{(2)} = \tilde{\chi}_{iK}^{(2)}, \quad \theta = \tilde{\theta}, \quad \text{on } \partial B \times I, \tag{4.13}$$

where $\tilde{x}_i$, $\tilde{y}_i$, $\tilde{\chi}_{iK}^{(1)}$, $\tilde{\chi}_{iK}^{(2)}$, and $\tilde{\theta}$ are prescribed functions. In the second boundary value problem, the boundary conditions are

$$\begin{gathered}(T_{Ki}^{(1)} + T_{Ki}^{(2)})N_K = \tilde{T}_i, \quad x_i - y_i = \tilde{d}_i, \quad (M_{Ki}^{(1)} + M_{Ki}^{(2)})N_K = \widetilde{M}_i, \\ \chi_{iK}^{(1)}\chi_{iL}^{(2)} = \tilde{\Delta}_{KL}, \quad Q_K N_K = \tilde{Q}, \quad \text{on } \partial B \times I,\end{gathered} \tag{4.14}$$

where $\tilde{T}_i$, $\tilde{d}_i$, $\widetilde{M}_i$, $\tilde{\Delta}_{KL}$, and $\tilde{Q}$ are given and $\tilde{\Delta}_{KL}$ is proper orthogonal.

The initial conditions are

$$\begin{gathered}x_i(\mathbf{X},0) = x_i^0(\mathbf{X}), \quad y_i(\mathbf{X},0) = y_i^0(\mathbf{X}), \quad \chi_{iK}^{(\alpha)}(\mathbf{X},0) = \chi_{iK}^{(\alpha)0}(\mathbf{X}), \\ \dot{x}_i(\mathbf{X},0) = v_i^{(1)0}(\mathbf{X}), \quad \dot{y}_i(\mathbf{X},0) = v_i^{(2)0}(\mathbf{X}), \quad v_i^{(\alpha)}(\mathbf{X},0) = v_i^{(\alpha)0}(\mathbf{X}), \\ \eta(\mathbf{X},0) = \eta^0(\mathbf{X}), \quad \mathbf{X} \in B,\end{gathered} \tag{4.15}$$

where x_i^0, y_i^0, $\chi_{iK}^{(\alpha)0}$, $v_i^{(\alpha)0}$, $\nu_i^{(\alpha)0}$, η^0 are prescribed functions.

5. The linear theory

In the following, we linearize the above equations. We use the notations

$$X_i = \delta_{iK}X_K, \qquad \chi_{ij}^{(\alpha)} = \delta_{jK}\chi_{iK}^{(\alpha)}, \qquad \frac{\partial f}{\partial X_i} = f_{,i}, \tag{5.1}$$

where δ_{iK} is the Kronecker delta. We have

$$x_i = X_i + u_i, \qquad y_i = X_i + w_i, \tag{5.2}$$

where u_i and w_i are the displacements vectors associated with s_1 and s_2, respectively. We denote

$$T = \theta - T_0, \tag{5.3}$$

where T_0 is the constant absolute temperature of the mixture in the reference configuration. Being concerned with first-order approximations, following [33] (see (1.2.19) and (1.6.6)), we take

$$\chi_{ij}^{(\alpha)} \simeq \delta_{ij} - \epsilon_{ijs}\varphi_s^{(\alpha)}, \tag{5.4}$$

where $\boldsymbol{\varphi}^{(\alpha)}$ is the microrotation vector associated with the constituent s_α. We suppose that

$$u_i = \varepsilon\tilde{u}_i, \qquad w_i = \varepsilon\tilde{w}_i, \qquad T = \varepsilon\tilde{T}, \qquad \varphi_i^{(\alpha)} = \varepsilon\tilde{\varphi}_i^{(\alpha)}, \qquad \alpha = 1,2, \tag{5.5}$$

where ε is a constant small enough for squares and higher powers to be neglected and $\widetilde{u}_i$, $\widetilde{w}_i$, $\widetilde{T}$, and $\widetilde{\varphi}_i^{(\alpha)}$ are independent of ε. It follows from (2.10), (5.4), and (5.5) that

