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Research Article

An Efficient Numerical Approach for Solving Nonlinear Coupled Hyperbolic Partial Differential Equations with Nonlocal Conditions

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One of the most important advantages of collocation method is the possibility of dealing with nonlinear partial differential equations (PDEs) as well as PDEs with variable coefficients. A numerical solution based on a Jacobi collocation method is extended to solve nonlinear coupled hyperbolic PDEs with variable coefficients subject to initial-boundary nonlocal conservation conditions. This approach, based on Jacobi polynomials and Gauss-Lobatto quadrature integration, reduces solving the nonlinear coupled hyperbolic PDEs with variable coefficients to a system of nonlinear ordinary differential equation which is far easier to solve. In fact, we deal with initial-boundary coupled hyperbolic PDEs with variable coefficients as well as initial-nonlocal conditions. Using triangular, soliton, and exponential-triangular solutions as exact solutions, the obtained results show that the proposed numerical algorithm is efficient and very accurate.

1. Introduction

For several decades, numerical methods have been developed to obtain more accurate solutions of differential and integral equations. Spectral method [1–9] is one of the family of weighted residual numerical methods for solving various problems, including variable coefficient and nonlinear differential equations [10, 11], integral equations [12, 13], integrodifferential equations [14, 15], fractional orders differential equations [16–19], and function approximation and variational problems [20]. The collocation method [21–28] can be classified as a special type of spectral methods. In the last few years, the collocation method has been introduced as a powerful approximation method for numerical solutions of all kinds of initial-boundary value problems.

Exact solutions for initial value problem for some nonconservative hyperbolic systems are presented in [29], while the analytical study of variable coefficient mixed hyperbolic partial differential problems is discussed in [30]. The solitary and periodic wave solutions have been studied for some kinds of hyperbolic Klein-Gordon equations in [31, 32]. Other numerical methods based on the boundary integral equation [33] and numerical integration techniques [34] are used to numerically solve different types of hyperbolic partial differential problems. In [35, 36], finite difference scheme is considered to numerically solve hyperbolic equations. Pseudospectral methods are used in [37–40] to solve Klein-Gordon equations. In [41], Dehghan and Shokri used the radial basis functions to solve a two-dimensional Sine-Gordon equation; moreover in [42] they developed numerical scheme to solve the one-dimensional nonlinear Klein-Gordon equation with quadratic and cubic nonlinearity using collocation points and approximating the solution using Thin Plate Splines and RBFs.

There are no results on Jacobi-Gauss-Lobatto collocation (J-GL-C) method for solving nonlinear coupled hyperbolic PDEs with variable coefficients subject to initial-boundary and nonlocal conditions. Therefore, the objective of this work is to present this method to numerically solve four nonlinear coupled hyperbolic PDEs with variable coefficients. By using collocation method, exponential convergence for the spatial variables can be achieved to approximate the solution of PDE.

The computerized mathematical algorithm is the main key to apply this method for solving the problem. Moreover, the nonlocal conservation conditions are efficiently treated by Jacobi-Gauss-Lobatto quadrature rule at $(N+1)$ nodes to obtain a system of ODEs in time and then proper initial value software can be applied to solve this system of ODEs. Several illustrative problems with various kinds of exact solutions such as triangular, soliton, and exponential-triangular solutions are presented for demonstrating the high accuracy of this scheme. Moreover, with the freedom of selecting the Jacobi indexes θ and ϑ, the scheme can be calibrated for a wide variety of problems. Finally, the accuracy of the proposed method is demonstrated by solving some test nonlinear problems.

A brief outline of this paper is as follows. We present some properties of Jacobi polynomials in the next section. The third section is divided into two subsections: the first one deals with coupled nonlinear hyperbolic PDE with initial-boundary conditions. The numerical treatment of solve initial-nonlocal conservation conditions is developed in Section 3.2. In Section 4 the proposed method is applied to four different test problems to show the accuracy of our method. In the last section, we present some observations and conclusions.

2. Jacobi Polynomials

Some basic properties of Jacobi polynomials have been recalled in this section. A basic property of the Jacobi polynomials is that they are the eigenfunctions to a singular Sturm-Liouville problem:

$$\begin{aligned}&\left(1-x^2\right)\phi''(x)+[\vartheta-\theta+(\theta+\vartheta+2)x]\phi'(x)\\&\quad+n(n+\theta+\vartheta+1)\phi(x)=0.\end{aligned}\tag{1}$$

We recall that the Jacobi polynomials satisfy the following recurrence relation:

$$\begin{aligned}&J_{k+1}^{(\theta,\vartheta)}(x)\\&\quad=\left(a_k^{(\theta,\vartheta)}x-b_k^{(\theta,\vartheta)}\right)J_k^{(\theta,\vartheta)}(x)-c_k^{(\theta,\vartheta)}J_{k-1}^{(\theta,\vartheta)}(x),\quad k\ge1,\\&J_0^{(\theta,\vartheta)}(x)=1,\qquad J_1^{(\theta,\vartheta)}(x)=\frac{1}{2}(\theta+\vartheta+2)x+\frac{1}{2}(\theta-\vartheta),\end{aligned}\tag{2}$$

where

$$\begin{aligned}a_k^{(\theta,\vartheta)}&=\frac{(2k+\theta+\vartheta+1)(2k+\theta+\vartheta+2)}{2(k+1)(k+\theta+\vartheta+1)},\\b_k^{(\theta,\vartheta)}&=\frac{\left(\vartheta^2-\theta^2\right)(2k+\theta+\vartheta+1)}{2(k+1)(k+\theta+\vartheta+1)(2k+\theta+\vartheta)},\\c_k^{(\theta,\vartheta)}&=\frac{(k+\theta)(k+\vartheta)(2k+\theta+\vartheta+2)}{(k+1)(k+\theta+\vartheta+1)(2k+\theta+\vartheta)}.\end{aligned}\tag{3}$$

Beside the following relations

$$\begin{aligned}J_k^{(\theta,\vartheta)}(-x)&=(-1)^kJ_k^{(\theta,\vartheta)}(x),\\J_k^{(\theta,\vartheta)}(-1)&=\frac{(-1)^k\Gamma(k+\vartheta+1)}{k!\Gamma(\vartheta+1)}.\end{aligned}\tag{4}$$

Moreover, the qth derivative of $J_k^{(\theta,\vartheta)}(x)$ can be obtained from

$$D^{(q)}J_k^{(\theta,\vartheta)}(x)=\frac{\Gamma(j+\theta+\vartheta+q+1)}{2^q\Gamma(j+\theta+\vartheta+1)}J_{k-q}^{(\theta+q,\vartheta+q)}(x).\tag{5}$$

Let $w^{(\theta,\vartheta)}(x)=(1-x)^\theta(1+x)^\vartheta$; then we define the weighted space $L^2_{w^{(\theta,\vartheta)}}$ as usual. The inner product and the norm of $L^2_{w^{(\theta,\vartheta)}}$ with respect to the weight function are defined as follows:

$$\begin{aligned}(u,v)_{w^{(\theta,\vartheta)}}&=\int_{-1}^{1}u(x)v(x)w^{(\theta,\vartheta)}(x)\,dx,\\\|u\|_{w^{(\theta,\vartheta)}}&=(u,u)^{1/2}_{w^{(\theta,\vartheta)}}.\end{aligned}\tag{6}$$

The set of Jacobi polynomials forms a complete $L^2_{w^{(\theta,\vartheta)}}$-orthogonal system, and

$$\begin{aligned}&\left\|J_k^{(\theta,\vartheta)}\right\|_{w^{(\theta,\vartheta)}}\\&\quad=h_k=\frac{2^{\theta+\vartheta+1}\Gamma(k+\theta+1)\Gamma(k+\vartheta+1)}{(2k+\theta+\vartheta+1)\Gamma(k+1)\Gamma(k+\theta+\vartheta+1)}.\end{aligned}\tag{7}$$

3. The Problem and the Numerical Algorithm

In this section, we approximate the solution of coupled nonlinear hyperbolic types equations with two different kinds of boundary conditions for space variable by using the Jacobi collocation method.

