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# Exact Determinants of Some Special Circulant Matrices Involving Four Kinds of Famous Numbers 

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#### Abstract

Circulant matrix family is used for modeling many problems arising in solving various differential equations. The RSFPLR circulant matrices and RSLPFL circulant matrices are two special circulant matrices. The techniques used herein are based on the inverse factorization of polynomial. The exact determinants of these matrices involving Perrin, Padovan, Tribonacci, and the generalized Lucas number are given, respectively.


## 1. Introduction

Circulant matrix family is used for modeling many problems arising in solving various differential equations. Therefore, studying algorithms and various properties of new classes of problems to such patterned matrices is crucial for applications. It is important to develop new theories and methods and to modify and refine the well-known techniques, for solving differential equations.

Lei and Sun [1] proposed the preconditioned CGNR (PCGNR) method with a circulant preconditioner to solve such Toeplitz-like systems. Pang et al. [2] used the normalized preconditioned conjugate gradient method with Strang's circulant preconditioner to solve a nonsymmetric Toeplitz system $A_{n} x=b$. By using a Strang-type block-circulant preconditioner, Zhang et al. [3] speeded up the convergent rate of boundary-value methods. Delgado et al. [4] developed some techniques to obtain global hyperbolicity for a certain class of endomorphisms of $\left(R^{p}\right)^{n}$ with $p, n \geq 2$; this kind of endomorphisms is obtained from vectorial difference equations where the mapping defining these equations satisfies a circulant matrix condition. The Strang-type preconditioner was also used to solve linear systems from differential-algebraic equations and delay differential equations; see [5-8]. In [9], a semicirculant preconditioner applied to a problem, subject to Dirichlet boundary conditions at the inflow boundaries,
was examined. A method was described for obtaining finite difference approximation solutions of multidimensional partial differential equations satisfying boundary conditions specified on irregularly shaped boundaries by using circulant matrices and fast Fourier transform (FFT) convolutions in [10]. Brockett and Willems [11] showed how the important problems of linear system theory can be solved concisely for a particular class of linear systems, namely, block-circulant systems, by exploiting the algebraic structure.

Circulant matrices have important applications in various disciplines including image processing, communications, and signal processing. The circulant matrices, a long fruitful subject of research, have in recent years been extended in many directions [12, 13]. The $f(x)$-circulant matrices are another natural extension of this well-studied class and can be found in [14-24]. The $f(x)$-circulant matrix has a wide application, especially on the generalized cyclic codes [14]. The properties and structures of the $x^{n}-x+1$-circulant matrices, which are called RSFPLR circulant matrices, are better than those of the general $f(x)$-circulant matrices, so there are good algorithms for determinants.

There are many interests in properties and generalization of some special matrices with famous numbers. Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [25]. Lin gave the determinant of the Fibonacci-Lucas quasi-cyclic matrices [26].

Lind presented the determinants of circulant and skewcirculant involving Fibonacci numbers in [27]. Shen et al. [28] discussed the determinant of circulant matrix involving Fibonacci and Lucas numbers. Akbulak and Bozkurt [29] gave the norms of Toeplitz involving Fibonacci and Lucas numbers. The authors [30,31] discussed some properties of Fibonacci and Lucas matrices. Stanimirović gave generalized Fibonacci and Lucas matrix in [32]. Z. Zhang and Y. Zhang [33] investigated the Lucas matrix and some combinatorial identities.

The determinant problems of the RSFPLR circulant matrices and RSLPFL circulant matrices involving the Perrin, Padovan, Tribonacci, and the generalized Lucas number are considered in this paper. The explicit determinants are presented by using some terms of these numbers. The techniques used herein are based on the inverse factorization of polynomial. Firstly, we introduce the definitions of the RSFPLR circulant matrices and RSLPFL circulant matrices and properties of the related famous numbers. Then, we present the main results and the detailed process.

Definition 1. A row skew first-plus-last right (RSFPLR) circulant matrix with the first row $\left(a_{0}, a_{1}, \ldots, a_{n-1}\right)$, denoted by $\operatorname{RSFPLRcircfr}\left(a_{0}, a_{1}, \ldots, a_{n-1}\right)$, is a square matrix of the form

$$
\left[\begin{array}{ccccc}
a_{0} & a_{1} & \cdots & a_{n-2} & a_{n-1}  \tag{1}\\
-a_{n-1} & a_{0}+a_{n-1} & a_{1} & \cdots & a_{n-2} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-a_{2} & -a_{3}+a_{2} & \cdots & a_{0}+a_{n-1} & a_{1} \\
-a_{1} & -a_{2}+a_{1} & \cdots & -a_{n-1}+a_{n-2} & a_{0}+a_{n-1}
\end{array}\right]
$$

The matrix with an arbitrary first row and the following rule for obtaining any other row from the previous one can be seen. Get the $i+1$ st row by adding the last element of the $i$ th row to the first element of the $i$ th row and -1 times the last element of the $i$ th row and then shifting the elements of the $i$ th row (cyclically) one position to the right.

Note that the RSFPLR circulant matrix is a $x^{n}-x+1$ circulant matrix [14] and that is neither the extension of skew circulant matrix [12, 13] nor its special case and they are two different kinds of special matrices. Moreover, it is a FLS $r$ circulant matrix [15-17] with $r=-1$.

We define $\boldsymbol{\Theta}_{(-1,1)}$ as the basic RSFPLR circulant matrix; that is,

$$
\boldsymbol{\Theta}_{(-1,1)}=\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0  \tag{2}\\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-1 & 1 & 0 & \cdots & 0
\end{array}\right)_{n \times n}
$$

It is easily verified that $g(x)=x^{n}-x+1$ has no repeated roots in its splitting field and $g(x)=x^{n}-x+1$ is both the minimal polynomial and the characteristic polynomial of the matrix $\boldsymbol{\Theta}_{(-1,1)}$. In addition, $\boldsymbol{\Theta}_{(-1,1)}$ is nonderogatory and satisfies $\Theta_{(-1,1)}^{j}=\operatorname{RSFPLRcircfr}(\underbrace{0, \ldots, 0}_{j}, 1, \underbrace{0, \ldots, 0}_{n-j-1})$ and
$\boldsymbol{\Theta}_{(-1,1)}^{n}=\mathbf{I}_{n}-\boldsymbol{\Theta}_{(-1,1)}$. Then a matrix $\mathbf{A}$ can be written in the form

$$
\begin{equation*}
\mathbf{A}=f\left(\boldsymbol{\Theta}_{(-1,1)}\right)=\sum_{i=0}^{n-1} a_{i} \boldsymbol{\Theta}_{(-1,1)}^{i} \tag{3}
\end{equation*}
$$

if and only if $\mathbf{A}$ is a RSFPLR circulant matrix, where the polynomial $f(x)=\sum_{i=0}^{n-1} a_{i} x^{i}$ is called the representer of the RSFPLR circulant matrix A.

It is clear that $\mathbf{A}$ is a RSFPLR circulant matrix if and only if $\mathbf{A}$ commutes with the $\boldsymbol{\Theta}_{(-1,1)}$; that is,

$$
\begin{equation*}
\mathbf{A} \boldsymbol{\Theta}_{(-1,1)}=\boldsymbol{\Theta}_{(-1,1)} \mathbf{A} \tag{4}
\end{equation*}
$$

Definition 2. A row skew last-plus-first left (RSLPFL) circulant matrix with the first row $\left(a_{0}, a_{1}, \ldots, a_{n-1}\right)$, denoted by RSLPFLcircfr $\left(a_{0}, a_{1}, \ldots, a_{n-1}\right)$, is a square matrix of the form

$$
\left[\begin{array}{cccc}
a_{0} & a_{1} & \cdots & a_{n-1}  \tag{5}\\
a_{1} & \cdots & a_{n-1}+a_{0} & -a_{0} \\
\vdots & \vdots & . & \vdots \\
a_{n-2} & a_{n-1}+a_{0} & \cdots & -a_{n-3} \\
a_{n-1}+a_{0} & \cdots & -a_{n-3}+a_{n-2} & -a_{n-2}
\end{array}\right]
$$

The matrix with an arbitrary first row and the following rule for obtaining any other row from the previous onecan be seen. Get the $i+1$ st row by adding the first element of the $i$ th row to the last element of the $i$ th row and -1 times the first element of the $i$ th row and then shifting the elements of the $i$ th row (cyclically) one position to the left.

Let $\mathbf{A}=$ RSFPLRcircfr $\left(a_{n}, a_{n-1}, \ldots, a_{1}\right)$ and $\mathbf{B}=$ RSLPFLcircfr $\left(a_{1}, a_{2}, \ldots, a_{n}\right)$. By explicit computation, we find

$$
\begin{equation*}
\mathbf{A}=\mathbf{B} \widehat{\mathbf{I}}_{n} \tag{6}
\end{equation*}
$$

where $\widehat{\mathbf{I}}_{n}$ is the backward identity matrix of the form

$$
\widehat{\mathbf{I}}_{n}=\left(\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 1  \tag{7}\\
0 & \cdots & 0 & 1 & 0 \\
\vdots & . & . & . & \vdots \\
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0
\end{array}\right)
$$

The Perrin [34, 35] and Padovan [34, 36] numbers $\left\{R_{n}\right\}$ and $\left\{\mathbb{P}_{n}\right\}$ are defined by a third-order recurrence

$$
\begin{array}{ll}
R_{n}=R_{n-2}+R_{n-3}, & n \geq 3 \\
\mathbb{P}_{n}=\mathbb{P}_{n-2}+\mathbb{P}_{n-3}, & n \geq 3 \tag{9}
\end{array}
$$

with the initial conditions $R_{0}=3, R_{1}=0, R_{2}=2$, and $\mathbb{P}_{0}=1$, $\mathbb{P}_{1}=1$, and $\mathbb{P}_{2}=1$.

The Tribonacci $[36,37]$ and the generalized Lucas numbers $\left\{T_{n}\right\}$ and $\left\{\mathbb{L}_{n}\right\}$ [37] are defined by a third-order recurrence

$$
\begin{array}{ll}
T_{n}=T_{n-1}+T_{n-2}+T_{n-3}, & n \geq 3,  \tag{10}\\
\mathbb{L}_{n}=\mathbb{L}_{n-1}+\mathbb{L}_{n-2}+\mathbb{L}_{n-3}, & n \geq 3,
\end{array}
$$

with the initial conditions $T_{0}=0, T_{1}=1, T_{2}=1$ and $\mathbb{L}_{0}=3$, $\mathbb{Z}_{1}=1$, and $\mathbb{L}_{2}=3$.