$$\dot{\varphi}_i^{(\alpha)} = \nu_i^{(\alpha)}. \tag{5.6}$$

Moreover, the strain measures E_{KL}, G_{KL}, D_K, $\Gamma_{KL}^{(\alpha)}$, Δ_{KL} defined by (2.14) reduce to

$$\begin{gathered} e_{ji} = u_{i,j} + \epsilon_{sij}\varphi_s^{(1)}, \qquad g_{ji} = w_{i,j} + \epsilon_{sij}\varphi_s^{(1)}, \qquad d_i = u_i - w_i, \\ \gamma_{ij}^{(\alpha)} = \varphi_{j,i}^{(\alpha)}, \qquad \Delta_{ij} = \epsilon_{ijs}\left(\varphi_s^{(1)} - \varphi_s^{(2)}\right), \end{gathered} \tag{5.7}$$

and the microinertia tensor $i_{ij}^{(\alpha)}$ is given by

$$i_{ij}^{(\alpha)} = I_{ij}^{(\alpha)}. \tag{5.8}$$

If we introduce the notations

$$\begin{gathered} t_{ji} = \delta_{jK} T_{Ki}^{(1)}, \qquad s_{ji} = \delta_{jK} T_{Ki}^{(2)}, \qquad m_{ji}^{(\alpha)} = \delta_{jK} M_{Ki}^{(\alpha)}, \\ p_i = P_i, \qquad \mathcal{R}_i = -R_i^{(2)}, \qquad q_i = \delta_{iK} Q_K, \end{gathered} \tag{5.9}$$

then the equations of motion (3.25) and the energy equation (4.12) can be written in the form

$$\begin{gathered} t_{ji,j} - p_i + \rho_1^0 F_i^{(1)} = \rho_1^0 \ddot{u}_i, \\ s_{ji,j} + p_i + \rho_2^0 F_i^{(2)} = \rho_2^0 \ddot{w}_i, \end{gathered} \tag{5.10}$$

$$\begin{gathered} m_{ji,j}^{(1)} + \epsilon_{ijk}\left[t_{jk} + s_{jk}\right] - \mathcal{R}_i + \rho_1^0 G_i^{(1)} = \rho_1^0 I_{ij}^{(1)} \ddot{\varphi}_j^{(1)}, \\ m_{ji,j}^{(2)} + \mathcal{R}_i + \rho_2^0 G_i^{(2)} = \rho_2^0 I_{ij}^{(2)} \ddot{\varphi}_j^{(2)}, \end{gathered} \tag{5.11}$$

$$\rho_0 T_0 \dot{\eta} = q_{i,i} + \rho_0 r. \tag{5.12}$$

Since Δ_{ij} is skew-symmetric, we may consider that σ depends on the following variables:

$$\sigma = \sigma\left(e_{ij}, g_{ij}, \gamma_{ij}^{(1)}, \gamma_{ij}^{(2)}, d_i, \pi_i, T; X_i\right), \tag{5.13}$$

where

$$\pi_i = \varphi_i^{(1)} - \varphi_i^{(2)}. \tag{5.14}$$

Collecting (4.10), (5.4), (5.5), (5.7), (5.9), (5.13), and (5.14), we deduce

$$\begin{gathered} t_{ij} = \frac{\partial \sigma}{\partial e_{ij}}, \qquad s_{ij} = \frac{\partial \sigma}{\partial g_{ij}}, \qquad m_{ij}^{(\alpha)} = \frac{\partial \sigma}{\partial \gamma_{ij}^{(\alpha)}}, \\ p_i = \frac{\partial \sigma}{\partial d_i}, \qquad \rho_0 \eta = -\frac{\partial \sigma}{\partial \theta}, \qquad \mathcal{R}_i = \epsilon_{ijk} \frac{\partial \sigma}{\partial \pi_s} \frac{\partial \pi_s}{\Delta_{jk}} = \frac{\partial \sigma}{\partial \pi_i}. \end{gathered} \tag{5.15}$$