3.1. Initial-Boundary Conditions. In what follows, we propose an efficient numerical algorithm to solve the coupled nonlinear hyperbolic types equations in the following form:

$$\begin{aligned}D_t^2u(y,t)&=\gamma u(y,t)v(y,t)\\&\quad\times\left(D_yu(y,t)+D_tu(y,t)\right.\\&\quad\left.+D_yv(y,t)+D_tv(y,t)\right)\\&\quad+g_1(y,t)D_y^2u(y,t)+g_2(y,t),\\D_t^2v(y,t)&=\delta u(y,t)v(y,t)\\&\quad\times\left(D_yu(y,t)+D_tu(y,t)\right.\\&\quad\left.+D_yv(y,t)+D_tv(y,t)\right)\\&\quad+g_3(y,t)D_y^2v(y,t)+g_4(y,t),\\&\qquad(y,t)\in[A,B]\times[0,T],\end{aligned}\tag{8}$$

related to the initial conditions,

$$\begin{aligned}&u(y,0)=f_1(y),\qquad v(y,0)=f_2(y),\\&D_tu(y,0)=f_3(y),\qquad D_tv(y,0)=f_4(y),\qquad y\in[A,B],\end{aligned}\tag{9}$$

and the boundary conditions,

$$u(A,t) = k_1(t), \qquad u(B,t) = k_2(t),$$
$$v(A,t) = k_3(t), \qquad v(B,t) = k_4(t), \quad t \in [0,T]. \tag{10}$$

Starting with the transformations $x = ((2/(B-A))y) + ((A+B)/(A-B))$, $w(x,t) = u(y,t)$, and $z(x,t) = v(y,t)$. Problem (8)–(10) will be a new problem in the spatial variable $x \in [-1,1]$. This transformation enable us to use the Jacobi collocation method on $[-1,1]$,

$$\begin{aligned} D_t^2 w(x,t) &= \gamma w(x,t)\, z(x,t) \left(\frac{2\left(D_y w(x,t) + D_y z(x,t)\right)}{B-A} + D_t w(x,t) + D_t z(x,t) \right) \\ &\quad + \frac{4 g_1(y,t)\, D_y^2 w(x,t)}{(B-A)^2} + g_2(y,t), \\ D_t^2 z(x,t) &= \delta w(x,t)\, z(x,t) \left(\frac{2\left(D_y w(x,t) + D_y z(x,t)\right)}{B-A} + D_t w(x,t) + D_t z(x,t) \right) \\ &\quad + \frac{4 g_3(y,t)\, D_y^2 z(x,t)}{(B-A)^2} + g_4(y,t), \\ &\quad (y,t) \in [A,B] \times [0,T], \end{aligned} \tag{11}$$

subject to a new set of initial and boundary conditions,

$$w(x,0) = f_5(x), \qquad D_t w(x,0) = f_7(x),$$
$$z(x,0) = f_6(x), \qquad D_t z(x,0) = f_4(x), \quad x \in [-1,1], \tag{12}$$

$$w(-1,t) = k_1(t), \qquad w(1,t) = k_2(t),$$
$$z(-1,t) = k_3(t), \qquad z(1,t) = k_4(t), \quad t \in [0,T]. \tag{13}$$

Now, we are interested in using the J-GL-C method to transform the previous coupled PDEs into system of ODEs. In order to do this, we approximate the spatial variable using J-GL-C method at some nodal points. The node points are the set of points in a specified domain where the dependent variable values are approximated. In general, the choice of the location of the node points is optional, but taking the roots of the Jacobi orthogonal polynomials referred to as Jacobi collocation points gives particularly accurate solutions for the spectral methods. Now, we outline the main step of the J-GL-C method for solving couples hyperbolic problem. Let us expand the dependent variable in a Jacobi series,

$$w(x,t) = \sum_{j=0}^{N} a_j(t)\, J_j^{(\theta,\vartheta)}(x),$$
$$z(x,t) = \sum_{j=0}^{N} b_j(t)\, J_j^{(\theta,\vartheta)}(x). \tag{14}$$

And, in virtue of (6)-(7), we evaluate $a_j(t)$ and $b_j(t)$ by

$$a_j(t) = \frac{1}{h_j} \int_{-1}^{1} w(x,t)\, w^{(\theta,\vartheta)}(x)\, J_j^{(\theta,\vartheta)}(x)\, dx,$$
$$b_j(t) = \frac{1}{h_j} \int_{-1}^{1} z(x,t)\, w^{(\theta,\vartheta)}(x)\, J_j^{(\theta,\vartheta)}(x)\, dx. \tag{15}$$

The Jacobi-Gauss-Lobatto quadrature has been used to evaluate the previous integrals accurately. For any $\phi \in S_{2N-1}[-1,1]$, we have that

$$\int_{-1}^{1} w^{(\theta,\vartheta)}(x)\, \phi(x)\, dx = \sum_{j=0}^{N} \varpi_{N,j}^{(\theta,\vartheta)}\, \phi\left(x_{N,j}^{(\theta,\vartheta)}\right). \tag{16}$$

For any positive integer N, $S_N[-1,1]$ stands for the set of polynomials of degree at most N, $x_{N,j}^{(\theta,\vartheta)}$ $(0 \le j \le N)$ and $\varpi_{N,j}^{(\theta,\vartheta)}$ $(0 \le j \le N)$ are used as the nodes and the corresponding Christoffel numbers in the interval $[-1,1]$, respectively. Thanks to (6), the coefficients $a_j(t)$ in terms of the solution at the collocation points can be approximated by

$$a_j(t) = \frac{1}{h_j} \sum_{i=0}^{N} J_j^{(\theta,\vartheta)}\left(x_{N,i}^{(\theta,\vartheta)}\right) \varpi_{N,i}^{(\theta,\vartheta)}\, w\left(x_{N,i}^{(\alpha,\beta)}, t\right),$$
$$b_j(t) = \frac{1}{h_j} \sum_{i=0}^{N} J_j^{(\theta,\vartheta)}\left(x_{N,i}^{(\theta,\vartheta)}\right) \varpi_{N,i}^{(\theta,\vartheta)}\, z\left(x_{N,i}^{(\alpha,\beta)}, t\right). \tag{17}$$

Due to (17), the approximate solution can be written as

$$\begin{aligned} w(x,t) &= \sum_{i=0}^{N} \left(\sum_{j=0}^{N} \frac{1}{h_j} J_j^{(\theta,\vartheta)}\left(x_{N,i}^{(\theta,\vartheta)}\right) J_j^{(\theta,\vartheta)}(x)\, \varpi_{N,i}^{(\theta,\vartheta)} \right) \times w\left(x_{N,i}^{(\theta,\vartheta)}, t\right), \\ z(x,t) &= \sum_{i=0}^{N} \left(\sum_{j=0}^{N} \frac{1}{h_j} J_j^{(\theta,\vartheta)}\left(x_{N,i}^{(\theta,\vartheta)}\right) J_j^{(\theta,\vartheta)}(x)\, \varpi_{N,i}^{(\theta,\vartheta)} \right) \times z\left(x_{N,i}^{(\theta,\vartheta)}, t\right). \end{aligned} \tag{18}$$

Furthermore, if we differentiate (18) once and evaluate it at the first $N+1$ Jacobi-Gauss-Lobatto collocation points, it is easy to compute the first spatial partial derivative of the numerical solution in terms of the values at these collocation points as

$$D_x w\left(x_{N,n}^{(\theta,\vartheta)},t\right)=\sum_{i=0}^{N}A_{ni}w\left(x_{N,i}^{(\theta,\vartheta)},t\right),$$
$$D_x z\left(x_{N,n}^{(\theta,\vartheta)},t\right)=\sum_{i=0}^{N}A_{ni}z\left(x_{N,i}^{(\theta,\vartheta)},t\right),\quad n=0,1,\ldots,N, \tag{19}$$

where

$$A_{ni}=\sum_{j=0}^{N}\frac{j+\theta+\vartheta+1}{2h_j}J_j^{(\theta,\vartheta)}\left(x_{N,i}^{(\theta,\vartheta)}\right)J_{j-1}^{(\theta+1,\vartheta+1)}\times\left(x_{N,n}^{(\theta,\vartheta)}\right)\varpi_{N,i}^{(\theta,\vartheta)}. \tag{20}$$

Accordingly, one can obtain the second spatial partial derivative as

$$D_x^2 w\left(x_{N,n}^{(\theta,\vartheta)},t\right)=\sum_{i=0}^{N}B_{ni}w\left(x_{N,i}^{(\theta,\vartheta)},t\right),$$
$$D_x^2 z\left(x_{N,n}^{(\theta,\vartheta)},t\right)=\sum_{i=0}^{N}B_{ni}z\left(x_{N,i}^{(\theta,\vartheta)},t\right),\quad n=0,1,\ldots,N, \tag{21}$$

where

$$B_{ni}=\sum_{j=0}^{N}\frac{(j+\theta+\vartheta+2)(j+\theta+\vartheta+1)}{4h_j}J_j^{(\theta,\vartheta)}\left(x_{N,i}^{(\theta,\vartheta)}\right)\times J_{j-2}^{(\theta+2,\vartheta+2)}\left(x_{N,n}^{(\theta,\vartheta)}\right)\varpi_{N,i}^{(\theta,\vartheta)}. \tag{22}$$