Recurrences (8) and (9) involve the characteristic equation $x^{3}-x-1=0$, the roots of which are denoted by $r_{1}, r_{2}$, and $r_{3}$. Then

$$
\begin{gather*}
r_{1}+r_{2}+r_{3}=0 \\
r_{1} r_{2}+r_{1} r_{3}+r_{2} r_{3}=-1  \tag{11}\\
r_{1} r_{2} r_{3}=1 .
\end{gather*}
$$

Moreover, the Binet form for the Perrin [34, 35] number is

$$
\begin{equation*}
R_{n}=r_{1}^{n}+r_{2}^{n}+r_{3}^{n} \tag{12}
\end{equation*}
$$

and the Binet form for Padovan $[34,36]$ number is

$$
\begin{equation*}
\mathbb{P}_{n}=a_{1} r_{1}^{n}+a_{2} r_{2}^{n}+a_{3} r_{3}^{n} \tag{13}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{i}=\prod_{j=1, j \neq i}^{3} \frac{r_{j}-1}{r_{i}-r_{j}}, \quad i=1,2,3 . \tag{14}
\end{equation*}
$$

Recurrences (10) as well imply the characteristic equation $x^{3}-$ $x^{2}-x-1=0$, and their roots are denoted by $t_{1}, t_{2}$, and $t_{3}$. Then

$$
\begin{gather*}
t_{1}+t_{2}+t_{3}=1 \\
t_{1} t_{2}+t_{1} t_{3}+t_{2} t_{3}=-1  \tag{15}\\
t_{1} t_{2} t_{3}=1
\end{gather*}
$$

Furthermore, the Binet form for the Tribonacci [36, 37] number is

$$
\begin{equation*}
T_{n}=b_{1} t_{1}^{n}+b_{2} t_{2}^{n}+b_{3} t_{3}^{n} \tag{16}
\end{equation*}
$$

where $b_{i}$ is the $i$ th root of the polynomial $44 y^{3}-2 y-1$, and the Binet form for the generalized Lucas [37] number is

$$
\begin{equation*}
\mathbb{L}_{n}=t_{1}^{n}+t_{2}^{n}+t_{3}^{n} . \tag{17}
\end{equation*}
$$

If the first row of a RSFPLR circulant matrix is $\quad\left(R_{1}, R_{2}, \ldots, R_{n}\right),\left(\mathbb{P}_{1}, \mathbb{P}_{2}, \ldots, \mathbb{P}_{n}\right), \quad\left(T_{1}, T_{2}, \ldots, T_{n}\right)$, $\left(\mathbb{L}_{1}, \mathbb{L}_{2}, \ldots, \mathbb{L}_{n}\right)$, then the matrix is called Perrin, Padovan, Tribonacci, and generalized Lucas RSFPLR circulant matrix, respectively.

If the first row of a RSLPFL circulant matrix is $\quad\left(R_{1}, R_{2}, \ldots, R_{n}\right), \quad\left(\mathbb{P}_{1}, \mathbb{P}_{2}, \ldots, \mathbb{P}_{n}\right), \quad\left(T_{1}, T_{2}, \ldots, T_{n}\right)$, $\left(\mathbb{L}_{1}, \mathbb{L}_{2}, \ldots, \mathbb{L}_{n}\right)$, then the matrix is called Perrin, Padovan, Tribonacci, and generalized Lucas RSLPFL circulant matrix, respectively.

## 2. Main Results

By Theorem 1.1 in [15], we deduce the following lemma.
Lemma 3. Let $\mathbf{A}=$ RSFPLRcircfr $\left(a_{1}, a_{2}, \ldots, a_{n}\right)$. Then the eigenvalues of $\mathbf{A}$ are given by

$$
\begin{equation*}
\lambda_{i}=f\left(\kappa_{i}\right)=\sum_{j=1}^{n} a_{j} \kappa_{i}^{j-1}, \quad i=1,2, \ldots, n \tag{18}
\end{equation*}
$$

and the determinant of $A$ is given by

$$
\begin{equation*}
\operatorname{det} \mathbf{A}=\prod_{i=1}^{n} \lambda_{i}=\prod_{i=1}^{n} \sum_{j=1}^{n} a_{j} \kappa_{i}^{j-1} \tag{19}
\end{equation*}
$$

where $\kappa_{i}(i=1, \ldots, n)$ are the roots of the equation $x^{n}-x+1=$ 0 .

Lemma 4. Suppose that $\kappa_{i}(i=1,2, \ldots, n)$ are the roots of the equation $x^{n}-x+1=0$. If $a=0$, then

$$
\begin{align*}
\prod_{i=1}^{n} & \left(a \kappa_{i}^{3}+b \kappa_{i}^{2}+c \kappa_{i}+d\right) \\
= & \prod_{i=1}^{n}\left(b \kappa_{i}^{2}+c \kappa_{i}+d\right)  \tag{20}\\
= & d^{n}+b^{n-1}(b+c+d)-d\left(x_{1}^{n-1}+x_{2}^{n-1}\right) \\
& \quad+\left(x_{1}^{n}+x_{2}^{n}\right)
\end{align*}
$$

where $a, b, c \in \mathbb{R}$ and

$$
\begin{equation*}
x_{1}=\frac{-c+\sqrt{c^{2}-4 b d}}{2} ; \quad x_{2}=\frac{-c-\sqrt{c^{2}-4 b d}}{2} . \tag{21}
\end{equation*}
$$

If $a \neq 0$, then

$$
\begin{align*}
\prod_{i=1}^{n} & \left(a \kappa_{i}^{3}+b \kappa_{i}^{2}+c \kappa_{i}+d\right) \\
= & \frac{(-1)^{n-1} a^{n-1}}{2} d\left(2 X_{n-1}+X_{n-1}^{2}-2 X_{2(n-1)}\right)  \tag{22}\\
& +\frac{(-1)^{n} a^{n}}{2}\left(2 X_{n}+2 X_{n+1}+X_{n}^{2}-X_{2 n}+2\right) \\
& +(-1)^{n} a^{n-1}(b+b+c+d)+d^{n}
\end{align*}
$$

where $X_{n}=\alpha_{1}^{n}+\alpha_{2}^{n}+\alpha_{3}^{n}$ and $\alpha_{1}, \alpha_{2}$, and $\alpha_{3}$ are the roots of the equation $a \kappa_{i}^{3}+b \kappa_{i}^{2}+c \kappa_{i}+d=0$.

Proof. Since $\kappa_{i}(i=1,2, \ldots, n)$ are the roots of the characteristic polynomial of $\boldsymbol{\Theta}_{(-1,1)}, g(x)=x^{n}-x+1$ can be factored as

$$
\begin{equation*}
x^{n}-x+1=\prod_{i=1}^{n}\left(x-\kappa_{i}\right) \tag{23}
\end{equation*}
$$

If $a=0$, please see [22] for details of the proof.

If $a \neq 0$, then

$$
\begin{align*}
& \prod_{i=1}^{n}\left(a \kappa_{i}^{3}+b \kappa_{i}^{2}+c \kappa_{i}+d\right) \\
& \begin{aligned}
&= a^{n} \prod_{i=1}^{n}\left(\kappa_{i}^{3}+\frac{b}{a} \kappa_{i}^{2}+\frac{c}{a} \kappa_{i}+\frac{d}{a}\right) \\
&= a^{n} \prod_{i=1}^{n}\left(\kappa_{i}-\alpha_{1}\right)\left(\kappa_{i}-\alpha_{2}\right)\left(\kappa_{i}-\alpha_{3}\right) \\
&=(-a)^{n} \prod_{i=1}^{n}\left(\alpha_{1}-\kappa_{i}\right) \prod_{i=1}^{n}\left(\alpha_{2}-\kappa_{i}\right) \prod_{i=1}^{n}\left(\alpha_{3}-\kappa_{i}\right) \\
&=(-a)^{n}\left(\alpha_{1}^{n}-\alpha_{1}+1\right)\left(\alpha_{2}^{n}-\alpha_{2}+1\right)\left(\alpha_{3}^{n}-\alpha_{3}+1\right) \\
&=(-a)^{n}\left\{\left(\alpha_{1} \alpha_{2} \alpha_{3}\right)^{n}+\alpha_{1} \alpha_{2} \alpha_{3}\right. \\
& \quad \quad\left[\left(\alpha_{1} \alpha_{2}\right)^{n-1}+\left(\alpha_{1} \alpha_{3}\right)^{n-1}+\left(\alpha_{2} \alpha_{3}\right)^{n-1}\right. \\
&\left.\quad+\alpha_{1}^{n-1}+\alpha_{2}^{n-1}+\alpha_{3}^{n-1}\right] \\
&+\left[\left(\alpha_{1} \alpha_{2}\right)^{n}+\left(\alpha_{1} \alpha_{3}\right)^{n}+\left(\alpha_{2} \alpha_{3}\right)^{n}\right. \\
&\left.\quad+\alpha_{1} \alpha_{2}+\alpha_{1} \alpha_{3}+\alpha_{2} \alpha_{3}\right] \\
&-\alpha_{1} \alpha_{2} \alpha_{3}+1-\left[\alpha_{1}^{n}\left(\alpha_{2}+\alpha_{3}\right)+\alpha_{2}^{n}\left(\alpha_{1}+\alpha_{3}\right)\right. \\
&\left.\quad+\alpha_{3}^{n}\left(\alpha_{1}+\alpha_{2}\right)\right] \\
&\left.+\left[\alpha_{1}^{n}+\alpha_{2}^{n}+\alpha_{3}^{n}-\left(\alpha_{1}+\alpha_{2}+\alpha_{3}\right)\right]\right\}
\end{aligned}
\end{align*}
$$