Assuming that the initial body is free from stress and couple stress, in the context of linear theory, we have

$$
\begin{aligned}
\sigma = {} & \frac{1}{2}A_{ijrs}e_{ij}e_{rs} + B_{ijrs}e_{ij}g_{rs} + \frac{1}{2}C_{ijrs}g_{ij}g_{rs} \\
& + \sum_{\alpha=1}^{2}\left(\frac{1}{2}D_{ijrs}^{(\alpha)}\gamma_{ij}^{(\alpha)}\gamma_{rs}^{(\alpha)} + F_{ijrs}^{(\alpha)}e_{ij}\gamma_{rs}^{(\alpha)} + H_{ijrs}^{(\alpha)}g_{ij}\gamma_{rs}^{(\alpha)}\right) \\
& + D_{ijrs}^{(3)}\gamma_{ij}^{(1)}\gamma_{rs}^{(2)} + \frac{1}{2}a_{ij}d_i d_j + \frac{1}{2}b_{ij}\pi_i\pi_j + c_{ij}d_i\pi_j + a_{ijk}e_{ij}d_k \\
& + b_{ijk}e_{ij}\pi_k + c_{ijk}g_{ij}d_k + d_{ijk}g_{ij}\pi_k + \alpha_{ijk}^{(1)}\gamma_{ij}^{(1)}d_k + \beta_{ijk}^{(1)}\gamma_{ij}^{(1)}\pi_k + \alpha_{ijk}^{(2)}\gamma_{ij}^{(2)}d_k \\
& + \beta_{ijk}^{(2)}\gamma_{ij}^{(2)}\pi_k - \frac{1}{2}a^{*}T^2 - \left(\alpha_{ij}^{*}e_{ij} + \beta_{ij}^{*}g_{ij} + \nu_{ij}^{*}\gamma_{ij}^{(1)} + \mu_{ij}^{*}\gamma_{ij}^{(1)} + \tau_i^{*}d_i + \sigma_i^{*}\pi_i\right)T, \\
q_i = {} & k_{ij}T_{,j},
\end{aligned}
\tag{5.16}
$$

where the constitutive coefficients have the following symmetries:

$$
\begin{gathered}
A_{ijrs} = A_{rsij}, \qquad C_{ijrs} = C_{rsij}, \qquad D_{ijrs}^{(\alpha)} = D_{rsij}^{(\alpha)}, \\
a_{ij} = a_{ji}, \qquad b_{ij} = b_{ji}.
\end{gathered}
\tag{5.17}
$$

From (5.13), (5.15), and (5.16), we obtain the following constitutive equations:

$$
\begin{aligned}
t_{ij} &= A_{ijrs}e_{rs} + B_{ijrs}g_{rs} + F_{ijrs}^{(1)}\gamma_{rs}^{(1)} + F_{ijrs}^{(2)}\gamma_{rs}^{(2)} + a_{ijk}d_k + b_{ijk}\pi_k - \alpha_{ij}^{*}T, \\
s_{ij} &= B_{rsij}e_{rs} + C_{ijrs}g_{rs} + H_{ijrs}^{(1)}\gamma_{rs}^{(1)} + H_{ijrs}^{(2)}\gamma_{rs}^{(2)} + c_{ijk}d_k + d_{ijk}\pi_k - \beta_{ij}^{*}T, \\
m_{ij}^{(1)} &= F_{rsij}^{(1)}e_{rs} + H_{rsij}^{(1)}g_{rs} + D_{ijrs}^{(1)}\gamma_{rs}^{(1)} + D_{ijrs}^{(3)}\gamma_{rs}^{(2)} + \alpha_{ijk}^{(1)}d_k + \beta_{ijk}^{(1)}\pi_k - \nu_{ij}^{*}T, \\
m_{ij}^{(2)} &= F_{rsij}^{(2)}e_{rs} + H_{rsij}^{(2)}g_{rs} + D_{rsij}^{(3)}\gamma_{rs}^{(1)} + D_{ijrs}^{(2)}\gamma_{rs}^{(2)} + \alpha_{ijk}^{(2)}d_k + \beta_{ijk}^{(2)}\pi_k - \mu_{ij}^{*}T, \\
p_i &= a_{ij}d_j + c_{ij}\pi_j + a_{jki}e_{jk} + c_{jki}g_{jk} + \alpha_{jki}^{(1)}\gamma_{jk}^{(1)} + \alpha_{jki}^{(2)}\gamma_{jk}^{(2)} - \tau_i^{*}T, \\
\mathcal{R}_i &= c_{ji}d_j + b_{ij}\pi_j + b_{jki}e_{jk} + d_{jki}g_{jk} + \beta_{jki}^{(1)}\gamma_{jk}^{(1)} + \beta_{jki}^{(2)}\gamma_{jk}^{(2)} - \sigma_i^{*}T, \\
\rho_0\eta &= a^{*}T + \alpha_{ij}^{*}e_{ij} + \beta_{ij}^{*}g_{ij} + \nu_{ij}^{*}\gamma_{ij}^{(1)} + \mu_{ij}^{*}\gamma_{ij}^{(2)} + \tau_i^{*}d_i + \sigma_i^{*}\pi_i, \\
q_i &= k_{ij}T_{,j}.
\end{aligned}
\tag{5.18}
$$