In the proposed J-GL-C method the residual of (11) is set to zero at $N-1$ of Jacobi-Gauss-Lobatto points; moreover, the boundary conditions (13) will be enforced at the two collocation points -1 and 1. Therefore, the approximation of (11)–(13) is

$$\begin{aligned}
&\ddot{w}_n(t)+\gamma w_n(t)z_n(t)\left(\dot{w}_n(t)+\dot{z}_n(t)\right)\\
&\quad=\frac{4g_5\left(x_{N,n}^{(\theta,\vartheta)},t\right)g_6\left(x_{N,n}^{(\theta,\vartheta)},t\right)\sum_{i=0}^{N}B_{ni}w_i(t)}{(B-A)^2}\\
&\qquad+\frac{2\gamma w_n(t)z_n(t)\left(\sum_{i=0}^{N}A_{ni}\left(w_i(t)+z_i(t)\right)\right)}{B-A},\\
&\ddot{z}_n(t)+\delta w_n(t)z_n(t)\left(\dot{w}_n(t)+\dot{z}_n(t)\right)\\
&\quad=\frac{4g_7\left(x_{N,n}^{(\theta,\vartheta)},t\right)g_8\left(x_{N,n}^{(\theta,\vartheta)},t\right)\sum_{i=0}^{N}B_{ni}z_i(t)}{(B-A)^2}\\
&\qquad+\frac{2\delta w_n(t)z_n(t)\left(\sum_{i=0}^{N}A_{ni}\left(w_i(t)+z_i(t)\right)\right)}{B-A},\\
&\qquad\qquad n=1,\ldots,N-1,
\end{aligned} \tag{23}$$

where

$$w_k(t)=w\left(x_{N,k}^{(\theta,\vartheta)},t\right),\qquad z_k(t)=z\left(x_{N,k}^{(\theta,\vartheta)},t\right),\qquad k=1,\ldots,N-1. \tag{24}$$

This approach provides a $(2N-2)$ system of second order ODEs in the expansion coefficients $a_j(t)$, $b_j(t)$,

$$\begin{aligned}
&\ddot{w}_n(t)+\gamma w_n(t)z_n(t)\left(\dot{w}_n(t)+\dot{z}_n(t)\right)\\
&\quad=\frac{4g_5\left(x_{N,n}^{(\theta,\vartheta)},t\right)g_6\left(x_{N,n}^{(\theta,\vartheta)},t\right)\sum_{i=0}^{N}B_{ni}w_i(t)}{(B-A)^2}\\
&\qquad+\frac{2\gamma w_n(t)z_n(t)\left(\sum_{i=0}^{N}A_{ni}\left(w_i(t)+z_i(t)\right)\right)}{B-A},\\
&\ddot{z}_n(t)+\delta w_n(t)z_n(t)\left(\dot{w}_n(t)+\dot{z}_n(t)\right)\\
&\quad=\frac{4g_7\left(x_{N,n}^{(\theta,\vartheta)},t\right)g_8\left(x_{N,n}^{(\theta,\vartheta)},t\right)\sum_{i=0}^{N}B_{ni}z_i(t)}{(B-A)^2}\\
&\qquad+\frac{2\delta w_n(t)z_n(t)\left(\sum_{i=0}^{N}A_{ni}\left(w_i(t)+z_i(t)\right)\right)}{B-A},
\end{aligned} \tag{25}$$

with the following initial conditions:

$$\begin{aligned}
&w_n(0)=f_5\left(x_{N,n}^{(\theta,\vartheta)}\right),\qquad \dot{w}_n(0)=f_7\left(x_{N,n}^{(\theta,\vartheta)}\right),\\
&z_n(0)=f_6\left(x_{N,n}^{(\theta,\vartheta)}\right),\qquad \dot{z}_n(0)=f_8\left(x_{N,n}^{(\theta,\vartheta)}\right),\\
&\qquad n=1,\ldots,N-1,
\end{aligned} \tag{26}$$

or in matrix notation as

$$\begin{pmatrix}
\ddot{w}_1(t)+\gamma w_1(t)z_n(t)\left(\dot{w}_1(t)+\dot{z}_1(t)\right)\\
\cdots\\
\cdots\\
\ddot{w}_{N-1}(t)+\gamma w_{N-1}(t)z_{N-1}(t)\left(\dot{w}_{N-1}(t)+\dot{z}_{N-1}(t)\right)\\
\ddot{z}_1(t)+\gamma w_1(t)z_n(t)\left(\dot{w}_1(t)+\dot{z}_1(t)\right)\\
\cdots\\
\cdots\\
\ddot{z}_{N-1}(t)+\gamma w_{N-1}(t)z_n(t)\left(\dot{w}_{N-1}(t)+\dot{z}_{N-1}(t)\right)
\end{pmatrix}
=\begin{pmatrix}
F_1(t,w(t),z(t))\\
\cdots\\
\cdots\\
F_{N-1}(t,w(t),z(t))\\
G_1(t,w(t),z(t))\\
\cdots\\
\cdots\\
G_{N-1}(t,w(t),z(t))
\end{pmatrix} \tag{27}$$

with

$$\begin{pmatrix} w_1(0) \\ \cdots \\ \cdots \\ w_{N-1}(0) \\ z_1(0) \\ \cdots \\ \cdots \\ z_{N-1}(0) \end{pmatrix} = \begin{pmatrix} f_5\left(x_{N,1}^{(\theta,\vartheta)}\right) \\ \cdots \\ \cdots \\ f_5\left(x_{N,N-1}^{(\theta,\vartheta)}\right) \\ f_6\left(x_{N,1}^{(\theta,\vartheta)}\right) \\ \cdots \\ \cdots \\ f_6\left(x_{N,N-1}^{(\theta,\vartheta)}\right) \end{pmatrix},$$
$$\begin{pmatrix} \dot{w}_1(0) \\ \cdots \\ \cdots \\ \dot{w}_{N-1}(0) \\ \dot{z}_1(0) \\ \cdots \\ \cdots \\ \dot{z}_{N-1}(0) \end{pmatrix} = \begin{pmatrix} f_7\left(x_{N,1}^{(\theta,\vartheta)}\right) \\ \cdots \\ \cdots \\ f_7\left(x_{N,N-1}^{(\theta,\vartheta)}\right) \\ f_8\left(x_{N,1}^{(\theta,\vartheta)}\right) \\ \cdots \\ \cdots \\ f_8\left(x_{N,N-1}^{(\theta,\vartheta)}\right) \end{pmatrix}, \tag{28}$$

where

$$\begin{aligned} F_n(t, w(t), z(t)) &= \frac{4g_5\left(x_{N,n}^{(\theta,\vartheta)}, t\right) g_6\left(x_{N,n}^{(\theta,\vartheta)}, t\right) \sum_{i=0}^{N} B_{ni} w_i(t)}{(B-A)^2} \\ &\quad + \frac{2\gamma w_n(t) z_n(t) \left(\sum_{i=0}^{N} A_{ni}\left(w_i(t) + z_i(t)\right)\right)}{B-A}, \\ G_n(t, w(t), z(t)) &= \frac{4g_7\left(x_{N,n}^{(\theta,\vartheta)}, t\right) g_8\left(x_{N,n}^{(\theta,\vartheta)}, t\right) \sum_{i=0}^{N} B_{ni} z_i(t)}{(B-A)^2} \\ &\quad + \frac{2\delta w_n(t) z_n(t) \left(\sum_{i=0}^{N} A_{ni}\left(w_i(t) + z_i(t)\right)\right)}{B-A}. \end{aligned} \tag{29}$$

The system of second order (27)-(28) can be solved by using diagonally implicit Runge-Kutta-Nyström (DIRKN).

3.2. Initial-Nonlocal Conservation Conditions. Here, we will implement the J-GL-C algorithm for the coupled nonlinear hyperbolic type equations with nonlocal conditions:

$$\begin{aligned} D_t^2 u(y,t) &= \gamma u(y,t) v(y,t) \\ &\quad \times \left(D_y u(y,t) + D_t u(y,t) + D_y v(y,t) + D_t v(y,t)\right) \\ &\quad + g_1(y,t) D_y^2 u(y,t) + g_2(y,t), \\ D_t^2 v(y,t) &= \delta u(y,t) v(y,t) \\ &\quad \times \left(D_y u(y,t) + D_t u(y,t) + D_y v(y,t) + D_t v(y,t)\right) \\ &\quad + g_3(y,t) D_y^2 v(y,t) + g_4(y,t), \\ &\quad (y,t) \in [A,B] \times [0,T], \end{aligned} \tag{30}$$

subject to the initial conditions,

$$\begin{gathered} u(y,0) = f_1(y), \qquad D_t u(y,0) = f_3(y), \\ v(y,0) = f_2(y), \qquad D_t v(y,0) = f_4(y), \quad y \in [A,B], \end{gathered} \tag{31}$$

and the boundary conditions,

$$u(A,t) = k_1(t), \qquad v(A,t) = k_3(t), \quad t \in [0,T], \tag{32}$$

while the other two boundary conditions have the nonlocal conservation form

$$\begin{gathered} \int_A^B u(y,t)\, dy = k_2(t), \\ \int_A^B v(y,t)\, dy = k_4(t), \quad t \in [0,T]. \end{gathered} \tag{33}$$