Let $X_{n}=\alpha_{1}^{n}+\alpha_{2}^{n}+\alpha_{3}^{n}$. We obtain $\left(\alpha_{1} \alpha_{2}\right)^{n}+\left(\alpha_{1} \alpha_{3}\right)^{n}+$ $\left(\alpha_{2} \alpha_{3}\right)^{n}=\left(\alpha_{n}^{2}-X_{2 n}\right) / 2$ from $\left(\alpha_{1}^{n}+\alpha_{2}^{n}+\alpha_{3}^{n}\right)^{2}=\alpha_{1}^{2 n}+\alpha_{2}^{2 n}+$ $\alpha_{3}^{2 n}+2\left[\left(\alpha_{1} \alpha_{2}\right)^{n}+\left(\alpha_{1} \alpha_{3}\right)^{n}+\left(\alpha_{2} \alpha_{3}\right)^{n}\right]$. Taking the relation of roots and coefficients

$$
\begin{gather*}
\alpha_{1}+\alpha_{2}+\alpha_{3}=-\frac{b}{a}, \\
\alpha_{1} \alpha_{2}+\alpha_{1} \alpha_{3}+\alpha_{2} \alpha_{3}=\frac{c}{a},  \tag{25}\\
\alpha_{1} \alpha_{2} \alpha_{3}=-\frac{d}{a}
\end{gather*}
$$

into account, we deduce that

$$
\begin{align*}
\prod_{i=1}^{n} & \left(a \kappa_{i}^{3}+b \kappa_{i}^{2}+c \kappa_{i}+d\right) \\
= & \frac{(-1)^{n-1} a^{n-1}}{2} d\left(2 X_{n-1}+X_{n-1}^{2}-2 X_{2(n-1)}\right)  \tag{26}\\
& +\frac{(-1)^{n} a^{n}}{2}\left(2 X_{n}+2 X_{n+1}+X_{n}^{2}-X_{2 n}+2\right) \\
& +(-1)^{n} a^{n-1}(b+b+c+d)+d^{n}
\end{align*}
$$

where $X_{n}=\alpha_{1}^{n}+\alpha_{2}^{n}+\alpha_{3}^{n}$ and $\alpha_{1}, \alpha_{2}, \alpha_{3}$ are the roots of the equation $a \kappa_{i}^{3}+b \kappa_{i}^{2}+c \kappa_{i}+d=0$.
2.1. Determinants of the RSFPLR Circulant Matrices and RSLPFL Circulant Matrices Involving Perrin Numbers

Theorem 5. Let $\mathbf{C}=$ RSFPLRcircfr $\left(R_{1}, R_{2}, \ldots, R_{n}\right)$ be a Perrin RSFPLR circulant matrix. Then

$$
\begin{equation*}
\operatorname{det} \mathbf{C}=\frac{R_{n}^{n-1} \Delta_{1}+R_{n}^{n} \Delta_{2}+2 R_{n+1}^{n}}{\Delta_{3}} \tag{27}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{1}= & 2 R_{n+1} Z_{n-1}+R_{n+1} Z_{n-1}^{2}-2 R_{n+1} Z_{2(n-1)} \\
& -16+2 R_{n+2}, \\
\Delta_{2}= & 2 Z_{n}+2 Z_{n+1}+Z_{n}^{2}-Z_{2 n}-2, \\
\Delta_{3}= & Y_{n}^{2}+Y_{n-1}^{2}-Y_{2 n}-2 Y_{2(n-1)}+2 Y_{n+1}  \tag{28}\\
& +2 Y_{n}+2 Y_{n-1}+6, \\
Z_{n}= & \gamma_{1}^{n}+\gamma_{2}^{n}+\gamma_{3}^{n}, \\
Y_{n}= & \beta_{1}^{n}+\beta_{2}^{n}+\beta_{3}^{n} .
\end{align*}
$$

$\gamma_{1}, \gamma_{2}$, and $\gamma_{3}$ are the roots of the equation $-R_{n} x^{3}+\left(3-R_{n+2}+\right.$ $\left.R_{n}\right) x^{2}+\left(2-R_{n+1}+R_{n+2}\right) x+R_{n+1}=0$, and $\beta_{1}, \beta_{2}, \beta_{3}$ are the roots of the equation $y^{3}+y^{2}-1=0$.

Proof. Obviously, C has the form

$$
\mathbf{C}=\left(\begin{array}{cccc}
R_{1} & R_{2} & \cdots & R_{n}  \tag{29}\\
-R_{n} & R_{1}+R_{n} & \cdots & R_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
-R_{3} & -R_{4}+R_{3} & \cdots & R_{2} \\
-R_{2} & -R_{3}+R_{2} & \cdots & R_{1}+R_{n}
\end{array}\right) .
$$

By Lemma 3, the Binet form (12), and (11), we have

$$
\begin{align*}
\operatorname{det} \mathbf{C}= & \prod_{i=1}^{n}\left(R_{1}+R_{2} \kappa_{i}+\cdots+R_{n} \kappa_{i}^{n-1}\right) \\
= & \prod_{i=1}^{n}\left[\sum_{k=1}^{n} \sum_{j=1}^{3} r_{j}^{k} \kappa_{i}^{k-1}\right] \\
= & \prod_{i=1}^{n}\left[\sum_{j=1}^{3} \frac{r_{j}\left(1-r_{j}^{n} \kappa_{i}^{n}\right)}{1-r_{j} \kappa_{i}}\right]  \tag{30}\\
= & \prod_{i=1}^{n}\left[\frac{-R_{n} \kappa_{i}^{3}+\left(3-R_{n+2}+R_{n}\right) \kappa_{i}^{2}}{-\kappa_{i}^{3}-\kappa_{i}^{2}+1}\right. \\
& \left.\quad+\frac{\left(2-R_{n+1}+R_{n+2}\right) \kappa_{i}+R_{n+1}}{-\kappa_{i}^{3}-\kappa_{i}^{2}+1}\right] .
\end{align*}
$$

By Lemma 4 and the recurrence (8), we obtain

$$
\begin{align*}
\prod_{i=1}^{n}[ & -R_{n} \kappa_{i}^{3}+\left(3-R_{n+2}+R_{n}\right) \kappa_{i}^{2} \\
& \left.+\left(2-R_{n+1}+R_{n+2}\right) \kappa_{i}+R_{n+1}\right] \\
= & \frac{R_{n}^{n-1}}{2}\left(2 R_{n+1} Z_{n-1}+R_{n+1} Z_{n-1}^{2}-2 R_{n+1} Z_{2(n-1)}\right.  \tag{31}\\
& \left.\quad 16+2 R_{n+2}\right)+R_{n+1}^{n} \\
& +\frac{R_{n}^{n}}{2}\left(2 Z_{n}+2 Z_{n+1}+Z_{n}^{2}-Z_{2 n}-2\right)
\end{align*}
$$

where $Z_{n}=\gamma_{1}^{n}+\gamma_{2}^{n}+\gamma_{3}^{n}$ and $\gamma_{1}, \gamma_{2}, \gamma_{3}$ are the roots of the equation $-R_{n} x^{3}+\left(3-R_{n+2}+R_{n}\right) x^{2}+\left(2-R_{n+1}+R_{n+2}\right) x+$ $R_{n+1}=0$ and

$$
\begin{align*}
\prod_{i=1}^{n} & \left(-\kappa_{i}^{3}-\kappa_{i}^{2}+1\right) \\
= & \frac{1}{2}\left(Y_{n}^{2}+Y_{n-1}^{2}-Y_{2 n}\right)+Y_{n-1}+Y_{n}+Y_{n+1}  \tag{32}\\
& -Y_{2(n-1)}+3
\end{align*}
$$

where $Y_{n}=\beta_{1}^{n}+\beta_{2}^{n}+\beta_{3}^{n}$ and $\beta_{1}, \beta_{2}, \beta_{3}$ are the roots of the equation $y^{3}+y^{2}-1=0$. Consequently,

$$
\begin{equation*}
\operatorname{det} \mathbf{C}=\frac{R_{n}^{n-1} \Delta_{1}+R_{n}^{n} \Delta_{2}+2 R_{n+1}^{n}}{\Delta_{3}} \tag{33}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{1}= & 2 R_{n+1} Z_{n-1}+R_{n+1} Z_{n-1}^{2}-2 R_{n+1} Z_{2(n-1)} \\
& -16+2 R_{n+2} \\
\Delta_{2}= & 2 Z_{n}+2 Z_{n+1}+Z_{n}^{2}-Z_{2 n}-2  \tag{34}\\
\Delta_{3}= & Y_{n}^{2}+Y_{n-1}^{2}-Y_{2 n}-2 Y_{2(n-1)}+2 Y_{n+1} \\
& +2 Y_{n}+2 Y_{n-1}+6
\end{align*}
$$

Theorem 6. Let $\mathbf{D}=\operatorname{RSFPLRcircfr}\left(R_{n}, R_{n-1}, \ldots, R_{1}\right)$ be a Perrin RSFPLR circulant matrix. Then

$$
\begin{align*}
\operatorname{det} \mathbf{D}=2[ & \frac{\left(R_{n}+3\right)^{n}+R_{n+5}\left(R_{n+1}-2\right)^{n-1}}{R_{n-1}^{2}+R_{n}^{2}-R_{2 n}-2 R_{2(n-1)}+2 R_{n+4}} \\
& \left.+\frac{-\left(R_{n}+3\right)\left(\rho_{1}^{n-1}+\rho_{2}^{n-1}\right)+\left(\rho_{1}^{n}+\rho_{2}^{n}\right)}{R_{n-1}^{2}+R_{n}^{2}-R_{2 n}-2 R_{2(n-1)}+2 R_{n+4}}\right], \tag{35}
\end{align*}
$$

where

$$
\begin{align*}
& \rho_{1}=\frac{1-R_{n+2}+\Delta_{4}}{2} \\
& \rho_{2}=\frac{1-R_{n+2}-\Delta_{4}}{2}  \tag{36}\\
& \Delta_{4}=\sqrt{\left(1-R_{n+2}\right)^{2}-4\left(R_{n+1}-2\right)\left(R_{n}+3\right)}
\end{align*}
$$

Proof. The matrix $\mathbf{D}$ has the form

$$
\mathbf{D}=\left(\begin{array}{cccc}
R_{n} & R_{n-1} & \cdots & R_{1}  \tag{37}\\
-R_{1} & R_{1}+R_{n} & \cdots & R_{2} \\
\vdots & \vdots & \ddots & \vdots \\
-R_{n-2} & -R_{n-3}+R_{n-2} & \cdots & R_{n-1} \\
-R_{n-1} & -R_{n-2}+R_{n-1} & \cdots & R_{1}+R_{n}
\end{array}\right)
$$