For isotropic solids, odd-order constitutive tensors vanish and even-order tensors can be constituted by the products of δ_{ij}. We also note that e_{ij}, g_{ij}, d_i, θ, t_{ij}, s_{ij}, p_i and η are polar tensors, while $\gamma_{ij}^{(\alpha)}$, π_i, $m_{ij}^{(\alpha)}$, and R_i are axial tensors. By examining (5.18), we find that the only surviving isotropic material constants are

$$
\begin{gathered}
A_{ijrs}=\lambda_1\delta_{ij}\delta_{rs}+(\mu_1+\kappa_1)\delta_{ir}\delta_{js}+\mu_1\delta_{is}\delta_{jr}, \qquad C_{ijrs}=\lambda_2\delta_{ij}\delta_{rs}+(\mu_2+\kappa_2)\delta_{ir}\delta_{js}+\mu_2\delta_{is}\delta_{jr},\\
D^{(1)}_{ijrs}=\alpha_1\delta_{ij}\delta_{rs}+\gamma_1\delta_{ir}\delta_{js}+\beta_1\delta_{is}\delta_{jr}, \qquad D^{(2)}_{ijrs}=\alpha_2\delta_{ij}\delta_{rs}+\gamma_2\delta_{ir}\delta_{js}+\beta_2\delta_{is}\delta_{jr},\\
B_{ijrs}=\nu\delta_{ij}\delta_{rs}+\xi\delta_{ir}\delta_{js}+\zeta\delta_{is}\delta_{jr}, \qquad D^{(3)}_{ijrs}=\alpha_3\delta_{ij}\delta_{rs}+\gamma_3\delta_{ir}\delta_{js}+\beta_3\delta_{is}\delta_{jr},\\
a_{ij}=a\delta_{ij}, \qquad b_{ij}=b\delta_{ij}, \qquad b_{ijk}=b^0\epsilon_{ijk}, \qquad d_{ijk}=d^0\epsilon_{ijk},\\
\alpha^{(1)}_{ijk}=\alpha^0\epsilon_{ijk}, \qquad \alpha^{(2)}_{ijk}=\beta^0\epsilon_{ijk}, \qquad \alpha^*_{ij}=\alpha^*\delta_{ij}, \qquad \beta^*=\beta^*\delta_{ij}.
\end{gathered}
\tag{5.19}
$$

Hence

$$
\begin{aligned}
\sigma={}&\frac{1}{2}\lambda_1e_{ii}e_{jj}+\frac{1}{2}(\mu_1+\kappa_1)e_{ij}e_{ij}+\frac{1}{2}\mu_1e_{ij}e_{ji}+\nu e_{ii}g_{jj}+\xi e_{ij}g_{ij}+\zeta e_{ij}g_{ji}\\
&+\frac{1}{2}\lambda_2g_{ii}g_{jj}+\frac{1}{2}(\mu_2+\kappa_2)g_{ij}g_{ij}+\frac{1}{2}\mu_2g_{ij}g_{ji}\\
&+\frac{1}{2}\alpha_1\gamma^{(1)}_{ii}\gamma^{(1)}_{jj}+\frac{1}{2}\gamma_1\gamma^{(1)}_{ij}\gamma^{(1)}_{ij}+\frac{1}{2}\beta_1\gamma^{(1)}_{ij}\gamma^{(1)}_{ji}+\alpha_3\gamma^{(1)}_{ii}\gamma^{(2)}_{jj}\\
&+\gamma_3\gamma^{(1)}_{ij}\gamma^{(2)}_{ij}+\beta_3\gamma^{(1)}_{ij}\gamma^{(2)}_{ji}+\frac{1}{2}\alpha_2\gamma^{(2)}_{ii}\gamma^{(2)}_{jj}+\frac{1}{2}\gamma_2\gamma^{(2)}_{ij}\gamma^{(2)}_{ij}+\frac{1}{2}\beta_2\gamma^{(2)}_{ij}\gamma^{(2)}_{ji}\\
&+\frac{1}{2}ad_id_i+\frac{1}{2}b\pi_i\pi_i+b^0\epsilon_{ijk}e_{ij}\pi_k+d^0\epsilon_{ijk}g_{ij}\pi_k\\
&+\alpha^0\epsilon_{ijk}\gamma^{(1)}_{ij}d_k+\beta^0\epsilon_{ijk}\gamma^{(2)}_{ij}d_k-\frac{1}{2}a^*T^2-(\alpha^*e_{ii}+\beta^*g_{ii})T.
\end{aligned}
\tag{5.20}
$$