Again, we used the change of variables $x = ((2/(B-A))y) + ((A+B)/(A-B))$, $w(x,t) = u(y,t)$, $z(x,t) = v(y,t)$, to reduce problem (30)–(33) into

$$\begin{aligned} D_t^2 w(x,t) &= \gamma w(x,t) z(x,t) \\ &\quad \times \left(\frac{2\left(D_y w(x,t) + D_y z(x,t)\right)}{B-A} + D_t w(x,t) + D_t z(x,t)\right) \\ &\quad + \frac{4g_1(y,t) D_y^2 w(x,t)}{(B-A)^2} + g_2(y,t), \\ D_t^2 z(x,t) &= \delta w(x,t) z(x,t) \left(\frac{2\left(D_y w(x,t) + D_y z(x,t)\right)}{B-A} + D_t w(x,t) + D_t z(x,t)\right) \\ &\quad + \frac{4g_3(y,t) D_y^2 z(x,t)}{(B-A)^2} + g_4(y,t), \\ &\quad (y,t) \in [A,B] \times [0,T] \end{aligned} \tag{34}$$

related to the new initial conditions,

$$\begin{gathered} w(x,0) = f_5(x), \qquad D_t w(x,0) = f_7(x), \\ z(x,0) = f_6(x), \qquad D_t z(x,0) = f_4(x), \quad x \in [-1,1], \end{gathered} \tag{35}$$

the boundary conditions,

$$w(-1,t)=k_1(t),\qquad z(-1,t)=k_3(t),\qquad t\in[0,T],\quad(36)$$

and the nonlocal conservation conditions,

$$\begin{aligned}&\frac{B-A}{2}\int_{-1}^{1}w(x,t)\,dx=k_2(t),\\&\frac{B-A}{2}\int_{-1}^{1}z(x,t)\,dx=k_4(t),\qquad t\in[0,T].\end{aligned}\quad(37)$$

The problem now is how to deal with the nonlocal conditions (37). For this purpose, let us introduce a collocation treatment for the integral conservation conditions (37) as

$$\begin{aligned}&\frac{B-A}{2}\\&\times\int_{-1}^{1}\sum_{i=0}^{N}\left(\sum_{j=0}^{N}\frac{1}{h_j}J_j^{\theta,\vartheta}\left(x_{N,i}^{\theta,\vartheta}\right)J_j^{\theta,\vartheta}(x)\,\varpi_{N,i}^{\theta,\vartheta}\right)w_i(t)\,dx=k_2(t),\\&\frac{B-A}{2}\\&\times\int_{-1}^{1}\sum_{i=0}^{N}\left(\sum_{j=0}^{N}\frac{1}{h_j}J_j^{\theta,\vartheta}\left(x_{N,i}^{\theta,\vartheta}\right)J_j^{\theta,\vartheta}(x)\,\varpi_{N,i}^{\theta,\vartheta}\right)z_i(t)\,dx=k_4(t).\end{aligned}\quad(38)$$

The above equations may be rearranged as

$$\begin{aligned}&\frac{B-A}{2}\sum_{i=0}^{N}\left(\sum_{j=0}^{N}\frac{1}{h_j}J_j^{\theta,\vartheta}\left(x_{N,i}^{\theta,\vartheta}\right)\varpi_{N,i}^{\theta,\vartheta}\left(\int_{-1}^{1}J_j^{\theta,\vartheta}(x)\,dx\right)\right)w_i(t)\\&\quad=k_2(t),\\&\frac{B-A}{2}\sum_{i=0}^{N}\left(\sum_{j=0}^{N}\frac{1}{h_j}J_j^{\theta,\vartheta}\left(x_{N,i}^{\theta,\vartheta}\right)\varpi_{N,i}^{\theta,\vartheta}\left(\int_{-1}^{1}J_j^{\theta,\vartheta}(x)\,dx\right)\right)z_i(t)\\&\quad=k_4(t),\end{aligned}\quad(39)$$

or briefly

$$\sum_{i=0}^{N}I_iw_i(t)=k_2(t),\qquad\sum_{i=0}^{N}I_iz_i(t)=k_4(t),\quad(40)$$

where

$$I_i=\frac{B-A}{2}\left(\sum_{j=0}^{N}\frac{1}{h_j}J_j^{\theta,\vartheta}\left(x_{N,i}^{\theta,\vartheta}\right)\varpi_{N,i}^{\theta,\vartheta}\left(\int_{-1}^{1}J_j^{\theta,\vartheta}(x)\,dx\right)\right).\quad(41)$$

Consequently, $w_N(t)$ and $z_N(t)$ are expressed as the following expansion of $w_i(t)$ and $z_i(t)$, $i=1,\ldots,N$:

$$\begin{aligned}w_N(t)&=\frac{1}{I_N}\left(k_2(t)-I_0w_0(t)-\sum_{i=1}^{N-1}I_iw_i(t)\right),\\&=\frac{1}{I_N}\left(k_2(t)-I_0k_1(t)-\sum_{i=1}^{N-1}I_iw_i(t)\right),\\z_N(t)&=\frac{1}{I_N}\left(k_4(t)-I_0z_0(t)-\sum_{i=1}^{N-1}I_iz_i(t)\right),\\&=\frac{1}{I_N}\left(k_4(t)-I_0k_3(t)-\sum_{i=1}^{N-1}I_iz_i(t)\right).\end{aligned}\quad(42)$$

Based on the information included in this subsection and the recent one, we obtain the following system of ODEs:

$$\begin{aligned}&\ddot{w}_n(t)+\gamma w_n(t)z_n(t)\left(\dot{w}_n(t)+\dot{z}_n(t)\right)\\&\quad=\frac{4g_5\left(x_{N,n}^{(\theta,\vartheta)},t\right)g_6\left(x_{N,n}^{(\theta,\vartheta)},t\right)\sum_{i=0}^{N}B_{ni}w_i(t)}{(B-A)^2}\\&\qquad+\frac{2\gamma w_n(t)z_n(t)\left(\sum_{i=0}^{N}A_{ni}\left(w_i(t)+z_i(t)\right)\right)}{B-A},\\&\ddot{z}_n(t)+\delta w_n(t)z_n(t)\left(\dot{w}_n(t)+\dot{z}_n(t)\right)\\&\quad=\frac{4g_7\left(x_{N,n}^{(\theta,\vartheta)},t\right)g_8\left(x_{N,n}^{(\theta,\vartheta)},t\right)\sum_{i=0}^{N}B_{ni}z_i(t)}{(B-A)^2}\\&\qquad+\frac{2\delta w_n(t)z_n(t)\left(\sum_{i=0}^{N}A_{ni}\left(w_i(t)+z_i(t)\right)\right)}{B-A},\end{aligned}\quad(43)$$

with the following initial conditions:

$$\begin{aligned}&w_n(0)=f_5\left(x_{N,n}^{(\theta,\vartheta)}\right),\qquad\dot{w}_n(0)=f_7\left(x_{N,n}^{(\theta,\vartheta)}\right),\\&z_n(0)=f_6\left(x_{N,n}^{(\theta,\vartheta)}\right),\qquad\dot{z}_n(0)=f_8\left(x_{N,n}^{(\theta,\vartheta)}\right),\\&\qquad\qquad n=1,\ldots,N-1,\end{aligned}\quad(44)$$

where w_0, w_N, z_0, and z_N are given in (36) and (42).

4. Test Problems

We test the numerical accuracy of the proposed method by introducing four test problems with different types of exact solutions.

4.1. Triangular Solution. As a first example, we consider the coupled nonlinear hyperbolic equation (8) with the following functions:

$$\begin{aligned}g_1(y,t)&=\left(1+e^t\cos(y)\right),\\g_2&=\frac{1}{2}\cos(t)\left(e^t-2\gamma\cos(t+y)\sin(t)\right)\sin(2y),\\g_3(y,t)&=\left(1+e^t\sin(y)\right),\\g_4&=\frac{1}{2}\left(e^t-2\delta\cos(t)\cos(t+y)\right)\sin(t)\sin(2y),\end{aligned}\quad(45)$$

Table 1

N	MAE_1	RMSE_1	N_{e_1}	MAE_2	RMSE_2	N_{e_2}
			$\alpha = 0, \beta = 0$			
4	3.73×10^{-4}	2.88×10^{-4}	7.93×10^{-4}	3.44×10^{-4}	1.47×10^{-4}	4.04×10^{-4}
8	2.02×10^{-8}	6.37×10^{-9}	1.74×10^{-8}	7.68×10^{-8}	2.22×10^{-8}	6.05×10^{-8}
12	1.52×10^{-8}	2.87×10^{-9}	7.81×10^{-9}	1.86×10^{-8}	1.19×10^{-8}	3.24×10^{-8}
			$\alpha = 0, \beta = 1/2$			
4	7.26×10^{-4}	4.15×10^{-4}	11.20×10^{-4}	5.05×10^{-4}	1.40×10^{-4}	3.76×10^{-4}
8	1.93×10^{-8}	9.64×10^{-9}	2.60×10^{-8}	5.64×10^{-8}	2.22×10^{-8}	5.97×10^{-8}
12	2.23×10^{-8}	4.33×10^{-9}	1.17×10^{-9}	1.86×10^{-8}	1.10×10^{-8}	2.96×10^{-8}
			$\alpha = 1/2, \beta = 0$			
4	6.81×10^{-4}	3.03×10^{-4}	8.46×10^{-4}	4.96×10^{-4}	2.57×10^{-4}	7.18×10^{-4}
8	2.02×10^{-8}	7.98×10^{-9}	2.19×10^{-8}	8.61×10^{-8}	3.72×10^{-8}	8.99×10^{-8}
12	2.49×10^{-8}	7.92×10^{-9}	2.16×10^{-8}	3.43×10^{-8}	1.86×10^{-8}	5.09×10^{-8}

subject to

$$\begin{gathered} k_1(t) = \sin(A)\cos(t), \qquad k_2(t) = \sin(B)\cos(t), \\ k_3(t) = \sin(t)\cos(A), \qquad k_4(t) = \cos(B)\sin(t), \\ f_1(t) = \sin(y), \qquad f_2(t) = f_3(t) = 0, \\ f_4(t) = \cos(y). \end{gathered} \tag{46}$$