According to Lemma 3, the Binet form (12), and (11), we have

$$
\begin{align*}
\operatorname{det} \mathbf{D} & =\prod_{i=1}^{n}\left(R_{n}+R_{n-1} \kappa_{i}+\cdots+R_{1} \kappa_{i}^{n-1}\right) \\
& =\prod_{i=1}^{n}\left[\sum_{k=0}^{n-1} \sum_{j=1}^{3} r_{j}^{n-k} \kappa_{i}^{k}\right] \\
& =\prod_{i=1}^{n}\left[\sum_{j=1}^{3} \frac{r_{j}^{n+1}-r_{j} \kappa_{i}^{n}}{r_{j}-\kappa_{i}}\right]  \tag{38}\\
& =\prod_{i=1}^{n} \frac{\left(R_{n+1}-2\right) \kappa_{i}^{2}+\left(R_{n+2}-1\right) \kappa_{i}+R_{n}+3}{-\kappa_{i}^{3}+\kappa_{i}+1} .
\end{align*}
$$

Using Lemma 4 and the recurrence (8), we obtain

$$
\begin{align*}
\prod_{i=1}^{n} & {\left[\left(R_{n+1}-2\right) \kappa_{i}^{2}+\left(R_{n+2}-1\right) \kappa_{i}+R_{n}+3\right] } \\
& =\left(R_{n}+3\right)^{n}+R_{n+5}\left(R_{n+1}-2\right)^{n-1}  \tag{39}\\
& -\left(R_{n}+3\right)\left(\rho_{1}^{n-1}+\rho_{2}^{n-1}\right)+\left(\rho_{1}^{n}+\rho_{2}^{n}\right)
\end{align*}
$$

where

$$
\begin{align*}
& \rho_{1}=\frac{1-R_{n+2}+\Delta_{4}}{2} \\
& \rho_{2}=\frac{1-R_{n+2}-\Delta_{4}}{2}  \tag{40}\\
& \Delta_{4}=\sqrt{\left(1-R_{n+2}\right)^{2}-4\left(R_{n+1}-2\right)\left(R_{n}+3\right)}, \\
& \prod_{i=1}^{n}\left(-\kappa_{i}^{3}+\kappa_{i}+1\right)  \tag{41}\\
&=\frac{1}{2}\left(R_{n-1}^{2}+R_{n}^{2}-R_{2 n}\right)-R_{2(n-1)}+R_{n+4}
\end{align*}
$$

Therefore,

$$
\begin{align*}
\operatorname{det} \mathbf{D}=2[ & \frac{\left(R_{n}+3\right)^{n}+R_{n+5}\left(R_{n+1}-2\right)^{n-1}}{R_{n-1}^{2}+R_{n}^{2}-R_{2 n}-2 R_{2(n-1)}+2 R_{n+4}} \\
& \left.+\frac{-\left(R_{n}+3\right)\left(\rho_{1}^{n-1}+\rho_{2}^{n-1}\right)+\left(\rho_{1}^{n}+\rho_{2}^{n}\right)}{R_{n-1}^{2}+R_{n}^{2}-R_{2 n}-2 R_{2(n-1)}+2 R_{n+4}}\right] \tag{42}
\end{align*}
$$

Theorem 7. Let $\mathbf{E}=\operatorname{RSLPFLcircfr}\left(R_{1}, R_{2}, \ldots, R_{n}\right)$ be a Perrin RSLPFL circulant matrix. Then

$$
\begin{align*}
\operatorname{det} \mathbf{E}= & 2\left[\frac{\left(R_{n}+3\right)^{n}+R_{n+5}\left(R_{n+1}-2\right)^{n-1}}{R_{n-1}^{2}+R_{n}^{2}-R_{2 n}-2 R_{2(n-1)}+2 R_{n+4}}\right. \\
& \left.\quad \frac{-\left(R_{n}+3\right)\left(\rho_{1}^{n-1}+\rho_{2}^{n-1}\right)+\left(\rho_{1}^{n}+\rho_{2}^{n}\right)}{2 R_{n-1}^{2}+R_{n}^{2}-R_{2 n}-2 R_{2(n-1)}+2 R_{n+4}}\right]  \tag{43}\\
& \times(-1)^{n(n-1) / 2},
\end{align*}
$$

where

$$
\begin{align*}
& \rho_{1}=\frac{1-R_{n+2}+\Delta_{4}}{2}, \\
& \rho_{2}=\frac{1-R_{n+2}-\Delta_{4}}{2},  \tag{44}\\
& \Delta_{4}=\sqrt{\left(1-R_{n+2}\right)^{2}-4\left(R_{n+1}-2\right)\left(R_{n}+3\right)} .
\end{align*}
$$

Proof. Since

$$
\begin{equation*}
\operatorname{det} \widehat{\mathbf{I}}_{n}=(-1)^{n(n-1) / 2} \tag{45}
\end{equation*}
$$

the result can be derived from Theorem 6 and (6).
2.2. Determinants of the RSFPLR Circulant Matrices and RSLPFL Circulant Matrices Involving Padovan Numbers

Theorem 8. Let $\mathbf{F}=\operatorname{RSFPLRcircfr}\left(\mathbb{P}_{1}, \mathbb{P}_{2}, \ldots, \mathbb{P}_{n}\right)$ be a Padovan RSFPLR circulant matrix. Then

$$
\begin{equation*}
\operatorname{det} \mathbf{F}=\frac{\mathbb{P}_{n}^{n} \Delta_{5}+\left(1+\mathbb{P}_{n+1}\right) \Delta_{6}+\Delta_{7}}{\Delta_{8}} \tag{46}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{5}= & U_{n}^{2}-U_{2 n}+2 U_{n+1}+2 U_{n} \\
\Delta_{6}= & \mathbb{P}_{n}^{n-1}\left(U_{n-1}^{2}-2 U_{2(n-1)}+2 U_{n-1}\right) \\
\Delta_{7}= & 2 \mathbb{P}_{n}^{n-1}\left(\mathbb{P}_{n+2}-\mathbb{P}_{n}-4\right)+2\left(1+\mathbb{P}_{n+1}\right)^{n}, \\
\Delta_{8}= & Y_{n}^{2}+Y_{n-1}^{2}-Y_{2 n}-2 Y_{2(n-1)}+2 Y_{n+1}+2 Y_{n}  \tag{47}\\
& +2 Y_{n-1}+6 \\
U_{n}= & \delta_{1}^{n}+\delta_{2}^{n}+\delta_{3}^{n} \\
Y_{n}= & \beta_{1}^{n}+\beta_{2}^{n}+\beta_{3}^{n} .
\end{align*}
$$

$\delta_{1}, \delta_{2}, \delta_{3}$ are the roots of the equation $-\mathbb{P}_{n} x^{3}+\left(1+\mathbb{P}_{n}-\mathbb{P}_{n+2}\right) x^{2}$ $+\left(1-\mathbb{P}_{n+1}+\mathbb{P}_{n+2}\right) x+1+\mathbb{P}_{n+1}=0$ and $\beta_{1}, \beta_{2}, \beta_{3}$ are the roots of the equation $y^{3}+y^{2}-1=0$.

Proof. The matrix $\mathbf{F}$ has the form

$$
\begin{align*}
& \mathbf{F}=\left(\begin{array}{cccc}
\mathbb{P}_{1} & \mathbb{P}_{2} & \cdots & \mathbb{P}_{n} \\
-\mathbb{P}_{n} & \mathbb{P}_{1}+\mathbb{P}_{n} & \cdots & \mathbb{P}_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
-\mathbb{P}_{3} & -\mathbb{P}_{4}+\mathbb{P}_{3} & \cdots & \mathbb{P}_{2} \\
-\mathbb{P}_{2} & -\mathbb{P}_{3}+\mathbb{P}_{2} & \cdots & \mathbb{P}_{1}+\mathbb{P}_{n}
\end{array}\right), \\
& \operatorname{det} \mathbf{F}=\prod_{i=1}^{n}\left(\mathbb{P}_{1}+\mathbb{P}_{2} \kappa_{i}+\cdots+\mathbb{P}_{n} \kappa_{i}^{n-1}\right) \\
&=\prod_{i=1}^{n}\left[\sum_{k=1}^{n} \sum_{j=1}^{3} a_{j} r_{j}^{k} \kappa_{i}^{k-1}\right]  \tag{48}\\
&=\prod_{i=1}^{n}\left[\sum_{j=1}^{3} \frac{a_{j} r_{j}\left(1-r_{j}^{n} \kappa_{i}^{n}\right)}{1-r_{j} \kappa_{i}}\right] \\
&=\prod_{i=1}^{n}\left[\frac{-\mathbb{P}_{n} \kappa_{i}^{3}+\left(1+\mathbb{P}_{n}-\mathbb{P}_{n+2}\right) \kappa_{i}^{2}}{-\kappa_{i}^{3}-\kappa_{i}^{2}+1}\right. \\
&\left.+\frac{\left(1-\mathbb{P}_{n+1}+\mathbb{P}_{n+2}\right) \kappa_{i}+1+\mathbb{P}_{n+1}}{-\kappa_{i}^{3}-\kappa_{i}^{2}+1}\right],
\end{align*}
$$

from Lemma 3, the Binet form (13), and (11).
Using Lemma 4 and the recurrence (9), we obtain

$$
\begin{align*}
\prod_{i=1}^{n} & {\left[-\mathbb{P}_{n} \kappa_{i}^{3}+\left(1+\mathbb{P}_{n}-\mathbb{P}_{n+2}\right) \kappa_{i}^{2}\right.} \\
& \left.+\left(1-\mathbb{P}_{n+1}+\mathbb{P}_{n+2}\right) \kappa_{i}+1+\mathbb{P}_{n+1}\right] \\
= & \frac{\mathbb{P}_{n}^{n}}{2}\left(U_{n}^{2}-U_{2 n}+2 U_{n+1}+2 U_{n}\right)  \tag{49}\\
& +\frac{\left(1+\mathbb{P}_{n+1}\right) \mathbb{P}_{n}^{n-1}}{2}\left(U_{n-1}^{2}-2 U_{2(n-1)}+2 U_{n-1}\right) \\
& +\left(1+\mathbb{P}_{n+1}\right)^{n}+\mathbb{P}_{n}^{n-1}\left(\mathbb{P}_{n+2}-\mathbb{P}_{n}-4\right),
\end{align*}
$$