The constitutive equations reduce to

$$
\begin{gathered}
t_{ij}=\lambda_1e_{rr}\delta_{ij}+(\mu_1+\kappa_1)e_{ij}+\mu_1e_{ji}+\nu g_{rr}\delta_{ij}+\xi g_{ij}+\zeta g_{ji}+b^0\epsilon_{ijk}\pi_k-\alpha^*T\delta_{ij},\\
s_{ij}=\nu e_{rr}\delta_{ij}+\xi e_{ij}+\zeta e_{ji}+\lambda_2g_{rr}\delta_{ij}+(\mu_2+\kappa_2)g_{ij}+\mu_2g_{ji}+d^0\epsilon_{ijk}\pi_k-\beta^*T\delta_{ij},\\
m^{(1)}_{ij}=\alpha_1\gamma^{(1)}_{rr}\delta_{ij}+\gamma_1\gamma^{(1)}_{ij}+\beta_1\gamma^{(1)}_{ji}+\alpha_3\gamma^{(2)}_{rr}\delta_{ij}+\gamma_3\gamma^{(2)}_{ij}+\beta_3\gamma^{(2)}_{ji}+\alpha^0\epsilon_{ijk}d_k,\\
m^{(2)}_{ij}=\alpha_3\gamma^{(1)}_{rr}\delta_{ij}+\gamma_3\gamma^{(1)}_{ij}+\beta_3\gamma^{(1)}_{ji}+\alpha_2\gamma^{(2)}_{rr}\delta_{ij}+\gamma_2\gamma^{(2)}_{ij}+\beta_2\gamma^{(2)}_{ji}+\beta^0\epsilon_{ijk}d_k,\\
p_i=ad_i+\alpha^0\epsilon_{jki}\gamma^{(1)}_{jk}+\beta^0\epsilon_{jki}\gamma^{(2)}_{jk},\\
\mathcal{R}_i=b\pi_i+b^0\epsilon_{jki}e_{jk}+d^0\epsilon_{jki}g_{jk},\\
\rho_0\eta=a^*T+\alpha^*e_{ii}+\beta^*g_{ii},\\
q_i=kT_{,i}.
\end{gathered}
\tag{5.21}
$$

From (4.8), we deduce

$$k \geq 0. \tag{5.22}$$

Thus, the basic equations in the linear theory are the equations of motions (5.10) and (5.11), the energy equation (5.12), the constitutive equations (5.18), and the geometric equations (5.7) and (5.14). In the case of first boundary value problem, the boundary conditions are

$$u_i = \widetilde{u}_i, \quad w_i = \widetilde{w}_i, \quad \varphi_i = \widetilde{\varphi}_i^{(\alpha)}, \quad T = \widetilde{T}, \quad \text{on } \partial B \times I, \tag{5.23}$$

where $\widetilde{u}_i$, $\widetilde{w}_i$, $\widetilde{\varphi}_i^{(\alpha)}$, and $\widetilde{T}$ are prescribed functions. In the second boundary value problem, the boundary conditions are