The exact solutions of this problem are

$$u(y,t) = \sin(y)\cos(t), \qquad v(y,t) = \sin(t)\cos(y). \tag{47}$$

The absolute errors in the given tables are

$$E(y,t) = \left|u(y,t) - \tilde{u}(y,t)\right|, \tag{48}$$

where $u(y,t)$ and $\tilde{u}(y,t)$ are the exact and approximate solutions at the point (y,t), respectively. Moreover, the maximum absolute error is given by

$$M_E = \operatorname{Max}\left\{E(y,t) : \forall (y,t) \in [A,B] \times [0,T]\right\}. \tag{49}$$

The root mean square (RMS) and N_e errors may be given by

$$\begin{gathered} \text{RMS} = \sqrt{\sum_{i=0}^{N} \frac{\left(u\left(x_{N,i}^{(\alpha,\beta)}, t_i\right) - \tilde{u}\left(x_{N,i}^{(\alpha,\beta)}, t_i\right)\right)^2}{N+1}}, \\ N_e = \sqrt{\frac{\sum_{i=0}^{N}\left(u\left(x_{N,i}^{(\alpha,\beta)}, t_i\right) - \tilde{u}\left(x_{N,i}^{(\alpha,\beta)}, t_i\right)\right)^2}{\sum_{i=0}^{N} u\left(x_{N,i}^{(\alpha,\beta)}, t_i\right)}}. \end{gathered} \tag{50}$$

Maximum absolute, root mean square, and N_e errors of (45) are introduced in Table 1 using J-GL-C method with three different choices of N, α, and β in the interval $[0,1]$. The approximate solutions $\tilde{u}$ and $\tilde{v}$ of problem (45) have been plotted in Figures 1 and 2, with values of parameters listed in their captions. Moreover, we plot the curves of approximate and exact solutions of $\tilde{u}$ at different values of x and t in Figures 3 and 4. Again, the curves of approximate and exact solutions of $\tilde{v}$ at different values of x and t are displayed in Figures 5 and 6.

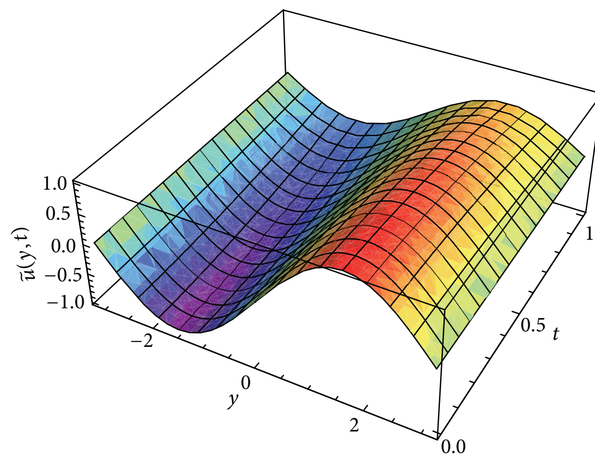

Figure 1: The approximate solution $\tilde{u}$ of problem (45), where $N = 16$ and $\alpha = \beta = 0$ in the interval $[-\pi, \pi]$.

4.2. Soliton Solution. Secondly, consider the coupled nonlinear hyperbolic equation (8) with the following functions:

$$\begin{aligned} & g_1(y,t) = \left(1 + e^t \cos(y)\right), \qquad g_3(y,t) = \left(1 + e^t \sin(y)\right), \\ & g_4 = -2\operatorname{sech}(y+t)^2\left(\operatorname{sech}(y+t) - e^t \sin(y) - \tanh(y+t)\right) \\ & \qquad \times \tanh(y+t), \\ & g_2 = \operatorname{sech}(y+t)\left(\operatorname{sech}(y+t) - \tanh(y+t)\right) \\ & \qquad \times \left(-2\operatorname{sech}(y+t)\tanh(y+t) + e^t \cos(y)\right. \\ & \qquad\qquad \left. \times \left(\operatorname{sech}(y+t) + \tanh(y+t)\right)\right), \end{aligned} \tag{51}$$

subject to

$$\begin{gathered} k_1(t) = \operatorname{sech}(A+t), \qquad k_2(t) = \operatorname{sech}(B+t), \\ k_3(t) = \tanh(A+t), \qquad k_4(t) = \tanh(B+t), \end{gathered}$$

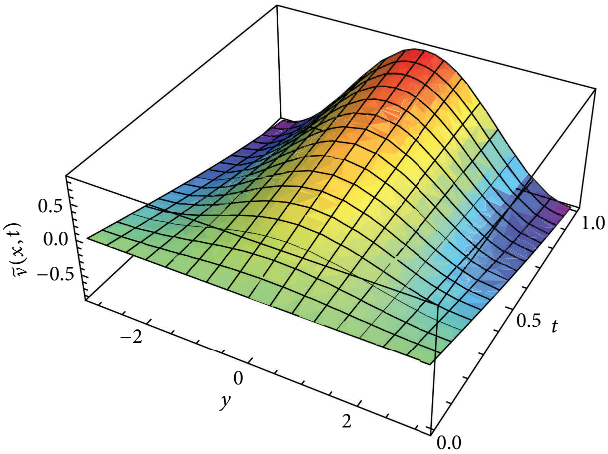

FIGURE 2: The approximate solution $\tilde{v}$ of problem (45), where $N = 16$ and $\alpha = \beta = 0$ in the interval $[-\pi, \pi]$.

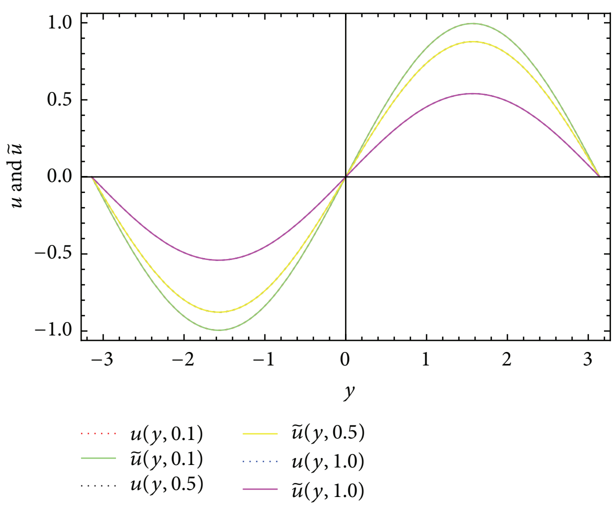

FIGURE 3: The approximate $\tilde{u}$ and the exact u solutions for different values of $t = 0.1, 0.5$, and 1.0 of problem (45), where $N = 16$ and $\alpha = \beta = 0$ in the interval $[-\pi, \pi]$.

$$f_1(t) = \operatorname{sech}(y), \qquad f_2(t) = \tanh(y),$$
$$f_3(t) = -\operatorname{sech}(y)\tanh(y), \qquad f_4(t) = \operatorname{sech}(y)^2. \tag{52}$$

The exact solutions are

$$u(y,t) = \operatorname{sech}(y+t), \qquad v(y,t) = \tanh(y+t). \tag{53}$$

Table 2 shows the accurate results for maximum absolute, root mean square, and N_e errors of (51) for various choices of N, α, and β in the interval $[0, 1]$. Figures 7 and 8 show that the absolute errors E_1 and E_2 are very small with values of parameters listed in their captions. We also plot the curves of approximate and exact solutions of $\tilde{u}$ at different values of x and t in Figures 9 and 10. Moreover, in Figures 11 and 12, the approximate and exact solutions of $\tilde{v}$ are plotted at different values of x and t.

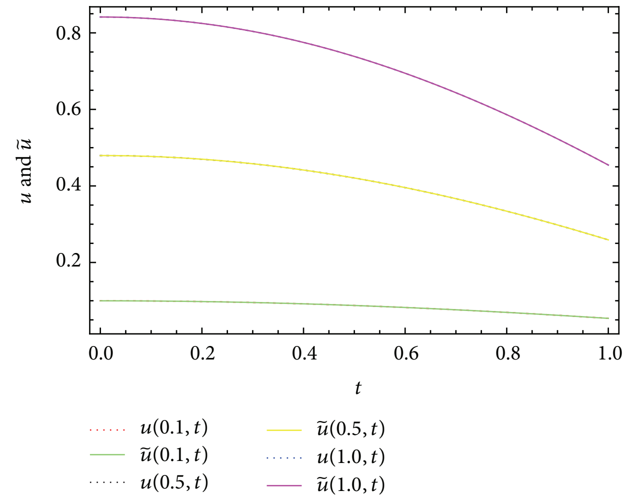

FIGURE 4: The approximate $\tilde{u}$ and the exact u solutions for different values of $y = 0.1, 0.5$, and 1.0 of problem (45), where $N = 16$ and $\alpha = \beta = 0$ in the interval $[-\pi, \pi]$.