where $U_{n}=\delta_{1}^{n}+\delta_{2}^{n}+\delta_{3}^{n}, \delta_{1}, \delta_{2}, \delta_{3}$ are the roots of the equation $-\mathbb{P}_{n} x^{3}+\left(1+\mathbb{P}_{n}-\mathbb{P}_{n+2}\right) x^{2}+\left(1-\mathbb{P}_{n+1}+\mathbb{P}_{n+2}\right) x+1+\mathbb{P}_{n+1}=0$.
According to (32), we have the following results:

$$
\begin{equation*}
\operatorname{det} \mathbf{F}=\frac{\mathbb{P}_{n}^{n} \Delta_{5}+\left(1+\mathbb{P}_{n+1}\right) \Delta_{6}+\Delta_{7}}{\Delta_{8}} \tag{50}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{5}= & U_{n}^{2}-U_{2 n}+2 U_{n+1}+2 U_{n} \\
\Delta_{6}= & \mathbb{P}_{n}^{n-1}\left(U_{n-1}^{2}-2 U_{2(n-1)}+2 U_{n-1}\right) \\
\Delta_{7}= & 2 \mathbb{P}_{n}^{n-1}\left(\mathbb{P}_{n+2}-\mathbb{P}_{n}-4\right)+2\left(1+\mathbb{P}_{n+1}\right)^{n}  \tag{51}\\
\Delta_{8}= & Y_{n}^{2}+Y_{n-1}^{2}-Y_{2 n}-2 Y_{2(n-1)}+2 Y_{n+1}+2 Y_{n} \\
& +2 Y_{n-1}+6
\end{align*}
$$

Theorem 9. Let $\mathbf{G}=$ RSFPLRcircfr $\left(\mathbb{P}_{n}, \mathbb{P}_{n-1}, \ldots, \mathbb{P}_{1}\right)$ be a Padovan RSFPLR circulant matrix. Then

$$
\begin{equation*}
\operatorname{det} \mathbf{G}=\frac{2\left(\mathbb{P}_{n}+1\right)^{n}+\left(\mathbb{P}_{n}+1\right) \Delta_{9}+\Delta_{10}}{2\left(R_{n+4}-R_{2(n-1)}\right)+R_{n-1}^{2}+R_{n}^{2}-R_{2 n}} \tag{52}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{9}= & V_{n-1}^{2}-2 V_{2(n-1)}+2 V_{n-1} \\
\Delta_{10}= & \left(2 V_{n+1}+V_{n}^{2}+2 V_{n}-V_{2 n}\right)  \tag{53}\\
& -2\left(1+\mathbb{P}_{n+1}+\mathbb{P}_{n+5}\right) \\
V_{n}= & \zeta_{1}^{n}+\zeta_{2}^{n}+\zeta_{3}^{n}
\end{align*}
$$

and $\zeta_{1}, \zeta_{2}, \zeta_{3}$ are the roots of the equation $-x^{3}+\mathbb{P}_{n+1} x^{2}+\left(\mathbb{P}_{n+2}+\right.$ 1) $x+\mathbb{P}_{n}+1=0$.

Proof. The matrix G has the form

$$
\mathbf{G}=\left(\begin{array}{cccc}
\mathbb{P}_{n} & \mathbb{P}_{n-1} & \cdots & \mathbb{P}_{1}  \tag{54}\\
-\mathbb{P}_{1} & \mathbb{P}_{1}+\mathbb{P}_{n} & \cdots & \mathbb{P}_{2} \\
\vdots & \vdots & \ddots & \vdots \\
-\mathbb{P}_{n-2} & -\mathbb{P}_{n-3}+\mathbb{P}_{n-2} & \cdots & \mathbb{P}_{n-1} \\
-\mathbb{P}_{n-1} & -\mathbb{P}_{n-2}+\mathbb{P}_{n-1} & \cdots & \mathbb{P}_{1}+\mathbb{P}_{n}
\end{array}\right) .
$$

According to Lemma 3, the Binet form (13), and (11), we have

$$
\begin{align*}
\operatorname{det} \mathbf{G} & =\prod_{i=1}^{n}\left(\mathbb{P}_{n}+\mathbb{P}_{n-1} \kappa_{i}+\cdots+\mathbb{P}_{1} \kappa_{i}^{n-1}\right) \\
& =\prod_{i=1}^{n}\left[\sum_{k=0}^{n-1} \sum_{j=1}^{3} a_{j} r_{j}^{n-k} \kappa_{i}^{k}\right]  \tag{55}\\
& =\prod_{i=1}^{n}\left[\sum_{j=1}^{3} \frac{a_{j} r_{j}^{n+1}-a_{j} r_{j} \kappa_{i}^{n}}{r_{j}-\kappa_{i}}\right] \\
& =\prod_{i=1}^{n} \frac{-\kappa_{i}^{3}+\mathbb{P}_{n+1} \kappa_{i}^{2}+\left(\mathbb{P}_{n+2}+1\right) \kappa_{i}+\mathbb{P}_{n}+1}{-\kappa_{i}^{3}+\kappa_{i}+1} .
\end{align*}
$$

Using Lemma 4 and (13), we obtain

$$
\left.\left.\begin{array}{rl}
\prod_{i=1}^{n} & {[ }
\end{array}-\kappa_{i}^{3}+\mathbb{P}_{n+1} \kappa_{i}^{2}+\left(\mathbb{P}_{n+2}+1\right) \kappa_{i}+\mathbb{P}_{n}+1\right]\right) \text { = } \begin{aligned}
& \left.\mathbb{P}_{n}+1\right)^{n}+\frac{\left(\mathbb{P}_{n}+1\right)}{2}\left(V_{n-1}^{2}-2 V_{2(n-1)}+2 V_{n-1}\right) \\
& +\frac{1}{2}\left(2 V_{n}+2 V_{n+1}+V_{n}^{2}-V_{2 n}\right)-\left(1+\mathbb{P}_{n+1}+\mathbb{P}_{n+5}\right),
\end{aligned}
$$

where $V_{n}=\zeta_{1}^{n}+\zeta_{2}^{n}+\zeta_{3}^{n}, \zeta_{1}, \zeta_{2}, \zeta_{3}$ are the roots of the equation $-x^{3}+\mathbb{P}_{n+1} x^{2}+\left(\mathbb{P}_{n+2}+1\right) x+\mathbb{P}_{n}+1=0$. Employing (41), we have the following results:

$$
\begin{equation*}
\operatorname{det} \mathbf{G}=\frac{2\left(\mathbb{P}_{n}+1\right)^{n}+\left(\mathbb{P}_{n}+1\right) \Delta_{9}+\Delta_{10}}{2\left(R_{n+4}-R_{2(n-1)}\right)+R_{n-1}^{2}+R_{n}^{2}-R_{2 n}} \tag{57}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{9}= & V_{n-1}^{2}-2 V_{2(n-1)}+2 V_{n-1} \\
\Delta_{10}= & \left(2 V_{n+1}+V_{n}^{2}+2 V_{n}-V_{2 n}\right)  \tag{58}\\
& -2\left(1+\mathbb{P}_{n+1}+\mathbb{P}_{n+5}\right), \\
V_{n}= & \zeta_{1}^{n}+\zeta_{2}^{n}+\zeta_{3}^{n}
\end{align*}
$$

and $\zeta_{1}, \zeta_{2}, \zeta_{3}$ are the roots of the equation $-x^{3}+\mathbb{P}_{n+1} x^{2}+$ $\left(\mathbb{P}_{n+2}+1\right) x+\mathbb{P}_{n}+1=0$.

Theorem 10. Let $\mathbf{H}=$ RSLPFLcircfr $\left(\mathbb{P}_{1}, \mathbb{P}_{2}, \ldots, \mathbb{P}_{n}\right)$ be a Padovan RSLPFL circulant matrix. Then

$$
\begin{align*}
\operatorname{det} \mathbf{H}= & \frac{2\left(\mathbb{P}_{n}+1\right)^{n}+\left(\mathbb{P}_{n}+1\right) \Delta_{9}+\Delta_{10}}{2\left(R_{n+4}-R_{2(n-1)}\right)+R_{n-1}^{2}+R_{n}^{2}-R_{2 n}}  \tag{59}\\
& \times(-1)^{n(n-1) / 2}
\end{align*}
$$

where

$$
\begin{align*}
\Delta_{9}= & V_{n-1}^{2}-2 V_{2(n-1)}+2 V_{n-1} \\
\Delta_{10}= & \left(2 V_{n+1}+V_{n}^{2}+2 V_{n}-V_{2 n}\right)  \tag{60}\\
& -2\left(1+\mathbb{P}_{n+1}+\mathbb{P}_{n+5}\right) \\
V_{n}= & \zeta_{1}^{n}+\zeta_{2}^{n}+\zeta_{3}^{n}
\end{align*}
$$

and $\zeta_{1}, \zeta_{2}, \zeta_{3}$ are the roots of the equation $-x^{3}+\mathbb{P}_{n+1} x^{2}+\left(\mathbb{P}_{n+2}+\right.$ 1) $x+\mathbb{P}_{n}+1=0$.

Proof. The theorem can be proved by using Theorem 9 and (6).

### 2.3. Determinants of RSFPLR Circulant Matrices and RSLPFL Circulant Matrices Involving Tribonacci Numbers

Theorem 11. Let $\mathbf{J}=\operatorname{RSFPLRcircfr}\left(T_{1}, T_{2}, \ldots, T_{n}\right)$ be a Tribonacci RSFPLR circulant matrix. Then

$$
\begin{equation*}
\operatorname{det} \mathbf{J}=\frac{T_{n}^{n} \Delta_{11}+2 T_{n}^{n-1}\left(T_{n+2}-T_{n}-2\right)+\Delta_{12}}{\Delta_{13}} \tag{61}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{11}= & W_{n}^{2}-W_{2 n}+2 W_{n+1}+2 W_{n} \\
\Delta_{12}= & T_{n}^{n-1}\left(1+T_{n+1}\right)\left(W_{n-1}^{2}-2 W_{2(n-1)}+2 W_{n-1}\right) \\
& +2\left(1+T_{n+1}\right)^{n} \\
\Delta_{13}= & 2 E_{n-1}+E_{n-1}^{2}-2 E_{2(n-1)}+E_{n}^{2}-E_{2 n}  \tag{62}\\
& +2 E_{n+1}+2 E_{n}+8 \\
W_{n}= & \eta_{1}^{n}+\eta_{2}^{n}+\eta_{3}^{n} \\
E_{n}= & \xi_{1}^{n}+\xi_{2}^{n}+\xi_{3}^{n}
\end{align*}
$$

and $\eta_{1}, \eta_{2}, \eta_{3}$ are the roots of the equation $-T_{n} x^{3}-\left(T_{n+2}-T_{n}\right) x^{2}$ $+\left(1-T_{n+1}+T_{n+2}\right) x+1+T_{n+1}=0$ and $\xi_{1}, \xi_{2}, \xi_{3}$ are the roots of the equation $y^{3}+y^{2}+y-1=0$.