$$\begin{aligned} (t_{ji} + s_{ji}) n_j &= \widetilde{t}_i, \qquad (m_{ji}^{(\alpha)} + m_{ji}^{(\alpha)}) n_j = \widetilde{m}_i, \\ d_i = \widetilde{d}_i, \qquad \pi_i &= \widetilde{\pi}_i, \qquad q_i n_i = \widetilde{q}, \quad \text{on } \partial B \times I, \end{aligned} \tag{5.24}$$

where $\widetilde{t}_i$, $\widetilde{m}_i$, $\widetilde{d}_i$, $\widetilde{\pi}_i$ and $\widetilde{q}$ are given. The initial conditions are

$$\begin{aligned} u_i(\mathbf{X},0) &= \widehat{a}_i(\mathbf{X}), \quad w_i(\mathbf{X},0) = \widehat{b}_i(\mathbf{X}), \quad \dot{u}_i(\mathbf{X},0) = \widehat{c}_i(\mathbf{X}), \quad \dot{w}_i(\mathbf{X},0) = \widehat{f}_i(\mathbf{X}), \\ \varphi_i^{(\alpha)}(\mathbf{X},0) &= \widehat{\psi}_i^{(\alpha)}(\mathbf{X}), \quad \dot{\varphi}_i^{(\alpha)}(\mathbf{X},0) = \widehat{\omega}_i^{(\alpha)}(\mathbf{X}), \quad T(\mathbf{X},0) = \widehat{T}(\mathbf{X}), \quad \mathbf{X} \in \overline{B}, \end{aligned} \tag{5.25}$$

where the functions $\widehat{a}_i$, $\widehat{b}_i$, $\widehat{c}_i$, $\widehat{f}_i$, $\widehat{\psi}_i^{(\alpha)}$, $\widehat{\omega}_i^{(\alpha)}$, and $\widehat{T}$ are prescribed.

6. Uniqueness theorem

In this section, we will establish a uniqueness result in the linear theory. It is indifferent which instant t_0 is selected as the initial one and hence we choose $t_0 = 0$. Let us introduce the following notations:

$$\begin{aligned} U = {} & \frac{1}{2} A_{ijrs} e_{ij} e_{rs} + B_{ijrs} e_{ij} g_{rs} + \frac{1}{2} C_{ijrs} g_{ij} g_{rs} \\ & + \sum_{\alpha=1}^{2} \left(\frac{1}{2} D_{ijrs}^{(\alpha)} \gamma_{ij}^{(\alpha)} \gamma_{rs}^{(\alpha)} + F_{ijrs}^{(\alpha)} e_{ij} \gamma_{rs}^{(\alpha)} + H_{ijrs}^{(\alpha)} g_{ij} \gamma_{rs}^{(\alpha)} \right) + D_{ijrs}^{(3)} \gamma_{ij}^{(1)} \gamma_{rs}^{(2)} \\ & + \frac{1}{2} a_{ij} d_i d_j + \frac{1}{2} b_{ij} \pi_i \pi_j + c_{ij} d_i \pi_j + a_{ijk} e_{ij} d_k + b_{ijk} e_{ij} \pi_k + c_{ijk} g_{ij} d_k \\ & + d_{ijk} g_{ij} \pi_k + \alpha_{ijk}^{(1)} \gamma_{ij}^{(1)} d_k + \beta_{ijk}^{(1)} \gamma_{ij}^{(1)} \pi_k + \alpha_{ijk}^{(2)} \gamma_{ij}^{(2)} d_k + \beta_{ijk}^{(2)} \gamma_{ij}^{(2)} \pi_k, \end{aligned} \tag{6.1}$$

$$E = \frac{1}{2} \int_B \left(\rho_1^0 \dot{u}_i \dot{u}_i + \rho_2^0 \dot{w}_i \dot{w}_i + \rho_1^0 I_{ij}^{(1)} \dot{\varphi}_i^{(1)} \dot{\varphi}_j^{(1)} + \rho_2^0 I_{ij}^{(2)} \dot{\varphi}_i^{(2)} \dot{\varphi}_j^{(2)} + a^* T^2 + 2U \right) dv. \tag{6.2}$$

Theorem 6.1. *Assume the following:*

(i) ρ_α^0 *and* a^* *are strictly positive and* $I_{ij}^{(\alpha)}$ *is positive definite;*

(ii) *the constitutive coefficients satisfy symmetry relations (5.17) and the inequality*

$$k_{ij}T_{,i}T_{,j} \geq 0; \tag{6.3}$$

(iii) *U is a positive semidefinite form.*

Then the initial boundary value problem defined by the equations of motion (5.10) and (5.11), the energy equation (5.12), the constitutive equations (5.18), the geometrical equations (5.7) and (5.14), the boundary conditions (5.23) (or (5.24)), and the initial conditions (5.25) has at most one solution.