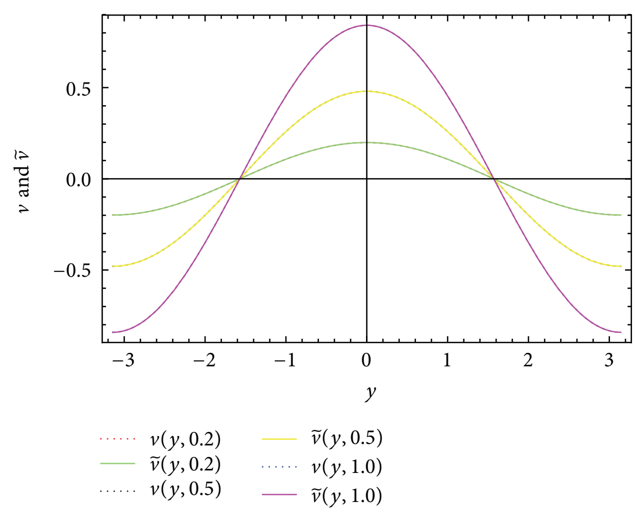

FIGURE 5: The approximate $\tilde{v}$ and the exact v solutions for different values of $t = 0.2, 0.5$, and 1.0 of problem (45), where $N = 16$ and $\alpha = \beta = 0$ in the interval $[-\pi, \pi]$.

4.3. Exponential-Triangular Solution. In the third example, consider the coupled nonlinear hyperbolic equations (8)-(9) with the following functions:

$$g_1(y,t) = \left(1 + e^t \cos(y)\right), \qquad g_3(y,t) = \left(1 + e^t \sin(y)\right),$$
$$g_2 = e^t \cos(y) + e^t \cos(y)\left(1 + e^t \cos(y)\right)$$
$$- 2e^{3t}\lambda \cos(y)^2 \sin(y),$$
$$g_4 = -e^t \sin(y)\left(-2 + 2e^{2t}\gamma \cos(y)^2 - e^t \sin(y)\right); \tag{54}$$

Table 2

N	MAE_1	RMSE_1	N_{e_1}	MAE_2	RMSE_2	N_{e_2}
			$\alpha=\beta=0$			
4	1.35×10^{-3}	4.35×10^{-4}	6.13×10^{-4}	2.63×10^{-3}	1.51×10^{-3}	2.15×10^{-3}
8	1.61×10^{-6}	9.88×10^{-7}	1.40×10^{-6}	9.83×10^{-7}	4.12×10^{-7}	5.84×10^{-7}
12	7.44×10^{-8}	3.33×10^{-8}	4.70×10^{-8}	9.30×10^{-8}	3.95×10^{-8}	5.59×10^{-8}
			$\alpha=\beta=1/2$			
4	1.95×10^{-3}	6.44×10^{-4}	9.10×10^{-4}	4.34×10^{-3}	2.27×10^{-3}	3.22×10^{-3}
8	2.85×10^{-6}	1.68×10^{-6}	2.38×10^{-6}	1.62×10^{-6}	6.59×10^{-7}	9.31×10^{-7}
12	1.34×10^{-7}	6.02×10^{-8}	8.52×10^{-8}	1.69×10^{-7}	6.79×10^{-8}	9.60×10^{-8}
			$\alpha=\beta=-1/2$			
4	7.59×10^{-4}	2.72×10^{-4}	3.82×10^{-4}	1.69×10^{-3}	7.67×10^{-4}	1.09×10^{-3}
8	6.01×10^{-7}	3.44×10^{-7}	4.85×10^{-8}	4.16×10^{-7}	1.19×10^{-7}	1.68×10^{-7}
12	3.06×10^{-8}	1.48×10^{-8}	2.09×10^{-8}	3.83×10^{-8}	1.49×10^{-8}	2.11×10^{-8}

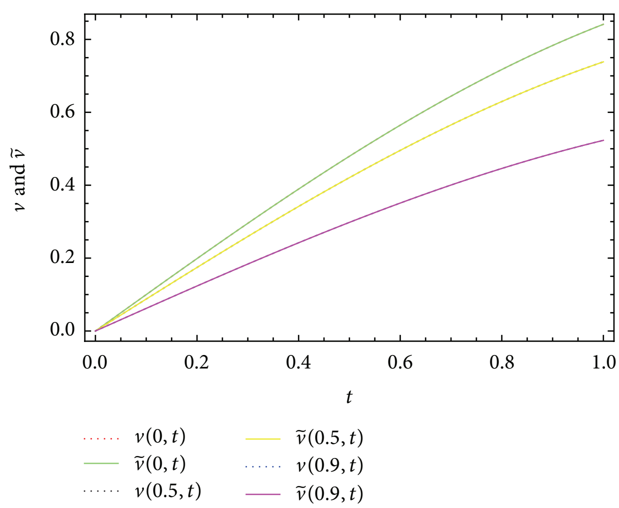

Figure 6: The approximate $\tilde{v}$ and the exact v solutions for different values of $y = 0.0$, 0.5, and 0.9 of problem (45), where $N = 16$ in the interval $[-\pi, \pi]$.

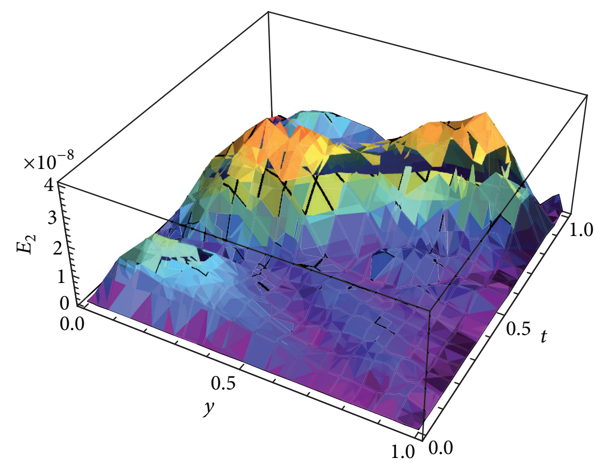

Figure 8: The absolute error E_2 of problem (51), where $N = 12$ and $\alpha = \beta = -(1/2)$ in the interval $[0, 1]$.

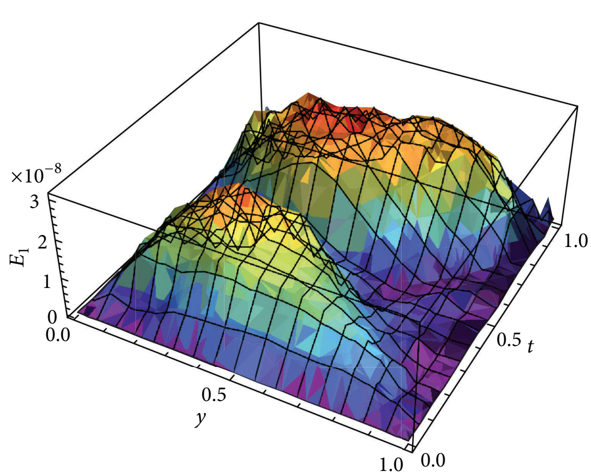

Figure 7: The absolute error E_1 of problem (51), where $N = 12$ and $\alpha = \beta = -(1/2)$ in the interval $[0, 1]$.

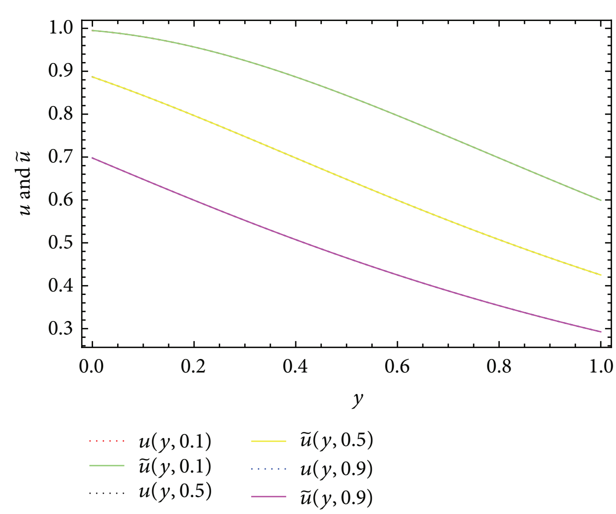

Figure 9: The approximate $\tilde{u}$ and the exact u solutions for different values of $t = 0.1$, 0.5, and 0.9 of problem (51), where $N = 12$ and $\alpha = \beta = -(1/2)$ in the interval $[0, 1]$.