Proof. Obviously, $\mathbf{J}$ has the form

$$
\mathbf{J}=\left(\begin{array}{cccc}
T_{1} & T_{2} & \cdots & T_{n}  \tag{63}\\
-T_{n} & T_{1}+T_{n} & \cdots & T_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
-T_{3} & -T_{4}+T_{3} & \cdots & T_{2} \\
-T_{2} & -T_{3}+T_{2} & \cdots & T_{1}+T_{n}
\end{array}\right) .
$$

According to Lemma 3, the Binet form (16), and (15), we have

$$
\begin{align*}
\operatorname{det} \mathbf{J}= & \prod_{i=1}^{n}\left(T_{1}+T_{2} \kappa_{i}+\cdots+T_{n} \kappa_{i}^{n-1}\right) \\
= & \prod_{i=1}^{n}\left[\sum_{k=1}^{n} \sum_{j=1}^{3} b_{j} t_{j}^{k} \kappa_{i}^{k-1}\right] \\
= & \prod_{i=1}^{n}\left[\sum_{i=1}^{3} \frac{b_{j} t_{j}\left(1-t_{j}^{n} \kappa_{i}^{n}\right)}{1-t_{j} \kappa_{i}}\right]  \tag{64}\\
= & \prod_{i=1}^{n}\left[\frac{-T_{n} \kappa_{i}^{3}-\left(-T_{n}+T_{n+2}\right) \kappa_{i}^{2}}{-\kappa_{i}^{3}-\kappa_{i}^{2}-\kappa_{i}+1}\right. \\
& \left.\quad+\frac{\left(1-T_{n+1}+T_{n+2}\right) \kappa_{i}+1+T_{n+1}}{-\kappa_{i}^{3}-\kappa_{i}^{2}-\kappa_{i}+1}\right]
\end{align*}
$$

From Lemma 4 it follows that

$$
\begin{align*}
\prod_{i=1}^{n}[ & -T_{n} \kappa_{i}^{3}-\left(-T_{n}+T_{n+2}\right) \kappa_{i}^{2} \\
& \left.+\left(1-T_{n+1}+T_{n+2}\right) \kappa_{i}+1+T_{n+1}\right] \\
= & \frac{1}{2} T_{n}^{n}\left(W_{n}^{2}-W_{2 n}+2 W_{n+1}+2 W_{n}\right)  \tag{65}\\
& +T_{n}^{n-1}\left(T_{n+2}-T_{n}-2\right)+\left(1+T_{n+1}\right)^{n} \\
& +\frac{1}{2} T_{n}^{n-1}\left(1+T_{n+1}\right)\left(W_{n-1}^{2}-2 W_{2(n-1)}+2 W_{n-1}\right)
\end{align*}
$$

where $W_{n}=\eta_{1}^{n}+\eta_{2}^{n}+\eta_{3}^{n}, \eta_{1}, \eta_{2}, \eta_{3}$ are the roots of the equation $-T_{n} x^{3}-\left(-T_{n}+T_{n+2}\right) x^{2}+\left(1-T_{n+1}+T_{n+2}\right) x+1+T_{n+1}=0$ and

$$
\begin{align*}
& \prod_{i=1}^{n}\left(-\kappa_{i}^{3}-\kappa_{i}^{2}-\kappa_{i}+1\right) \\
&=\frac{1}{2} {\left[2 E_{n-1}+E_{n-1}^{2}-2 E_{2(n-1)}\right.}  \tag{66}\\
&\left.+\left(E_{n}^{2}-E_{2 n}+2 E_{n+1}+2 E_{n}+2\right)\right]+3
\end{align*}
$$

where $E_{n}=\xi_{1}^{n}+\xi_{2}^{n}+\xi_{3}^{n}, \xi_{1}, \xi_{2}, \xi_{3}$ are the roots of the equation $y^{3}+y^{2}+y-1=0$. Consequently, we have the following results:

$$
\begin{equation*}
\operatorname{det} \mathbf{J}=\frac{T_{n}^{n} \Delta_{11}+2 T_{n}^{n-1}\left(T_{n+2}-T_{n}-2\right)+\Delta_{12}}{\Delta_{13}}, \tag{67}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{11}= & W_{n}^{2}-W_{2 n}+2 W_{n+1}+2 W_{n} \\
\Delta_{12}= & T_{n}^{n-1}\left(1+T_{n+1}\right)\left(W_{n-1}^{2}-2 W_{2(n-1)}+2 W_{n-1}\right) \\
& +2\left(1+T_{n+1}\right)^{n}  \tag{68}\\
\Delta_{13}= & 2 E_{n-1}+E_{n-1}^{2}-2 E_{2(n-1)}+E_{n}^{2}-E_{2 n} \\
& +2 E_{n+1}+2 E_{n}+8 .
\end{align*}
$$

Theorem 12. Let $\mathbf{K}=\operatorname{RSFPLRcircfr}\left(T_{n}, T_{n-1}, \ldots, T_{1}\right)$ be a Tribonacci RSFPLR circulant matrix. Then

$$
\begin{equation*}
\operatorname{det} \mathbf{K}=\frac{2 T_{n}^{n}+T_{n} \Delta_{14}+\Delta_{15}}{\mathbb{L}_{n}^{2}+\mathbb{\mathbb { L }}_{n-1}^{2}+\mathbb{\mathbb { L }}_{2 n}-2 \mathbb{L}_{2(n-1)}+2 \mathbb{L}_{n+2}-4}, \tag{69}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{14}= & G_{n-1}^{2}-2 G_{2(n-1)}+2 G_{n-1}-2, \\
\Delta_{15}= & -4 T_{n+1}-T_{n+2}+2 G_{n}-2 G_{n+1}  \tag{70}\\
& +G_{n}^{2}-G_{2 n}, \\
G_{n}= & v_{1}^{n}+v_{2}^{n}+v_{3}^{n}
\end{align*}
$$

and $v_{1}, v_{2}, v_{3}$ are the roots of the equation $-x^{3}+T_{n+1} x^{2}+\left(T_{n+2}+\right.$ 1) $x+T_{n}=0$.

Proof. The matrix $\mathbf{K}$ has the form

$$
\mathbf{K}=\left(\begin{array}{cccc}
T_{n} & T_{n-1} & \cdots & T_{1}  \tag{71}\\
-T_{1} & T_{1}+T_{n} & \cdots & T_{2} \\
\vdots & \vdots & \ddots & \vdots \\
-T_{n-2} & -T_{n-3}+T_{n-2} & \cdots & T_{n-1} \\
-T_{n-1} & -T_{n-2}+T_{n-1} & \cdots & T_{1}+T_{n}
\end{array}\right) .
$$

According to Lemma 3, the Binet form (16), and (15), we have

$$
\begin{align*}
\operatorname{det} \mathbf{K} & =\prod_{i=1}^{n}\left(T_{n}+T_{n-1} \kappa_{i}+\cdots+T_{1} \kappa_{i}^{n-1}\right) \\
& =\prod_{i=1}^{n}\left[\sum_{k=0}^{n-1} \sum_{j=1}^{3} b_{j} t_{j}^{n-k} \kappa_{i}^{k}\right]  \tag{72}\\
& =\prod_{i=1}^{n}\left[\sum_{j=1}^{3} \frac{b_{j} t_{j}^{n+1}-b_{j} t_{j} \kappa_{i}^{n}}{t_{j}-\kappa_{i}}\right] \\
& =\prod_{i=1}^{n} \frac{-\kappa_{i}^{3}+T_{n+1} \kappa_{i}^{2}+\left(T_{n+2}+1\right) \kappa_{i}+T_{n}}{-\kappa_{i}^{3}+\kappa_{i}^{2}+\kappa_{i}+1} .
\end{align*}
$$

Considering Lemma 4 and (17), we obtain

$$
\begin{align*}
\prod_{i=1}^{n} & {\left[-\kappa_{i}^{3}+T_{n+1} \kappa_{i}^{2}+\left(T_{n+2}+1\right) \kappa_{i}+T_{n}\right] } \\
= & T_{n}^{n}+\frac{T_{n}}{2}\left(G_{n-1}^{2}-2 G_{2(n-1)}+2 G_{n-1}-2\right)  \tag{73}\\
& +\frac{1}{2}\left(-4 T_{n+1}-T_{n+2}+2 G_{n}-2 G_{n+1}+G_{n}^{2}-G_{2 n}\right)
\end{align*}
$$

where $G_{n}=v_{1}^{n}+v_{2}^{n}+v_{3}^{n}, v_{1}, v_{2}, v_{3}$ are the roots of the equation $-x^{3}+T_{n+1} x^{2}+\left(T_{n+2}+1\right) x+T_{n}=0$ and

$$
\begin{align*}
\prod_{i=1}^{n}( & \left.-\kappa_{i}^{3}+\kappa_{i}^{2}+\kappa_{i}+1\right) \\
= & \frac{1}{2}  \tag{74}\\
{[ } & \left(2 \mathbb{L}_{n}+2 \mathbb{L}_{n+1}+\mathbb{C}_{n}^{2}+\mathbb{L}_{2 n}-4\right) \\
& \left.+\mathbb{L}_{n-1}^{2}-2 \mathbb{L}_{2(n-1)}+2 \mathbb{L}_{n-1}\right] \\
= & \frac{1}{2}\left[\mathbb{L}_{n}^{2}+\mathbb{L}_{2 n}+\mathbb{Q}_{n-1}^{2}-2 \mathbb{L}_{2(n-1)}+2 \mathbb{L}_{n+2}-4\right] .
\end{align*}
$$

Consequently,

$$
\begin{equation*}
\operatorname{det} \mathbf{K}=\frac{2 T_{n}^{n}+T_{n} \Delta_{14}+\Delta_{15}}{\mathbb{L}_{n}^{2}+\mathbb{\mathbb { L }}_{n-1}^{2}+\mathbb{\mathbb { L }}_{2 n}-2 \mathbb{L}_{2(n-1)}+2 \mathbb{L}_{n+2}-4} \tag{75}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{14}= & G_{n-1}^{2}-2 G_{2(n-1)}+2 G_{n-1}-2 \\
\Delta_{15}= & -4 T_{n+1}-T_{n+2}+2 G_{n}-2 G_{n+1} \\
& +G_{n}^{2}-G_{2 n}  \tag{76}\\
G_{n}= & v_{1}^{n}+v_{2}^{n}+v_{3}^{n}
\end{align*}
$$

and $v_{1}, v_{2}, v_{3}$ are the roots of the equation $-x^{3}+T_{n+1} x^{2}+$ $\left(T_{n+2}+1\right) x+T_{n}=0$.