Proof. In view of the (5.17), (5.18), and (6.1) we deduce

$$t_{ij}\dot{e}_{ij} + s_{ij}\dot{g}_{ij} + m_{ij}^{(1)}\dot{\gamma}_{ij}^{(1)} + m_{ij}^{(2)}\dot{\gamma}_{ij}^{(2)} + p_i\dot{d}_i + \mathcal{R}_i\pi_i + \rho_0\dot{\eta}T = \frac{\partial}{\partial t}\left(\frac{1}{2}a^*T^2 + U\right). \tag{6.4}$$

On the other hand, in view of (5.10), (5.11), (5.12), (5.7), and (5.14), we obtain

$$\begin{aligned}
&t_{ij}\dot{e}_{ij} + s_{ij}\dot{g}_{ij} + m_{ij}^{(1)}\dot{\gamma}_{ij}^{(1)} + m_{ij}^{(2)}\dot{\gamma}_{ij}^{(2)} + p_i\dot{d}_i + \mathcal{R}_i\pi_i + \rho_0\dot{\eta}T \\
&\quad = \left(t_{ji}\dot{u}_i + s_{ji}\dot{w}_i + m_{ji}^{(1)}\dot{\varphi}_i^{(1)} + m_{ji}^{(2)}\dot{\varphi}_i^{(2)} + \frac{1}{T_0}q_jT\right)_{,j} + \rho_1^0F_i^{(1)}\dot{u}_i + \rho_2^0F_i^{(2)}\dot{w}_i + \rho_1^0G_i^{(1)}\dot{\varphi}_i^{(1)} \\
&\qquad + \rho_2^0G_i^{(2)}\dot{\varphi}_i^{(2)} + \frac{1}{T_0}\rho_0rT - \frac{1}{2}\frac{\partial}{\partial t}\left(\rho_1^0\dot{u}_i\dot{u}_i + \rho_2^0\dot{w}_i\dot{w}_i + \rho_1^0I_{ij}^{(1)}\dot{\varphi}_i^{(1)}\dot{\varphi}_j^{(1)} + \rho_2^0I_{ij}^{(2)}\dot{\varphi}_i^{(2)}\dot{\varphi}_j^{(2)}\right) \\
&\qquad - \frac{1}{T_0}k_{ij}T_{,i}T_{,j}.
\end{aligned} \tag{6.5}$$

Then (6.4) and (6.5) imply

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left(\rho_1^0\dot{u}_i\dot{u}_i + \rho_2^0\dot{w}_i\dot{w}_i + \rho_1^0I_{ij}^{(1)}\dot{\varphi}_i^{(1)}\dot{\varphi}_j^{(1)} + \rho_2^0I_{ij}^{(2)}\dot{\varphi}_i^{(2)}\dot{\varphi}_j^{(2)} + a^*T^2 + 2U\right) \\
&\quad + \frac{1}{T_0}k_{ij}T_{,i}T_{,j} = \rho_1^0F_i^{(1)}\dot{u}_i + \rho_2^0F_i^{(2)}\dot{w}_i + \rho_1^0G_i^{(1)}\dot{\varphi}_i^{(1)} + \rho_2^0G_i^{(2)}\dot{\varphi}_i^{(2)} + \frac{1}{T_0}\rho_0rT \\
&\quad + \frac{1}{2}\Big[\left(t_{ji} + s_{ji}\right)\left(\dot{u}_i + \dot{w}_i\right) + \left(t_{ji} - s_{ji}\right)\left(\dot{u}_i - \dot{w}_i\right) \\
&\qquad + \left(m_{ji}^{(1)} + m_{ji}^{(2)}\right)\left(\dot{\varphi}_i^{(1)} + \dot{\varphi}_i^{(1)}\right) + \left(m_{ji}^{(1)} - m_{ji}^{(2)}\right)\left(\dot{\varphi}_i^{(1)} - \dot{\varphi}_i^{(2)}\right) + \frac{2}{T_0}q_jT\Big]_{,j}.
\end{aligned} \tag{6.6}$$