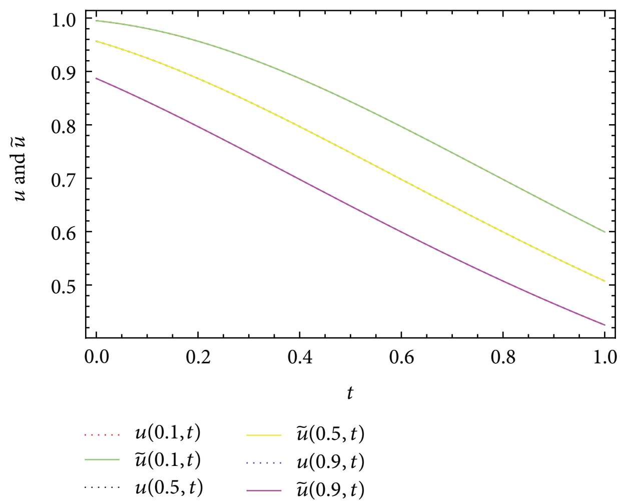

FIGURE 10: The approximate $\tilde{u}$ and the exact u solutions for different values of $y = 0.1, 0.5$, and 0.9 of problem (51), where $N = 12$ and $\alpha = \beta = -(1/2)$ in the interval $[0, 1]$.

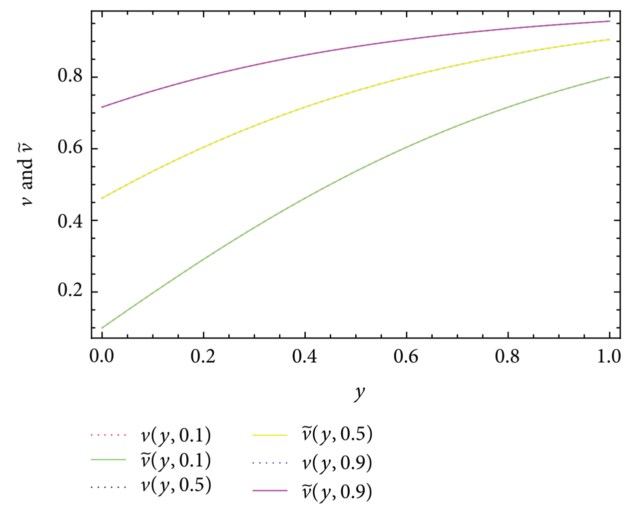

FIGURE 11: The approximate $\tilde{v}$ and the exact v solutions for different values of $t = 0.1, 0.5$, and 0.9 of problem (51), where $N = 12$ and $\alpha = \beta = -(1/2)$ in the interval $[0, 1]$.

the initial-boundary conditions (9) and (10) may be given by

$$\begin{aligned} k_1(t) &= e^t \cos(A), & k_2(t) &= e^t \cos(B), \\ k_3(t) &= e^t \sin(A), & k_4(t) &= e^t \sin(B), \\ f_1(t) &= \cos(y), & f_2(t) &= \sin(y), \\ f_3(t) &= \cos(y), & f_4(t) &= \sin(y). \end{aligned} \tag{55}$$

The exact solutions of this problem are

$$u(y,t) = e^t \cos(y), \qquad v(y,t) = e^t \sin(y). \tag{56}$$

More accurate results for maximum absolute, root mean square, and N_e errors of (55) are given in Table 3, for different choices of Jacobi parameters; even we use limited values of N. The approximate solutions $\tilde{u}$ and $\tilde{v}$ of problem (55) are plotted in Figures 13 and 14 with values of parameters listed in their captions. In addition, Figures 15 and 16 present the approximate and exact solutions of $\tilde{u}(y,t)$; moreover, the corresponding figures for $\tilde{v}(y,t)$ at parameters listed in their captions are displayed in Figures 17 and 18.

4.4. Triangular Solution. In the last example, consider the coupled nonlinear hyperbolic equation (30) with the following functions:

$$\begin{aligned} g_1(y,t) &= \left(1 + e^t \cos(y)\right), \\ g_2 &= \frac{1}{2}\cos(t)\left(e^t - 2\gamma\cos(t+y)\sin(t)\right)\sin(2y), \\ g_3(y,t) &= \left(1 + e^t \sin(y)\right), \\ g_4 &= \frac{1}{2}\left(e^t - 2\delta\cos(t)\cos(t+y)\right)\sin(t)\sin(2y), \end{aligned} \tag{57}$$

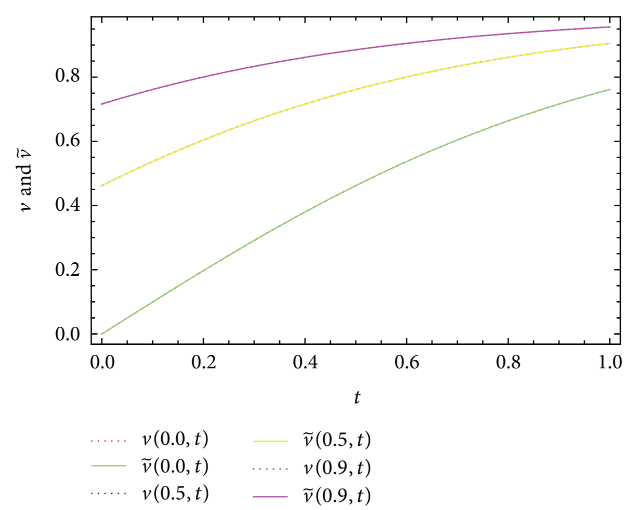

FIGURE 12: The approximate $\tilde{v}$ and the exact v solutions for different values of $y = 0.0, 0.5$, and 0.9 of problem (51), where $N = 12$ and $\alpha = \beta = -(1/2)$ in the interval $[0, 1]$.

related to the initial conditions (31),

$$\begin{gathered} f_1(t) = \sin(y), \qquad f_2(t) = f_3(t) = 0, \\ f_4(t) = \cos(y), \end{gathered} \tag{58}$$

and the boundary conditions (32),

$$k_1(t) = \sin(A)\cos(t), \qquad k_3(t) = \sin(t)\cos(A), \tag{59}$$

while the nonlocal conservation conditions are (33)

$$\begin{aligned} k_2(t) &= (\cos(A) - \cos(B))\cos(t), \\ k_4(t) &= (\sin(B) - \sin(A))\sin(t). \end{aligned} \tag{60}$$

Table 3

N	MAE_1	RMSE_1	N_{e_1}	MAE_2	RMSE_2	N_{e_2}
			$\alpha = 0, \beta = 0$			
4	2.71×10^{-3}	1.05×10^{-3}	7.78×10^{-4}	2.47×10^{-3}	4.77×10^{-3}	3.61×10^{-4}
6	6.84×10^{-6}	1.86×10^{-6}	1.38×10^{-6}	6.82×10^{-6}	1.47×10^{-6}	1.13×10^{-6}
8	8.31×10^{-8}	1.19×10^{-8}	8.80×10^{-9}	9.64×10^{-8}	6.91×10^{-9}	5.35×10^{-9}
			$\alpha = 1/2, \beta = 0$			
4	3.74×10^{-3}	1.43×10^{-3}	1.07×10^{-3}	3.79×10^{-3}	6.02×10^{-4}	4.70×10^{-4}
6	1.03×10^{-5}	2.91×10^{-6}	2.17×10^{-6}	1.15×10^{-5}	1.81×10^{-6}	1.43×10^{-6}
8	1.26×10^{-7}	2.28×10^{-8}	1.69×10^{-8}	1.10×10^{-7}	1.01×10^{-8}	8.04×10^{-9}
			$\alpha = 1, \beta = 1$			
4	5.00×10^{-3}	1.88×10^{-3}	1.39×10^{-3}	5.33×10^{-3}	1.20×10^{-3}	9.27×10^{-4}
6	1.55×10^{-5}	4.76×10^{-6}	3.51×10^{-6}	1.64×10^{-5}	2.88×10^{-6}	2.26×10^{-6}
8	2.45×10^{-7}	4.06×10^{-8}	2.99×10^{-8}	2.39×10^{-7}	2.53×10^{-8}	2.01×10^{-8}

Table 4

N	MAE_1	RMSE_1	N_{e_1}	MAE_2	RMSE_2	N_{e_2}
			$\alpha = 0, \beta = 0$			
4	7.46×10^{-4}	2.34×10^{-4}	6.44×10^{-4}	2.30×10^{-4}	1.14×10^{-4}	3.13×10^{-4}
8	7.84×10^{-7}	1.13×10^{-7}	3.08×10^{-7}	3.02×10^{-6}	7.51×10^{-7}	2.05×10^{-6}
			$\alpha = \beta = -1/2$			
4	8.06×10^{-4}	2.55×10^{-4}	7.07×10^{-4}	1.17×10^{-4}	6.09×10^{-5}	1.69×10^{-4}
8	4.74×10^{-7}	1.08×10^{-7}	2.97×10^{-7}	5.27×10^{-7}	1.18×10^{-7}	4.97×10^{-7}
			$\alpha = 0, \beta = -1/2$			
4	7.54×10^{-4}	2.81×10^{-4}	7.90×10^{-4}	9.02×10^{-4}	4.16×10^{-4}	1.17×10^{-4}
8	2.32×10^{-7}	9.26×10^{-8}	2.56×10^{-7}	1.41×10^{-6}	2.21×10^{-7}	6.11×10^{-7}