Theorem 13. Let $\mathbf{L}=$ RSLPFLcircfr $\left(T_{1}, T_{2}, \ldots, T_{n}\right)$ be a Tribonacci RSLPFL circulant matrix. Then

$$
\begin{align*}
\operatorname{det} \mathbf{L}= & \frac{2 T_{n}^{n}+T_{n} \Delta_{14}+\Delta_{15}}{\mathbb{L}_{n}^{2}+\mathbb{L}_{n-1}^{2}+\mathbb{L}_{2 n}-2 \mathbb{L}_{2(n-1)}+2 \mathbb{L}_{n+2}-4}  \tag{77}\\
& \times(-1)^{n(n-1) / 2},
\end{align*}
$$

where

$$
\begin{align*}
\Delta_{14}= & G_{n-1}^{2}-2 G_{2(n-1)}+2 G_{n-1}-2, \\
\Delta_{15}= & -4 T_{n+1}-T_{n+2}+2 G_{n}-2 G_{n+1} \\
& +G_{n}^{2}-G_{2 n},  \tag{78}\\
G_{n}= & v_{1}^{n}+v_{2}^{n}+v_{3}^{n}
\end{align*}
$$

and $v_{1}, v_{2}, v_{3}$ are the roots of the equation $-x^{3}+T_{n+1} x^{2}+\left(T_{n+2}+\right.$ 1) $x+T_{n}=0$.

Proof. The theorem can be proved by using Theorem 12 and (6).
2.4. Determinants of the RSFPLR Circulant Matrices and RSLPFL Circulant Matrices Involving Generalized Lucas Numbers

Theorem 14. Let $\mathbf{M}=\operatorname{RSFPLRcircfr}\left(\mathbb{L}_{1}, \mathbb{L}_{2}, \ldots, \mathbb{L}_{n}\right)$ be a generalized Lucas RSFPLR circulant matrix. Then

$$
\begin{equation*}
\operatorname{det} \mathbf{M}=\frac{\mathbb{L}_{n}^{n} \Delta_{16}+2 \mathbb{L}_{n}^{n-1}\left(\mathbb{L}_{n+2}-\mathbb{L}_{n}-11\right)+\Delta_{17}}{\Delta_{18}} \tag{79}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{16}= & H_{n}^{2}-H_{2 n}+2 H_{n+1}+2 H_{n} \\
\Delta_{17}= & \mathbb{L}_{n}^{n-1}\left(3+\mathbb{L}_{n+1}\right)\left(2 H_{n-1}+H_{n-1}^{2}-2 H_{2(n-1)}\right) \\
& +2\left(3+\mathbb{L}_{n+1}\right)^{n}, \\
\Delta_{18}= & 2 E_{n}+2 E_{n+1}+E_{n}^{2}-E_{2 n}+E_{n-1}^{2}-2 E_{2(n-1)}  \tag{80}\\
& +2 E_{n-1}+8 \\
H_{n}= & \mu_{1}^{n}+\mu_{2}^{n}+\mu_{3}^{n} \\
E_{n}= & \xi_{1}^{n}+\xi_{2}^{n}+\xi_{3}^{n}
\end{align*}
$$

$\mu_{1}, \mu_{2}, \mu_{3}$ are the roots of the equation $-\mathbb{L}_{n} x^{3}+\left(3-\mathbb{L}_{n+2}+\mathbb{L}_{n}\right) x^{2}$ $+\left(2-\mathbb{L}_{n+1}+\mathbb{L}_{n+2}\right) x+3+\mathbb{\unrhd}_{n+1}=0$, and $\xi_{1}, \xi_{2}, \xi_{3}$ are the roots of the equation $y^{3}+y^{2}+y-1=0$.

Proof. The matrix $\mathbf{M}$ has the form

$$
\mathbf{M}=\left(\begin{array}{cccc}
\mathbb{L}_{1} & \mathbb{L}_{2} & \cdots & \mathbb{L}_{n}  \tag{81}\\
-\mathbb{L}_{n} & \mathbb{L}_{1}+\mathbb{L}_{n} & \cdots & \mathbb{L}_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
-\mathbb{L}_{3} & -\mathbb{L}_{4}+\mathbb{L}_{3} & \cdots & \mathbb{L}_{2} \\
-\mathbb{L}_{2} & -\mathbb{L}_{3}+\mathbb{L}_{2} & \cdots & \mathbb{L}_{1}+\mathbb{L}_{n}
\end{array}\right) .
$$

According to Lemma 3, the Binet form (17), and (15), we have

$$
\begin{align*}
\operatorname{det} \mathbf{M}= & \prod_{i=1}^{n}\left(\mathbb{L}_{1}+\mathbb{L}_{2} \kappa_{i}+\cdots+\mathbb{L}_{n} \kappa_{i}^{n-1}\right) \\
= & \prod_{i=1}^{n}\left[\sum_{k=1}^{n} \sum_{j=1}^{3} t_{j}^{k} \kappa_{i}^{k-1}\right] \\
= & \prod_{i=1}^{n}\left[\sum_{j=1}^{3} \frac{t_{j}\left(1-t_{j}^{n} \kappa_{i}^{n}\right)}{1-t_{j} \kappa_{i}}\right]  \tag{82}\\
= & \prod_{i=1}^{n}\left[\frac{-\mathbb{L}_{n} \kappa_{i}^{3}+\left(3-\mathbb{L}_{n+2}+\mathbb{L}_{n}\right) \kappa_{i}^{2}}{-\kappa_{i}^{3}-\kappa_{i}^{2}-\kappa_{i}+1}\right. \\
& \left.\quad+\frac{\left(2-\mathbb{L}_{n+1}+\mathbb{L}_{n+2}\right) \kappa_{i}+3+\mathbb{L}_{n+1}}{-\kappa_{i}^{3}-\kappa_{i}^{2}-\kappa_{i}+1}\right] .
\end{align*}
$$

From Lemma 4 and (17), we obtain

$$
\begin{align*}
\prod_{i=1}^{n}[ & -\mathbb{L}_{n} \kappa_{i}^{3}+\left(3-\mathbb{L}_{n+2}+\mathbb{L}_{n}\right) \kappa_{i}^{2} \\
& \left.+\left(2-\mathbb{L}_{n+1}+\mathbb{L}_{n+2}\right) \kappa_{i}+3+\mathbb{L}_{n+1}\right] \\
= & \frac{1}{2} \mathbb{L}_{n}^{n}\left(H_{n}^{2}-H_{2 n}+2 H_{n+1}+2 H_{n}\right)  \tag{83}\\
& +\mathbb{Q}_{n}^{n-1}\left(\mathbb{L}_{n+2}-\mathbb{\mathbb { L }}_{n}-11\right) \\
& +\frac{1}{2} \mathbb{L}_{n}^{n-1}\left(3+\mathbb{\mathbb { L }}_{n+1}\right)\left(H_{n-1}^{2}-2 H_{2(n-1)}+2 H_{n-1}\right) \\
& +\left(3+\mathbb{L}_{n+1}\right)^{n},
\end{align*}
$$

where $H_{n}=\mu_{1}^{n}+\mu_{2}^{n}+\mu_{3}^{n}, \mu_{1}, \mu_{2}, \mu_{3}$ are the roots of the equation $-\mathbb{L}_{n} x^{3}+\left(3-\mathbb{L}_{n+2}+\mathbb{L}_{n}\right) x^{2}+\left(2-\mathbb{\unrhd}_{n+1}+\mathbb{L}_{n+2}\right) x+3+\mathbb{L}_{n+1}=0$ and

$$
\begin{align*}
& \prod_{i=1}^{n}\left(-\kappa_{i}^{3}-\kappa_{i}^{2}-\kappa_{i}+1\right) \\
&= \frac{1}{2}[ \tag{84}
\end{align*} E_{n}+2 E_{n+1}+E_{n}^{2}-E_{2 n} .
$$

where $E_{n}=\xi_{1}^{n}+\xi_{2}^{n}+\xi_{3}^{n}, \xi_{1}, \xi_{2}, \xi_{3}$ are the roots of the equation $y^{3}+y^{2}+y-1=0$. Hence,

$$
\begin{equation*}
\operatorname{det} \mathbf{M}=\frac{\mathbb{L}_{n}^{n} \Delta_{16}+2 \mathbb{L}_{n}^{n-1}\left(\mathbb{L}_{n+2}-\mathbb{L}_{n}-11\right)+\Delta_{17}}{\Delta_{18}}, \tag{85}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{16}= & H_{n}^{2}-H_{2 n}+2 H_{n+1}+2 H_{n}, \\
\Delta_{17}= & \mathbb{L}_{n}^{n-1}\left(3+\mathbb{L}_{n+1}\right)\left(2 H_{n-1}+H_{n-1}^{2}-2 H_{2(n-1)}\right) \\
& +2\left(3+\mathbb{L}_{n+1}\right)^{n}, \\
\Delta_{18}= & 2 E_{n}+2 E_{n+1}+E_{n}^{2}-E_{2 n}+E_{n-1}^{2}-2 E_{2(n-1)}  \tag{86}\\
& +2 E_{n-1}+8 \\
H_{n}= & \mu_{1}^{n}+\mu_{2}^{n}+\mu_{3}^{n} \\
E_{n}= & \xi_{1}^{n}+\xi_{2}^{n}+\xi_{3}^{n}
\end{align*}
$$

$\mu_{1}, \mu_{2}, \mu_{3}$ are the roots of the equation $-\mathbb{L}_{n} x^{3}+\left(3-\mathbb{\unrhd}_{n+2}+\right.$ $\left.\mathbb{L}_{n}\right) x^{2}+\left(2-\mathbb{-}_{n+1}+\mathbb{L}_{n+2}\right) x+3+\mathbb{L}_{n+1}=0$, and $\xi_{1}, \xi_{2}, \xi_{3}$ are the roots of the equation $y^{3}+y^{2}+y-1=0$.