By an integration of the above relation over B and by using the divergence theorem and (6.2), we obtain

$$\begin{aligned}
\dot{E} + \int_B \frac{1}{T_0}k_{ij}T_{,i}T_{,j}dV &= \int_B\left(\rho_1^0F_i^{(1)}\dot{u}_i + \rho_2^0F_i^{(2)}\dot{w}_i + \rho_1^0G_i^{(1)}\dot{\varphi}_i^{(1)} + \rho_2^0G_i^{(2)}\dot{\varphi}_i^{(2)} + \frac{1}{T_0}\rho_0rT\right)dV \\
&\quad + \frac{1}{2}\int_{\partial B}\Big[\left(t_{ji} + s_{ji}\right)\left(\dot{u}_i + \dot{w}_i\right) + \left(t_{ji} - s_{ji}\right)\dot{d}_i + \left(m_{ji}^{(1)} + m_{ji}^{(2)}\right) \\
&\qquad \times \left(\dot{\varphi}_i^{(1)} + \dot{\varphi}_i^{(1)}\right) + \left(m_{ji}^{(1)} - m_{ji}^{(2)}\right)\dot{\pi}_i + \frac{2}{T_0}q_jT\Big]n_jdA.
\end{aligned} \tag{6.7}$$

Let us suppose that there are two solutions $\{{}^1u_i, {}^1w_i, {}^1\varphi_i^{(1)}, {}^1\varphi_i^{(2)}, {}^1T\}$ and $\{{}^2u_i, {}^2w_i, {}^2\varphi_i^{(1)}, {}^2\varphi_i^{(2)}, {}^2T\}$. Because of the linearity of the considered initial boundary value problem, their difference $\mathcal{P} = \{\overline{u}_i, \overline{w}_i, \overline{\varphi}_i^{(1)}, \overline{\varphi}_i^{(2)}, \overline{T}\}$ corresponds to the null data. It follows from (5.23), (5.24), and (6.7) that

$$\dot{\overline{E}} + \int_B \frac{1}{T_0} k_{ij} \overline{T}_{,i} \overline{T}_{,j} dV = 0, \tag{6.8}$$

where $\overline{E}$ is the function defined by (6.2) associated with the process $\mathcal{P}$. In view of the inequality (6.3), we deduce $\dot{\overline{E}} \le 0$ on $[0, t_1)$, so that $\overline{E}(t) \le \overline{E}(0)$, $t \in [0, t_1)$. From the initial conditions, we find $\overline{E}(t) = 0$ and therefore $\overline{E}(0) = 0$, $t \in [0, t_1)$. The hypotheses (i) and (iii) imply $\dot{\overline{\mathbf{u}}} = \mathbf{0}$, $\dot{\overline{\mathbf{w}}} = \mathbf{0}$, $\dot{\overline{\boldsymbol{\varphi}}}^{(1)} = \mathbf{0}$, $\dot{\overline{\boldsymbol{\varphi}}}^{(2)} = \mathbf{0}$, $\overline{T} = 0$ on $B \times [0, t_1)$. Since $\overline{\mathbf{u}}$, $\overline{\mathbf{w}}$, $\overline{\boldsymbol{\varphi}}^{(1)}$, $\overline{\boldsymbol{\varphi}}^{(2)}$ vanish initially, we conclude that $\overline{\mathbf{u}} = \mathbf{0}$, $\overline{\mathbf{w}} = \mathbf{0}$, $\overline{\boldsymbol{\varphi}}^{(1)} = \mathbf{0}$, $\overline{\boldsymbol{\varphi}}^{(2)} = \mathbf{0}$, $\overline{T} = 0$ on $B \times [0, t_1)$ and the proof is complete. □

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C. Galeş: Faculty of Mathematics, University of Iaşi, 700506 Iaşi, Romania
Email address: cgales@uaic.ro