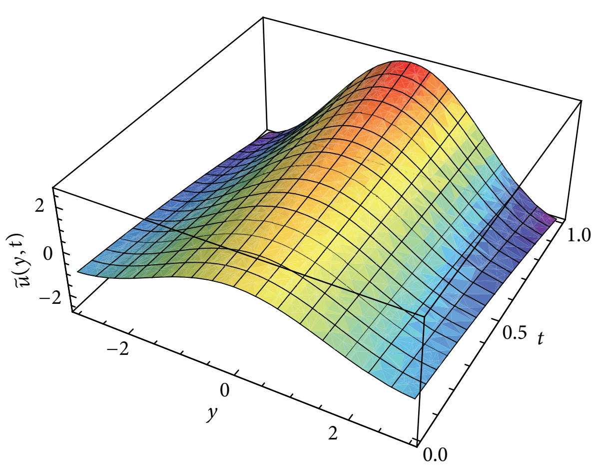

Figure 13: The approximate solution $\tilde{u}$ of problem (55), where $N = 10$ and $\alpha = \beta = 1/2$ in the interval $[0, 1]$.

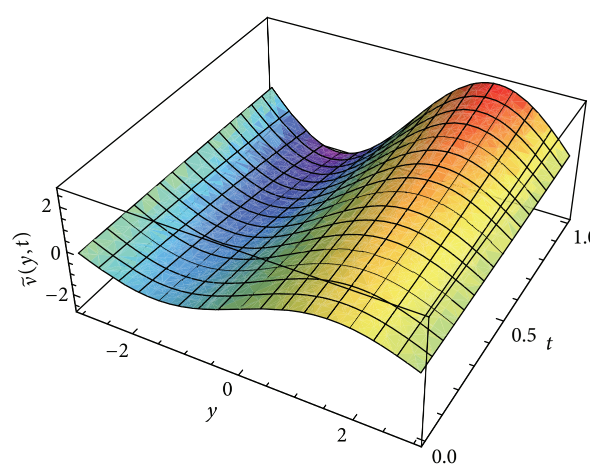

Figure 14: The approximate solution $\tilde{v}$ of problem (55), where $N = 10$ and $\alpha = \beta = 1/2$ in the interval $[0, 1]$.

The exact solutions of (30) are

$$u(y,t) = \sin(y)\cos(t), \qquad v(y,t) = \sin(t)\cos(y). \tag{61}$$

Maximum absolute, root mean square, and N_e errors of (57) are introduced in Table 4 using J-GL-C method with various choices of N, α, and β in the interval $[0, 1]$. From numerical results of this table, it can be concluded that the numerical solutions for problems with nonlocal conservation conditions are in good agreement with the exact solutions.

5. Conclusion

For boundary and nonlocal conditions, we have proposed an efficient and accurate numerical algorithm based on Jacobi-Gauss-Lobatto spectral method to get high accurate solutions

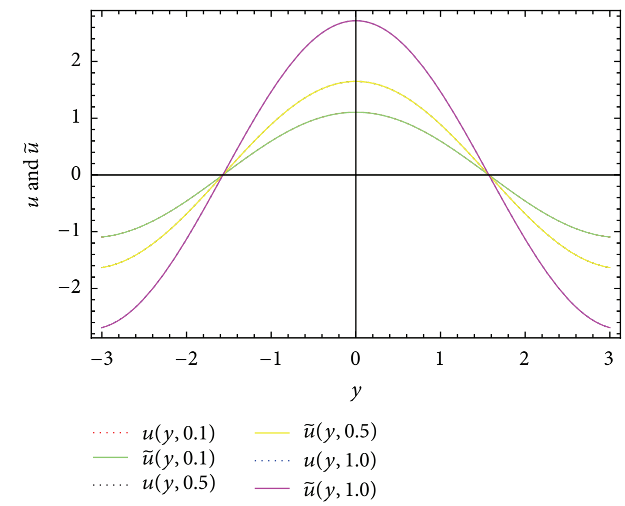

FIGURE 15: The approximate $\tilde{u}$ and the exact u solutions for different values of $t = 0.1, 0.5$, and 1.0 of problem (55), where $N = 10$ and $\alpha = \beta = 1/2$ in the interval $[0, 1]$.

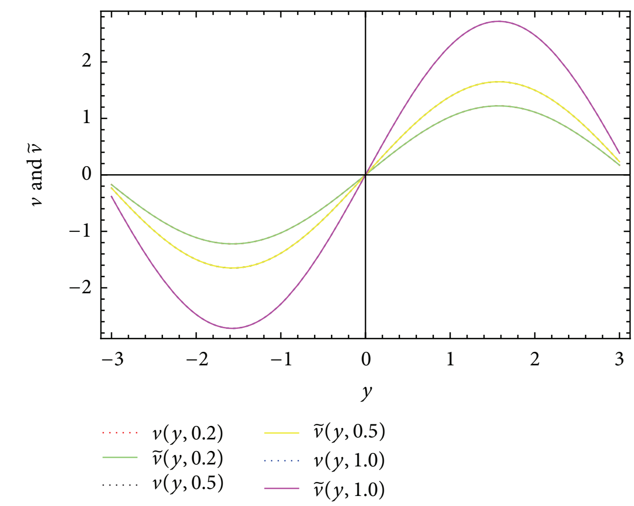

FIGURE 17: The approximate $\tilde{v}$ and the exact v solutions for different values of $t = 0.2, 0.5$, and 1.0 of problem (55), where $N = 10$ and $\alpha = \beta = 1/2$ in the interval $[0, 1]$.

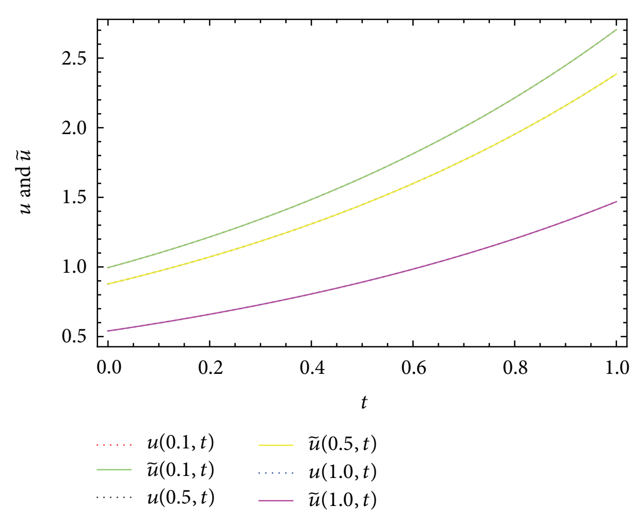

FIGURE 16: The approximate $\tilde{u}$ and the exact u solutions for different values of $y = 0.1, 0.5$, and 1.0 of problem (55), where $N = 10$ and $\alpha = \beta = 1/2$ in the interval $[0, 1]$.

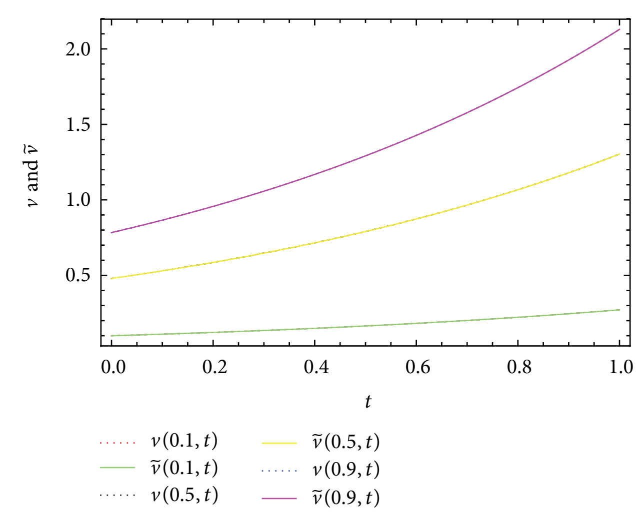

FIGURE 18: The approximate $\tilde{v}$ and the exact v solutions for different values of $y = 0.0, 0.5$, and 0.9 of problem (55), where $N = 10$ and $\alpha = \beta = 1/2$ in the interval $[0, 1]$.

for nonlinear coupled hyperbolic equations. The method is based upon reducing the mentioned problem into a system of second order ODEs in the expansion coefficient of the solution. The use of the Jacobi-Gauss-Lobatto points as collocation nodes saves the spectral convergence for the spatial variable in the approximate solution. Numerical examples were also provided to illustrate the effectiveness of the derived algorithm. The numerical experiments show that the Jacobi collocation approximation is very accurate with a limited number of collocation nodes.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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