Theorem 15. Let $\mathbf{N}=\operatorname{RSFPLRcircfr}\left(\mathbb{L}_{n}, \mathbb{L}_{n-1}, \ldots, \mathbb{L}_{1}\right)$ be a generalized Lucas RSFPLR circulant matrix. Then

$$
\begin{equation*}
\operatorname{det} \mathbf{N}=\frac{\Delta_{19}+\left(\mathbb{L}_{n}+3\right) \Delta_{20}+2\left(\mathbb{L}_{n}+3\right)^{n}-2 \mathbb{L}_{n}}{\mathbb{\mathbb { L }}_{n-1}^{2}-2 \mathbb{L}_{2(n-1)}+\mathbb{L}_{n}^{2}+\mathbb{L}_{2 n}+2 \mathbb{L}_{n+2}-4}, \tag{87}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{19}= & K_{n}^{2}-K_{2 n}+2 K_{n+1}+2 K_{n}-4 \mathbb{L}_{n+1} \\
& -2 \mathbb{L}_{n+2}+2,  \tag{88}\\
\Delta_{20}= & K_{n-1}^{2}-2 K_{2(n-1)}+2 K_{n-1},
\end{align*}
$$

$K_{n}=v_{1}^{n}+v_{2}^{n}+v_{3}^{n}, v_{1}, v_{2}, v_{3}$ are the roots of the equation $-x^{3}+$ $\left(\mathbb{L}_{n+1}-1\right) x^{2}+\left(\mathbb{L}_{n+2}-1\right) x+\mathbb{L}_{n}+3=0$.

Proof. The matrix $\mathbf{N}$ has the form

$$
\mathbf{N}=\left(\begin{array}{cccc}
\mathbb{L}_{n} & \mathbb{L}_{n-1} & \cdots & \mathbb{L}_{1}  \tag{89}\\
-\mathbb{L}_{1} & \mathbb{L}_{1}+\mathbb{L}_{n} & \cdots & \mathbb{L}_{2} \\
\vdots & \vdots & \ddots & \vdots \\
-\mathbb{L}_{n-2} & -\mathbb{L}_{n-3}+\mathbb{L}_{n-2} & \cdots & \mathbb{L}_{n-1} \\
-\mathbb{L}_{n-1} & -\mathbb{L}_{n-2}+\mathbb{L}_{n-1} & \cdots & \mathbb{L}_{1}+\mathbb{L}_{n}
\end{array}\right)
$$

According to Lemma 3, (17), and (15), we have

$$
\begin{align*}
\operatorname{det} \mathbf{N}= & \prod_{i=1}^{n}\left(\mathbb{L}_{n}+\mathbb{Q}_{n-1} \kappa_{i}+\cdots+\mathbb{Q}_{1} \kappa_{i}^{n-1}\right) \\
= & \prod_{i=1}^{n}\left[\sum_{k=0}^{n-1} \sum_{j=1}^{3} t_{j}^{n-k} \kappa_{i}^{k}\right] \\
= & \prod_{i=1}^{n}\left[\sum_{j=1}^{3} \frac{t_{j}^{n+1}-t_{j} \kappa_{i}^{n}}{t_{j}-\kappa_{i}}\right]  \tag{90}\\
= & \prod_{i=1}^{n}\left[\frac{-\kappa_{i}^{3}+\left(\mathbb{L}_{n+1}-1\right) \kappa_{i}^{2}}{-\kappa_{i}^{3}+\kappa_{i}^{2}+\kappa_{i}+1}\right. \\
& \left.\quad+\frac{\left(\mathbb{L}_{n+2}-1\right) \kappa_{i}+\mathbb{L}_{n}+3}{-\kappa_{i}^{3}+\kappa_{i}^{2}+\kappa_{i}+1}\right] .
\end{align*}
$$

By Lemma 4 and the Binet form (17), we obtain

$$
\begin{align*}
\prod_{i=1}^{n}[ & \left.-\kappa_{i}^{3}+\left(\mathbb{L}_{n+1}-1\right) \kappa_{i}^{2}+\left(\mathbb{L}_{n+2}-1\right) \kappa_{i}+\mathbb{\mathbb { L }}_{n}+3\right] \\
= & \frac{1}{2}\left(K_{n}^{2}-K_{2 n}+2 K_{n+1}+2 K_{n}\right. \\
& \left.\quad-2 \mathbb{L}_{n+1}-2 \mathbb{L}_{n+1}-2 \mathbb{L}_{n+2}+2\right)  \tag{91}\\
& +\frac{1}{2}\left(\mathbb{L}_{n}+3\right)\left(K_{n-1}^{2}-2 K_{2(n-1)}+2 K_{n-1}\right) \\
& +\left(\mathbb{L}_{n}+3\right)^{n}-\mathbb{L}_{n},
\end{align*}
$$

where $K_{n}=\nu_{1}^{n}+v_{2}^{n}+v_{3}^{n}, v_{1}, v_{2}, v_{3}$ are the roots of the equation $-x^{3}+\left(\mathbb{L}_{n+1}-1\right) x^{2}+\left(\mathbb{L}_{n+2}-1\right) x+\mathbb{L}_{n}+3=0$ and

$$
\begin{align*}
\prod_{i=1}^{n}( & \left.-\kappa_{i}^{3}+\kappa_{i}^{2}+\kappa_{i}+1\right) \\
= & \frac{1}{2}\left(2 \mathbb{L}_{n-1}+\mathbb{\mathbb { L }}_{n-1}^{2}-2 \mathbb{L}_{2(n-1)}\right)  \tag{92}\\
& +\frac{1}{2}\left(2 \mathbb{L}_{n}+\mathbb{L}_{n}^{2}+\mathbb{Q}_{2 n}+2 \mathbb{L}_{n+1}-4\right) \\
= & \frac{1}{2}\left[\mathbb{L}_{n-1}^{2}-2 \mathbb{L}_{2(n-1)}+\mathbb{\mathbb { L }}_{n}^{2}+\mathbb{\mathbb { L }}_{2 n}+2 \mathbb{L}_{n+2}-4\right]
\end{align*}
$$

Thus,

$$
\begin{equation*}
\operatorname{det} \mathbf{N}=\frac{\Delta_{19}+\left(\mathbb{L}_{n}+3\right) \Delta_{20}+2\left(\mathbb{L}_{n}+3\right)^{n}-2 \mathbb{L}_{n}}{\mathbb{L}_{n-1}^{2}-2 \mathbb{L}_{2(n-1)}+\mathbb{L}_{n}^{2}+\mathbb{\unrhd}_{2 n}+2 \mathbb{L}_{n+2}-4} \tag{93}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta_{19}= & K_{n}^{2}-K_{2 n}+2 K_{n+1}+2 K_{n}-4 \mathbb{\unrhd}_{n+1} \\
& -2 \mathbb{\square}_{n+2}+2  \tag{94}\\
\Delta_{20}= & K_{n-1}^{2}-2 K_{2(n-1)}+2 K_{n-1}, \\
K_{n}= & \nu_{1}^{n}+\nu_{2}^{n}+\nu_{3}^{n}
\end{align*}
$$

and $v_{1}, v_{2}, v_{3}$ are the roots of the equation $-x^{3}+\left(\mathbb{L}_{n+1}-1\right) x^{2}+$ $\left(\mathbb{L}_{n+2}-1\right) x+\mathbb{L}_{n}+3=0$.

Theorem 16. Let $\mathbf{P}=\operatorname{RSLPFLcircfr}\left(\mathbb{L}_{1}, \mathbb{L}_{2}, \ldots, \mathbb{L}_{n}\right)$ be a generalized Lucas RSLPFL circulant matrix. Then

$$
\begin{align*}
\operatorname{det} \mathbf{P}= & \frac{\Delta_{19}+\left(\mathbb{L}_{n}+3\right) \Delta_{20}+2\left(\mathbb{L}_{n}+3\right)^{n}-2 \mathbb{L}_{n}}{\mathbb{L}_{n-1}^{2}-2 \mathbb{L}_{2(n-1)}+\mathbb{L}_{n}^{2}+\mathbb{L}_{2 n}+2 \mathbb{L}_{n+2}-4}  \tag{95}\\
& \times(-1)^{n(n-1) / 2},
\end{align*}
$$

where

$$
\begin{align*}
& \Delta_{19}= K_{n}^{2}-K_{2 n}+2 K_{n+1}+2 K_{n}-4 \mathbb{\unrhd}_{n+1} \\
&-2 \mathbb{L}_{n+2}+2 \\
& \Delta_{20}= K_{n-1}^{2}-2 K_{2(n-1)}+2 K_{n-1}  \tag{96}\\
& K_{n}=\nu_{1}^{n}+\nu_{2}^{n}+v_{3}^{n}
\end{align*}
$$

and $\nu_{1}, v_{2}, v_{3}$ are the roots of the equation $-x^{3}+\left(\mathbb{L}_{n+1}-1\right) x^{2}$ $+\left(\mathbb{\square}_{n+2}-1\right) x+\mathbb{\square}_{n}+3=0$.

Proof. The theorem can be proved by using Theorem 15 and (6).

## 3. Conclusion

The determinant problems of the RSFPLR circulant matrices and RSLPFL circulant matrices involving the Perrin, Padovan, Tribonacci, and the generalized Lucas number
are considered in this paper. The explicit determinants are presented by using some terms of these numbers. The techniques used herein are based on the inverse factorization of polynomial. It is important to develop new theories and methods and to modify and refine the well-known techniques, for solving differential equations. On the basis of existing application situation [1-11], we will exploit solving some differential equations based on the RSFPLR circulant matrices and RSLPFL circulant matrices.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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